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ELLIPTIC INTEGRAL REPRESENTATION OF AXIALLY SYMMETRIC FLOWS*

BY

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1. Introduction. In a recent paper[1] Alexander Weinstein reconsidered axially symmetric potential flows of an ideal, incompressible fluid and in particular discussed the flow generated by various axially symmetric distributions of sources. Weinstein based his analysis on the classical[2] representation of the corresponding velocity potentials and stream functions in terms of improper integrals involving Bessel functions. In the present paper, which was suggested by and is complementary to Weinstein's work, the potentials, stream functions, and velocity components for basic axially symmetric distributions of sources or vorticity over the circumference or the interior of a circle, are established in terms of elliptic integrals of the first and second kind.

The relative merit of the alternative representation used here appears to be twofold. First, the approach via elliptic integrals yields an analytically more transparent description than that afforded by the representation through discontinuous integrals of Bessel functions. In particular, further insight is gained into the cyclic character of Stoke's stream function or the velocity potential in cases where the distribution of singularities is such as to give rise to a multiply connected domain of regularity in the meridional half-plane. The results here referred to are of course in complete agreement with those of Weinstein who originally clarified this aspect in connection with the stream function for the source ring. Secondly, the use of functions which have been tabulated exhaustively facilitates the physical interpretation of the results and renders possible the complete determination of the corresponding streamline patterns. By superposition of the foregoing basic axially symmetric flows and appropriately chosen uniform streams one may obtain a variety of technically significant flows around solid and annular shaped bodies and half-bodies of revolution. A comprehensive study of the various body shapes and associated streamline patterns so obtainable is currently in progress in cooperation with V. L. Streeter and P. C. Chu, of Illinois Institute of Technology, who have independently completed by aid of numerical integrations several flow patterns based on the homogeneous source disc.

It should be recalled at this place that, historically, elliptic integrals were introduced early in connection with problems of rotational symmetry in potential theory. Thus the simple formula for the velocity potential of a homogeneous source ring appears in classical treatises as the gravitational potential of a homogeneous circumference, and the electro-magnetic analogue of the stream function for the vortex ring was given by

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[1]Alexander Weinstein, *On axially symmetric flows*, Quart. of Applied Math. 4, 429-444, 1948.

[2]E. Beltrami, *Opere Matematiche*, Vol. 3, U. Hoepli, Milano, 1911.

Maxwell[3] in terms of the complete integrals of the first and second kind. It appears, however, that the complete integral of the third kind and its relation to the incomplete integral of the second kind and Jacobian elliptic functions, have been neglected in the problems under consideration. In this connection it is interesting to note that G. M. Minchin[4] obtained an expression involving integrals of the third kind for what corresponds to the velocity potential of the vortex ring, but failing to apply the foregoing reductions, gives preference to the open representation in terms of a series of spherical harmonics.

Finally, the authors understand that A. Van Tuyl, of the Naval Ordinance Laboratory, by formal transformation of the corresponding Bessel integrals has recently reached results which partly overlap with those given here.

2. The governing equations. For the sake of convenience we cite here the basic equations governing the steady irrotational flow of an ideal incompressible fluid in the presence of axial symmetry. Referring the motion to the cylindrical coordinates (x, ρ, θ) where the x-axis is assumed coincident with the axis of symmetry, there exists a velocity potential $\phi(x, \rho)$ which satisfies Laplace's equation

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) = 0 \tag{2.1}$$

at all non-singular points of the field. Equation (2.1) may be considered as the condition of integrability assuring the existence of Stokes' stream function $\psi(x, \rho)$ as defined by

$$\frac{\partial \psi}{\partial x} = -\rho \frac{\partial \phi}{\partial \rho}, \qquad \frac{\partial \psi}{\partial \rho} = \rho \frac{\partial \phi}{\partial x} \tag{2.2}$$

Choosing the additive constant in $\psi(x, \rho)$ such that

$$\psi(-\infty, 0) = 0$$

we can represent $\psi(x, \rho)$ by the line integral

$$\psi(x, \rho) = \int_{(-\infty,0)}^{(x,\rho)} \left[-\rho \frac{\partial \phi}{\partial \rho}\, dx + \rho \frac{\partial \phi}{\partial x}\, d\rho\right] \tag{2.3}$$

The choice of the path of integration is immaterial in simply connected fields.

From (2.2) follows

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \rho}\right) = 0 \tag{2.4}$$

The axial and radial velocity components are then given by

$$v_x = \frac{\partial \phi}{\partial x} = \frac{1}{\rho}\frac{\partial \psi}{\partial \rho}, \qquad v_\rho = \frac{\partial \phi}{\partial \rho} = \frac{-1}{\rho}\frac{\partial \psi}{\partial x} \tag{2.5}$$

The curves $\psi(x, \rho) =$ constant are the streamlines in the meridional half-plane and $2\pi(\psi_2 - \psi_1)$ constitutes the total flow between the streamsurfaces generated by $\psi = \psi_2$, $\psi = \psi_1$.

As was emphasized by Weinstein and is apparent from the definition (2.3), a given

[3]J. C. Maxwell, *Electricity and magnetism*, Clarendon Press, Oxford, 1892, p. 339.

[4]G. M. Minchin, *The magnetic field of a circular current*, Phil. Mag., **35**, 354-365, 1893.

velocity potential $\phi(x, \rho)$ whose derivatives are single-valued guarantees the single-valuedness of the associated stream function $\psi(x, \rho)$ if the domain of regularity in the half-plane $\rho \geq 0$ is simply connected. This condition, however, is in general not satisfied for source distributions off the axis of symmetry.

