WALL EFFECTS IN CAVITY FLOW—I*

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1. Introduction. We consider below high-speed *cavity flow* of a liquid past a solid, in which a long *cavity*, filled with gas (air or water vapor), is formed behind the solid. For such a flow, the cavity pressure p_c is presumably nearly constant, and the *cavitation parameter* Q or K is defined as†

$$Q = K = (p_f - p_c)/\tfrac{1}{2}\rho v_f^2 . \tag{1}$$

Here p_f is the free stream pressure, and v_f the free stream velocity. There is considerable experimental evidence[1] that for small K (say $K < 0.15$), the physical conditions assumed in classical "wake" theory are approximately fulfilled. We present various theoretical considerations, which indicate strongly that a *free jet* is preferable to an ordinary water tunnel with *fixed walls* in studying such flows, because of two related *wall effects.*

First, we deduce in Sec. 2 the existence of a *blocking constant* for each tunnel and model, below which K cannot fall.

Second, we show that a large *wall correction* must be made, for drag coefficients made in a water tunnel with fixed walls. The case of "infinitely long" cavities is covered in Secs. 7-8 below; the case of "finite cavities" will be dealt with in