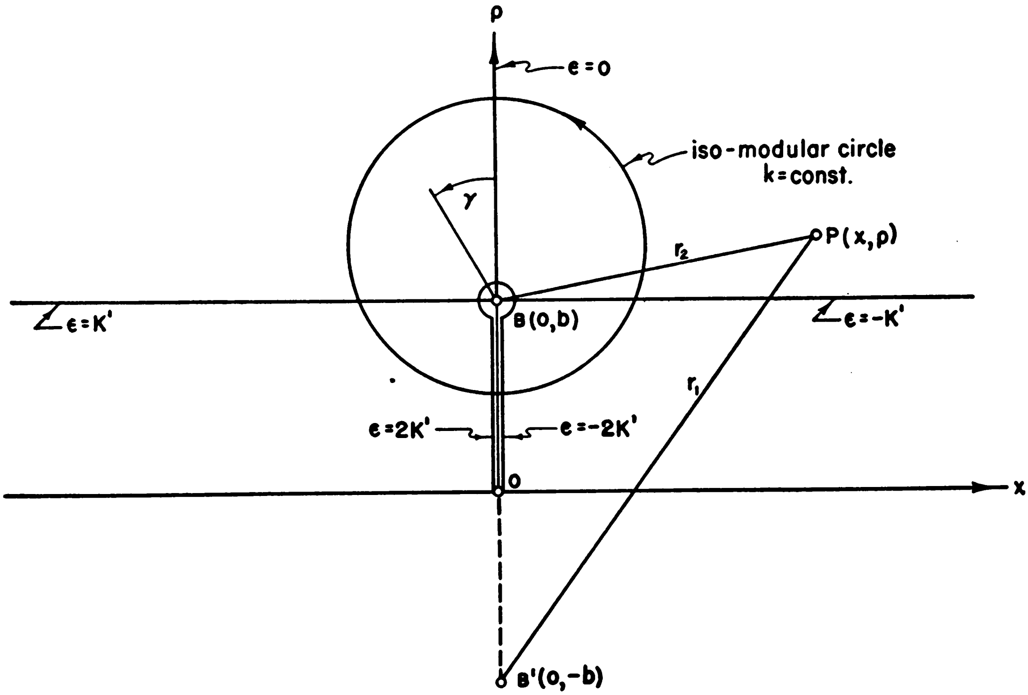

FIG. 1: Position Parameters

3. Velocity potential and velocity components of the source ring. Consider a homogeneous distribution of sources of total strength m (total flux $4\pi m$) along the circle $x = 0$, $\rho = b$. The distance R from a point $P(x, \rho, 0)$ of the field to a point $Q(0, b, \theta)$ on the source ring is

$$R = (x^2 + b^2 + \rho^2 - 2b\rho \cos \theta)^{1/2} \tag{3.1}$$

$$= r_1(1 - \mathrm{k}^2 \sin^2 \varphi)^{1/2} \tag{3.2}$$

in which

$$\varphi = \frac{\pi - \theta}{2}, \qquad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \tag{3.3}$$

$$\mathrm{k}^2 = \frac{4b\rho}{r_1^2} = 1 - \frac{r_2^2}{r_1^2} \tag{3.4}$$

$$\begin{aligned} r_1^2 &= x^2 + (\rho + b)^2 \\ r_2^2 &= x^2 + (\rho - b)^2 \end{aligned} \tag{3.5}$$

so that r_1, r_2 denote the distances $B'P$ and BP respectively in Fig. 1.

The velocity potential at P is given by

$$\phi(x, \rho) = -\frac{m}{2\pi} \int_0^{2\pi} \frac{d\theta}{R} \tag{3.6}$$

To rationalize R, we introduce u by means of

$$\left.\begin{aligned} \sin \varphi &= \mathrm{sn}\,(u, \mathrm{k}) \\ \cos \varphi &= \mathrm{cn}\,(u, \mathrm{k}) \end{aligned}\right. \qquad -\mathrm{K} \leq u \leq \mathrm{K} \tag{3.7}$$

which gives

$$R = r_1 \,\mathrm{dn}\,(u,\ \mathrm{k}), \qquad \frac{d\varphi}{R} = \frac{du}{r_1} = -\frac{1}{2}\frac{d\theta}{R} \tag{3.8}$$

and thus

$$\phi(x, \rho) = -\frac{m}{\pi r_1} \int_{-\mathrm{K}}^{\mathrm{K}} du = -\frac{2m\mathrm{K}}{\pi r_1} \tag{3.9}$$

where K denotes the complete elliptic integral of the first kind for the modulus k. Proceeding to the limit as $\mathrm{k} \to 0$, we confirm that along the x-axis

$$\phi(x, 0) = \frac{-m}{r_0} \tag{3.10}$$

where,

$$r_0 = r_1(at \rho = 0) = r_2(at \rho = 0) = (x^2 + b^2)^{1/2} \tag{3.11}$$

The components of the velocity field generated by the potential (3.9) are obtained from (2.5) by aid of the identity[5]

$$\frac{d\mathrm{K}}{d\mathrm{k}} = \frac{1}{\mathrm{k}(\mathrm{k}')^2}\,[\mathrm{E} - (\mathrm{k}')^2\, \mathrm{K}] \tag{3.12}$$

Here k′ is the complementary modulus,

$$\mathrm{k}' = (1 - \mathrm{k}^2)^{1/2} = \frac{r_2}{r_1} \tag{3.13}$$

and E stands for the complete elliptic integral of the second kind referred to the modulus k. The computation yields

$$\left.\begin{aligned} v_x(x, \rho) &= \frac{2mx}{\pi r_1 r_2^2}\,\mathrm{E} \\ v_\rho(x, \rho) &= \frac{m}{\pi \rho r_1}\left(\mathrm{K} + \frac{\rho^2 - x^2 - b^2}{r_2^2}\,\mathrm{E}\right) \end{aligned}\right\} \tag{3.14}$$

and, in particular,

$$v_x(x, 0) = \frac{mx}{r_0^3}, \qquad v_\rho(x, 0) = 0 \tag{3.15}$$

[5]For this and subsequent identities associated with elliptic functions and integrals, see for example E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed., University Press, Cambridge, 1935, Ch. XXII.

The behavior of ϕ, v_x, v_ρ in the vicinity of the source ring, i.e., in the neighborhood of the singular point $B(0, b)$, is characterized by the following limits as r_2 and hence k′ tend to zero:

$$\phi + \frac{m}{\pi b} \log \frac{4r_1}{r_2} \to 0, \qquad v_x + \frac{m \sin \gamma}{\pi b r_2} \to 0, \qquad v_\rho - \frac{m \cos \gamma}{\pi b r_2} \to 0. \tag{3.16}$$

Here γ denotes the polar angle at B (Fig. 1). Thus,

$$\sin \gamma = \frac{-x}{r_2}, \qquad \cos \gamma = \frac{\rho - b}{r_2}, \qquad -\pi \leq \gamma \leq \pi \tag{3.17}$$

4. Stream function of the source ring. The stream function associated with the velocity potential (3.9) may be determined directly by virtue of its intrinsic hydrodynamic significance. Recalling that $2\pi[\psi(x, \rho) - \psi(x, 0)]$ represents the total flow through the circle of radius ρ centered on the x-axis and lying in a plane perpendicular to the axis, we have within an unessential additive constant

$$\psi(x, \rho) = \frac{m}{2\pi} \int_0^{2\pi} \int_0^{\rho} \frac{x}{r^3} \mu \, d\mu \, d\theta \tag{4.1}$$

where

$$r = (x^2 + \mu^2 + b^2 - 2\mu b \cos \theta)^{1/2} \tag{4.2}$$

denotes the distance of the point $B(0, b)$ in the plane $\theta = 0$ from the point $Q(x, \mu, \theta)$. The surface integral in (4.1) is the solid angle Ω subtended at $B(0, b)$ by the foregoing circle whose trace in the meridional half-plane is $P(x, \rho)$. We write,

$$\psi(x, \rho) = \frac{m}{2\pi} \Omega(0, b; x, \rho) \tag{4.3}$$

It is, however, more expedient to determine $\psi(x, \rho)$ from (2.5) and (3.6). Accordingly,

$$\frac{\partial \psi}{\partial x} = -\rho \frac{\partial \phi}{\partial \rho} = \frac{m}{\pi} \int_0^{\pi} \frac{b\rho \cos \theta - \rho^2}{R^3} \, d\theta \tag{4.4}$$

so that

$$\Delta\psi = \psi(x, \rho) - \psi(0, \rho) = \frac{m}{\pi} \int_0^x \int_0^{\pi} \frac{b\rho \cos \theta - \rho^2}{R^3} \, d\theta \, dx \tag{4.5}$$

Performing the elementary integration with respect to x,

$$\Delta\psi = \frac{m}{\pi} \int_0^{\pi} \frac{(b\rho \cos \theta - \rho^2)x}{(b^2 + \rho^2 - 2b\rho \cos \theta)R} \, d\theta \tag{4.6}$$

and using the transformations (3.3), (3.7),

$$\Delta\psi = \frac{-2mx\rho}{\pi(\rho + b)r_1} \int_0^{\mathrm{K}} du + \frac{4mbx\rho(b - \rho)}{\pi r_1(b + \rho)} \int_0^{\mathrm{K}} \frac{\mathrm{sn}^2 (u, \mathrm{k}) \, du}{(b + \rho)^2 - \mathrm{k}^2 r_1^2 \, \mathrm{sn}^2 (u, \mathrm{k})} \tag{4.7}$$

With a view toward transforming the last integral in (4.7) into the standard form of the elliptic integral of the third kind, we introduce the complex parameter a through

$$\mathrm{sn}\,(\mathrm{a}, \mathrm{k}) = \frac{r_1}{\rho + b}, \qquad \mathrm{cn}\,(\mathrm{a}, \mathrm{k}) = \frac{-ix}{\rho + b}, \qquad \mathrm{dn}(\mathrm{a}, \mathrm{k}) = \frac{\rho - b}{\rho + b} \tag{4.8}$$

and reach

$$\Delta\psi = \frac{-2m}{\pi}\left[\frac{x\rho}{(\rho + b)r_1}\,\mathrm{K} + \frac{i}{2}\,\Pi(\mathrm{K},\,\mathrm{a})\right] \tag{4.9}$$

where

$$\Pi(\mathrm{K},\,\mathrm{a}) = \int_0^{\mathrm{K}} \frac{\mathrm{k}^2\,\mathrm{sn\,a\;cn\,a\;dn\,a\,sn}^2\,u}{1 - \mathrm{k}^2\,\mathrm{sn}^2\,\mathrm{a}\,\mathrm{sn}^2\,u}\,du \tag{4.10}$$

is the standard form of the complete elliptic integral of the third kind.

From (4.8), sn (a, k) > 1 which characterizes the integral as being of circular type according to Legendre's classification. Here an explicitly real representation in terms of a real parameter ϵ is obtained by putting

$$\begin{gathered} \mathrm{a} = \mathrm{K} + i\epsilon \\ -\,2\mathrm{K}' \leq \epsilon \leq 2\mathrm{K}' \end{gathered} \tag{4.11^6}$$

The integral (4.10) can now be expressed as follows:

$$\begin{aligned} \Pi(\mathrm{K},\,\mathrm{a}) &= \Pi(\mathrm{K},\,\mathrm{K} + i\epsilon) \\ &= \mathrm{K}Z(\mathrm{K} + i\epsilon,\,\mathrm{k}) + n\pi i \\ &= \mathrm{KE}(\mathrm{K} + i\epsilon,\,\mathrm{k}) - (\mathrm{K} + i\epsilon)\mathrm{E} + n\pi i \end{aligned} \tag{4.12}$$

valid within the open ranges

$$(2n - 1)\mathrm{K}' < \epsilon < (2n + 1)\mathrm{K}' \tag{4.13}$$

in which n denotes an integer. For the values

$$\epsilon = (2n + 1)\mathrm{K}' \qquad \text{we have} \qquad \Pi(\mathrm{K},\,\mathrm{a}) = 0\,. \tag{4.14}$$

$Z(\mathrm{a}, \mathrm{k})$ and $\mathrm{E}(\mathrm{a}, \mathrm{k})$ stand for Jacobi's Zeta function and the incomplete elliptic integral of the second kind of the argument a. By addition theorems and Jacobi's imaginary transformation for sn u, cn u, dn u, $\mathrm{E}(u)$ as well as Legendre's relation

$$\mathrm{KE}' + \mathrm{EK}' - \mathrm{KK}' = \frac{\pi}{2} \tag{4.15}$$

equations (4.8) to (4.13) lead to

$$\psi(x,\,\rho) = \frac{-m}{\pi}\left[\pi + \frac{x}{r_1}\,\mathrm{K} - \Lambda(\epsilon,\,\mathrm{k}')\right] \tag{4.16}$$

with

$$\Lambda(\epsilon,\,\mathrm{k}') = \mathrm{KE}(\epsilon,\,\mathrm{k}') + (\mathrm{E} - \mathrm{K})\epsilon = \mathrm{K}Z(\epsilon,\,\mathrm{k}') + \frac{\pi\epsilon}{2\mathrm{K}'} \tag{4.17}$$

and

$$\mathrm{sn}\,(\epsilon,\,\mathrm{k}') = \frac{-x}{r_2}, \qquad \mathrm{cn}\,(\epsilon,\,\mathrm{k}') = \frac{\rho - b}{r_2}, \qquad \mathrm{dn}(\epsilon,\,\mathrm{k}') = \frac{\rho + b}{r_1} \tag{4.18}$$

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[6] K′ and E′ designate the complete elliptic integrals of the first and second kind respectively for the modulus k′.

The appearance of n in (4.12) and its disappearance from (4.16) and (4.17) is explained as follows: The integral (4.10), having as path of integration the straight line segment (0, K), is single-valued in the neighborhood of $x = 0$, $\rho = b$. The quantity a has the form given in (4.11), $a = K + i\epsilon$, where ϵ increases by $4K'$ in a circuit around $(0, b)$, and $\Pi(K, K + i\epsilon)$ is periodic in ϵ with the period $2K'$. On the other hand, $KZ(K + i\epsilon, k)$ is infinitely many-valued around $(0, b)$. Now, in the relations (4.12), n has to appear to produce single-valued right-hand members, the left-hand member being single-valued. However, since ψ varies continuously in a circuit around $(0, b)$, it is correct to replace $\Pi(K, a)$ in the process of computation of ψ by $KZ(a, k)$ plus a single additive constant for the entire range $-2K' < \epsilon < 2K'$. That is what has been done to obtain the results as expressed by (4.16) and (4.17) from which n has disappeared.[7]

5. Discussion. Comparing (4.18) and (3.17), we note that

$$\begin{gathered} \gamma = am(\epsilon, k') \\ \epsilon = F(\gamma, k') = \int_0^\gamma \frac{dt}{(1 - (k')^2 \sin^2 t)^{1/2}} \end{gathered} \tag{5.1}$$

where $F(\gamma, k')$ is the incomplete elliptic integral of the first kind. The functions $k'(x, \rho)$ defined by (3.13), (3.5) and $\epsilon(x, \rho)$ given by (4.18) or equivalently by (3.17) and (5.1) may be regarded as curvilinear coordinates in the half-plane $\rho \geq 0$. The coordinate lines $k' =$ constant ($k =$ constant) are the "iso-modular" circles

$$\left(\frac{x}{b}\right)^2 + \left[\frac{\rho}{b} - \frac{1 + (k')^2}{k^2}\right]^2 = \frac{4(k')^2}{k^4} \tag{5.2}$$

As any iso-modular circle is traversed in a counterclockwise sense, ϵ increases by $4K'$. In view of (5.1),

$$\left.\begin{aligned} \epsilon &= 0 &&\text{for} \quad \gamma = 0 \\ \epsilon &= \pm K' &&\text{for} \quad \gamma = \pm\frac{\pi}{2} \\ \epsilon &= \pm 2K' &&\text{for} \quad \gamma = \pm\pi \\ &\lim_{k' \to 0} \epsilon = \gamma \end{aligned}\right\} \tag{5.3}$$

so that ϵ tends toward the polar angle γ as $P(x, \rho) \to B(0, b)$ (see Fig. 1).

The auxiliary function $\Lambda(\epsilon, k')$, defined by (4.15), is of basic significance for what follows. $\Lambda(\epsilon, k')$ is multiple valued for $-\infty < \epsilon < \infty$. For a closed path encircling B once we obtain for the circulation of Λ

$$\Lambda(\epsilon + 4K', k') - \Lambda(\epsilon, k') = 2\pi \tag{5.4}$$

Here we face the alternative of either admitting a many-valued Λ and permitting a circulation, or of cutting the semi-plane along $x = 0 \quad 0 \leq \rho \leq b$ so that Λ becomes single-valued but discontinuous along the cut. The ranges for ϵ and γ as previously announced by (4.11) and (3.17) correspond to such a cut.

[7]The arbitrary additive constant was chosen consistent with $\psi(0, \rho) = -m$ for $\rho > b$.

The local behavior of Λ in the vicinity of point B is characterized by

$$\lim_{k'\to 0} \Lambda(\epsilon, k') = \gamma \tag{5.5}$$

Confining the discussion to the range $-2K' \leq \epsilon \leq 2K'$, we record the values:

$$\left.\begin{array}{ll} x > 0,\ \rho = 0 & \Lambda = \pi\left(\dfrac{x}{2r_0} - 1\right) \\ x < 0,\ \rho = 0 & \Lambda = \pi\left(\dfrac{x}{2r_0} + 1\right) \\ x = +0,\ \rho < b & \Lambda = -\pi \\ x = -0,\ \rho < b & \Lambda = \pi \\ x = 0,\ \rho > b & \Lambda = 0 \end{array}\right\} \tag{5.6}$$

Equations (5.6) reveal the discontinuity of Λ along $x = 0,\ 0 \leq \rho < b$ in presence of a cut.

It follows from (5.5) and (5.6) that in absence of a cut the stream function $\psi(x, \rho)$, given by (4.16), is also a multiple valued function whose circulation is twice the total strength m of the source ring.

$$\psi(\epsilon + 4K', k') - \psi(\epsilon, k') = 2m \tag{5.7}$$

Again, for the cut domain, $-2K' \leq \epsilon \leq 2K'$, we find in further agreement with Weinstein[8]

$$\left.\begin{array}{lll} \psi(x, 0) = \begin{cases} -2m & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} & & \\ \psi(+0, \rho) = -2m & \text{for} & \rho < b \\ \psi(-0, \rho) = 0 & \text{for} & \rho < b \\ \psi(0, \rho) = -m & \text{for} & \rho > b \end{array}\right\} \tag{5.8}$$

Observing

$$\lim_{r_2\to 0} \psi = m\left(\frac{\gamma}{\pi} - 1\right) \tag{5.9}$$

we recognize the local significance of ψ in the neighborhood of P as essentially that of the polar angle γ.

As pointed out earlier, the multiple-valued or, alternatively, discontinuous character of $\psi(x, \rho)$ was to be anticipated in the presence of a multiply connected domain of regularity of $\phi(x, \rho)$ whose boundary here consists of the x-axis and the point $B(0, b)$. It should be pointed out that the cyclic nature of the stream function is also immediately evident from the solid angle interpretation underlying (4.3).

[8]See *loc. cit.*

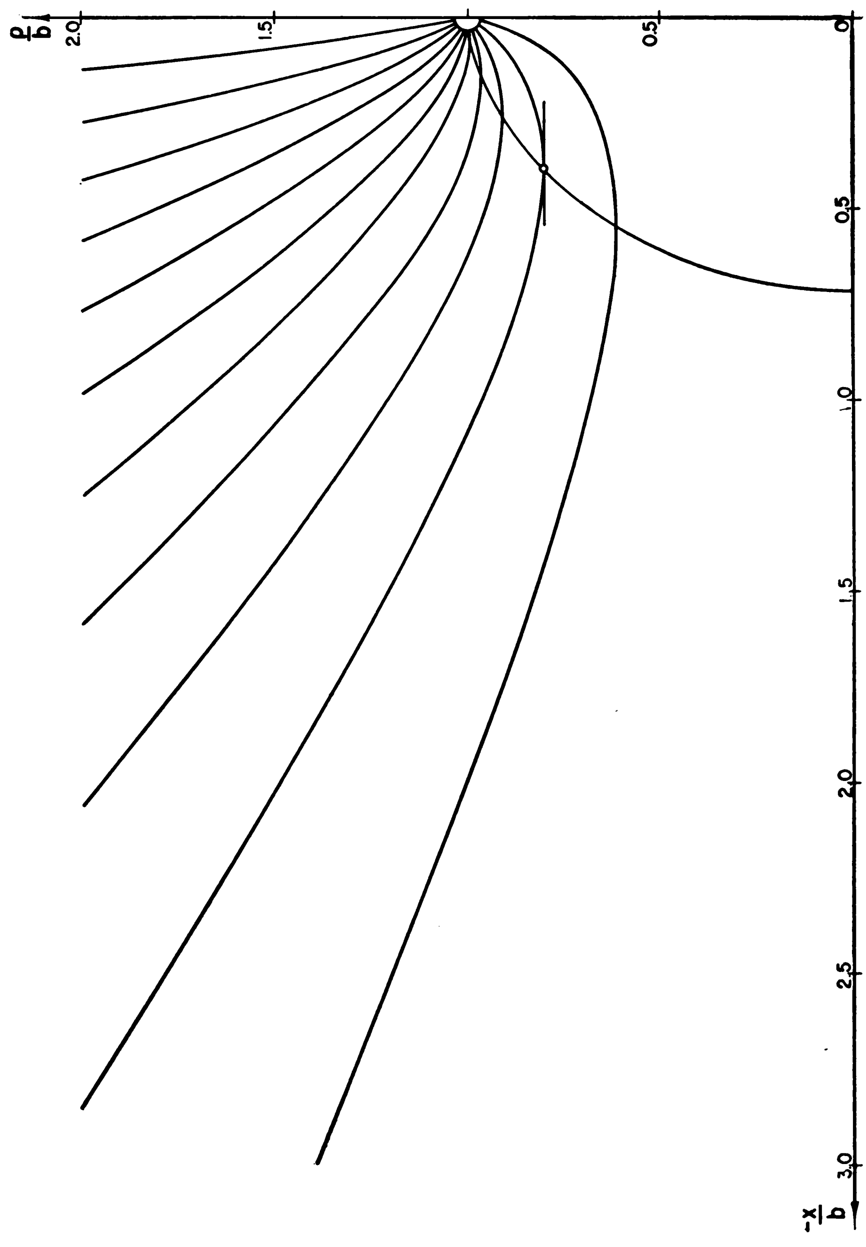

FIG. 2: Streamlines for the Source Ring

Figure 2 shows the streamline pattern of the source ring, i.e., the curves $\psi(x, \rho) =$ constant, for the quadrant $x \leq 0$, $\rho \geq 0$ of the meridional half-plane. The curves given correspond to equal increments in ψ. The diagram also shows the locus $v_\rho(x, \rho) = 0$.

6. Results for the source disc. We consider next a homogeneous distribution of sources of total strength m over the circular region $x = 0$, $0 \leq \rho \leq b$. The determination of the velocity potential $\phi(x, \rho)$ and of the associated stream function $\psi(x, \rho)$ may be achieved by integration with respect to b of the corresponding functions for the source ring. We cite directly the results obtained.

$$\left.\begin{aligned}
\phi(x, \rho) &= \frac{2m}{\pi b^2}\left[\frac{\rho^2 - b^2}{r_1}\mathrm{K} - r_1\mathrm{E} - x\Lambda\right] \\
\phi(x, 0) &= \frac{-2m}{|x| + r_0} \\
\phi &\to \frac{-4m}{\pi b} \qquad \text{as} \qquad r_2 \to 0
\end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned}
v_x(x, \rho) &= \frac{-2m}{\pi b^2}\left[\frac{x}{r_1}\mathrm{K} + \Lambda\right] \\
v_x(x, 0) &= \frac{\pm 2m}{r_0(r_0 + |x|)} \qquad \text{for} \qquad x \gtrless 0 \\
v_x(\pm 0, \rho) &= \pm\frac{2m}{b^2} \qquad \text{for} \qquad \rho < b \\
v_x(0, \rho) &= 0 \qquad \text{for} \qquad \rho > b \\
v_x &\to \frac{-2m\gamma}{\pi b^2} \qquad \text{as} \qquad r_2 \to 0
\end{aligned}\right\} \tag{6.2}$$

$$\left.\begin{aligned}
v_\rho(x, \rho) &= \frac{2mr_1}{\pi b^2 \rho}\left[\frac{x^2 + b^2 + \rho^2}{r_1^2}\mathrm{K} - \mathrm{E}\right] \\
v_\rho(x, 0) &= 0 \\
v_\rho - \frac{2m}{\pi b^2}&\left[\log\frac{4r_1}{r_2} - 2\right] \to 0 \qquad \text{as} \qquad r_2 \to 0
\end{aligned}\right\} \tag{6.3}$$

$$\left.\begin{aligned}
\psi(x, \rho) &= \frac{m}{\pi b^2}\left[xr_1\mathrm{E} - \frac{x(x^2 + 2\rho^2 + 2b^2)}{r_1}\mathrm{K} + (b^2 - \rho^2)\Lambda - \pi b^2\right] \\
\psi(x, 0) &= \begin{cases} -2m & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \\
\psi(+0, \rho) &= m\left(\frac{\rho^2}{b^2} - 2\right) \qquad \text{for} \qquad \rho \le b \\
\psi(-0, \rho) &= -\frac{m\rho^2}{b^2} \qquad \text{for} \qquad \rho \le b \\
\psi(0, \rho) &= -m \qquad \text{for} \qquad \rho \ge b \\
\psi &\to -m \qquad \text{as} \qquad r_2 \to 0
\end{aligned}\right\} \tag{6.4}$$

In agreement with Weinstein, we note that Stoke's stream function is single valued in the simply connected domain of definition whose boundary consists of the entire x-axis and of the cut along the segment $x = 0$, $0 \le \rho \le b$. Moreover, ϕ, v_x, v_ρ, and ψ are single-valued throughout the foregoing region which corresponds to $0 \le k' \le 1$, $-2K' \le \epsilon \le 2K'$. We observe that v_x is constant over each one of the two faces of the disc and that

$$v_x(+0,\ \rho) - v_x(-0,\ \rho) = \frac{4m}{b^2} = \sigma, \qquad 0 \le \rho \le b \tag{6.5}$$

where σ is the flux per unit area of the source disc. The velocity component v_ρ, on the other hand, has no discontinuity as we pass from one face of the disc to the other, but has a logarithmic singularity at point B, i.e., at the edge of the disc. These results are in agreement with the general theory of source sheets.

The solution corresponding to a homogeneous source distribution over an annular region bounded by two concentric circles is of course at once obtainable from the results just given by application of the principle of superposition.

7. The vortex ring. We now turn to the flow generated by a vortex filament of circulation Γ along the circle $x = 0$, $\rho = b$. According to a theorem of vortex theory,[9] the velocity potential at any point of the space due to a single closed vortex filament is proportional to the solid angle which it subtends at that point. Thus for the case under consideration, using the notation introduced in (4.3),

$$\phi(x,\ \rho) = \frac{-\Gamma}{4\pi}\,\Omega(x,\ \rho;\ 0,\ b) \tag{7.1}$$

Comparing (4.3) and (7.1), we note a reciprocal relationship between the stream function of the source ring and the potential of the vortex ring. By virtue of (7.1), (4.3) and (4.16), the explicit formula for $\phi(x, \rho)$ is immediate.

Alternatively, we recall[10] that the flow corresponding to a closed vortex filament is identical with that induced by a uniform distribution of doublets over any surface bounded by it. The axes of the doublets are assumed to be everywhere perpendicular to the surface, and the surface density (strength per unit area) of the doublet sheet is equal to $\Gamma/4\pi$, where Γ is the circulation of the vortex. The solution for the vortex ring therefore coincides with that for a homogeneous doublet disc, which in turn is obtainable by differentiating with respect to x the solution for the uniform source disc given in the preceding section. Thus,

$$\phi\ (\text{Vortex Ring}) = \frac{-b^2\Gamma}{4m}\,\frac{\partial}{\partial x}\,\phi\ (\text{Source disc}) \tag{7.1}$$

etc.

Carrying out the required differentiations, we reach:

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[9]See, for example, H. Lamb, *Hydrodynamics*, 5th ed., University Press, Cambridge, 1924, p. 195.
[10]See H. Lamb, *loc. cit.*, p. 195.

$$\left.\begin{aligned}
\phi(x, \rho) &= \frac{\Gamma}{2\pi}\left[\frac{x}{r_1}\mathrm{K} + \Lambda\right] \\
\phi(x, 0) &= \mp\frac{\Gamma}{2}\left(1 - \frac{|x|}{r_0}\right) \quad \text{as} \quad x > 0 \quad \text{or} \quad x < 0 \\
\phi(\pm 0, \rho) &= \mp\frac{\Gamma}{2} \quad \text{for} \quad \rho < b \\
\phi(0, \rho) &= 0 \quad \text{for} \quad \rho > b \\
\phi &\to \frac{\Gamma\gamma}{2\pi} \quad \text{as} \quad r_2 \to 0
\end{aligned}\right\} \tag{7.2}$$

$$\left.\begin{aligned}
v_x(x, \rho) &= \frac{\Gamma}{2\pi r_1}\left[\mathrm{K} + \frac{b^2 - x^2 - \rho^2}{r_2^2}\mathrm{E}\right] \\
v_x(x, 0) &= \frac{\Gamma b^2}{2r_0^3} \\
v_x + \frac{\Gamma}{2\pi}&\left[\frac{\cos\gamma}{r_2} - \frac{1}{2b}\left(\log\frac{8r_1}{r_2} - \sin^2\gamma\right)\right] \to 0 \quad \text{as} \quad r_2 \to 0
\end{aligned}\right\} \tag{7.3}$$

$$\left.\begin{aligned}
v_\rho(x, \rho) &= \frac{\Gamma x}{2\pi\rho r_1}\left[\frac{x^2 + \rho^2 + b^2}{r_2^2}\mathrm{E} - \mathrm{K}\right] \\
v_\rho(x, 0) &= v_\rho(0, \rho) = 0 \\
v_\rho - \frac{\Gamma\sin\gamma}{2\pi r_2} &\to 0 \quad \text{as} \quad r_2 \to 0
\end{aligned}\right\} \tag{7.4}$$

$$\left.\begin{aligned}
\psi(x, \rho) &= \frac{\Gamma}{2\pi}\left[\frac{x^2 + \rho^2 + b^2}{r_1}\mathrm{K} - r_1\mathrm{E}\right] \\
\psi(x, 0) &= 0 \\
\psi - \frac{\Gamma b}{2\pi}\left[\log\frac{8r_1}{r_2} - 2\right] &\to 0 \quad \text{as} \quad r_2 \to 0
\end{aligned}\right\} \tag{7.5}$$

The formula for $\psi(x, \rho)$ in (7.5) coincides with that given by Maxwell.[11] In contrast to the results for the source ring, the stream function here remains single valued if the cut is removed whereas the potential $\phi(x, \rho)$ in (7.2) becomes cyclic having a circulation equal to the circulation of the vortex ring:

$$\phi(\epsilon + 4\mathrm{K}', \mathrm{k}') - \phi(\epsilon, \mathrm{k}') = \Gamma \tag{7.6}$$

Equation (7.6) follows also directly from (7.1) and agrees with the general theory of vortex filaments.

[11]See H. Lamb, *loc. cit.*, p. 219. The corresponding streamline pattern appears on p. 221.

8. The vortex disc. We consider, finally, a distribution of concentric circular vortex filaments over the circular region $x = 0, 0 \leq \rho \leq b$ in which the radial circulation density be given by

$$c(\rho) = \frac{2\Gamma\rho}{b^2} \tag{8.1}$$

so that the circulation around the annular region bounded by the circumference of the disc and the circle of radius ρ is

$$C(\rho) = \int_\rho^b c(t)\, dt = \Gamma\left[1 - \left(\frac{\rho}{b}\right)^2\right] \tag{8.2}$$

The total circulation of this disc-vortex is

$$C(0) = \Gamma \tag{8.3}$$

It follows from a theorem on vortex sheets[12] that the foregoing vortex disc is equivalent to a non-uniform distribution of doublets over the same circular region. The axes of the doublets are again parallel to the x-axis and the surface density (strength per unit area) of the doublet sheet is $C(\rho)/4\pi$.

The solution for the vortex disc under consideration, however, is obtained most conveniently by integration with respect to b of the solution for the vortex ring discussed in the preceding section. We merely state the results of these lengthy computations.

$$\left.\begin{aligned}
\phi(x, \rho) &= \frac{\Gamma}{2\pi b^2}\left[3xr_1\mathrm{E} - \frac{x}{r_1}(x^2 + 4\rho^2)\mathrm{K} + (2x^2 - \rho^2 + b^2)\Lambda\right] \\
\phi(x, 0) &= \mp\frac{\Gamma}{2b^2}(r_0 - |\, x\,|)^2 \quad \text{as} \quad x > 0 \quad \text{or} \quad x < 0 \\
\phi(\pm 0, \rho) &= \mp\frac{\Gamma r_1 r_2}{2b^2} \quad \text{for} \quad \rho \leq b \\
\phi(x, \rho) &= 0 \quad \text{for} \quad \rho \geq b \\
\phi &\to 0 \quad \text{as} \quad r_2 \to 0
\end{aligned}\right\} \tag{8.4}$$

$$\left.\begin{aligned}
v_x(x, \rho) &= \frac{\Gamma}{\pi b^2}\left[2r_1\mathrm{E} - \frac{2\rho^2}{r_1}\mathrm{K} + 2x\Lambda\right] \\
v_x(x, 0) &= \frac{\Gamma}{b^2 r_0}(r_0 - |\, x\,|)^2 \\
v_x - \frac{\Gamma}{\pi b}\left(4 - \log\frac{8r_1}{r_2}\right) &\to 0 \quad \text{as} \quad r_2 \to 0
\end{aligned}\right\} \tag{8.5}$$

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[12]See, for example, *Handbuch der Physik*, vol. 7, Julius Springer, Berlin, 1927, p. 43.

$$\left.\begin{aligned} v_\rho(x,\,\rho) &= \frac{\Gamma}{\pi b^2}\left[\frac{xr_1}{\rho}\mathrm{E} - \frac{x}{\rho r_1}(x^2 + b^2 + 2\rho^2)\mathrm{K} - \rho\Lambda\right] \\ v_\rho(x,\,0) &= 0 \\ v_\rho(\pm 0,\,\rho) &= \pm\frac{\Gamma\rho}{b^2} \qquad \text{for} \qquad \rho < b \\ v_\rho(0,\,\rho) &= 0 \qquad\qquad \text{for} \qquad \rho > b \\ v_\rho + \frac{\Gamma\gamma}{\pi b} &\to 0 \qquad \text{as} \qquad r_2 \to 0 \end{aligned}\right\} \tag{8.6}$$

$$\left.\begin{aligned} \psi(x,\,\rho) &= \frac{\Gamma}{\pi b^2}\left[\left(\frac{r_1^2 r_2^2}{3} + b^2\rho^2 - \rho^4\right)\frac{\mathrm{K}}{r_1} + \frac{2\rho^2 - x^2 - b^2}{3}r_1\mathrm{E} + x\rho^2\Lambda\right] \\ \psi(x,\,0) &= 0 \\ \psi &\to \frac{2\Gamma b}{3\pi} \qquad \text{as} \qquad r_2 \to 0 \end{aligned}\right\} \tag{8.7}$$

It is seen that ϕ, v_x , v_ρ , and ψ are single-valued in the simply connected region $0 \leq \mathrm{k}' \leq 1$, $-2\mathrm{K}' \leq \epsilon \leq 2\mathrm{K}'$. In contrast to the solution for the source disc, v_x has a logarithmic singularity at the edge of the disc but is continuous along $x = 0$, $0 \leq \rho < b$. The velocity component v_ρ has a finite jump discontinuity as the disc is traversed. By (8.6) and (8.2), and in agreement with the theory of vortex sheets, we obtain

$$v_\rho(+0,\,\rho) - v_\rho(-0,\,\rho) = \frac{2\Gamma\rho}{b^2} = -\frac{dC(\rho)}{d\rho}, \qquad 0 \leq \rho < b \tag{8.8}$$

9. Remark on numerical evaluations of results. The velocity potentials, velocity components, and stream functions occurring in the four solutions given in this paper, involve beyond elementary functions exclusively the complete elliptic integrals K and E and the cyclic function $\Lambda(\epsilon, \mathrm{k}')$. In view of (4.15), (4.16) and (5.1), the values of Λ are readily obtained by aid of tables of complete and incomplete elliptic integrals of the first and second kind. Alternatively, tables of Jacobi's Zeta function and of Jacobian elliptic functions may be used without recourse to tabulations of incomplete elliptic integrals. An extensive numberical tabulation of Λ values has been completed in the process of work now in progress at Illinois Institute of Technology.

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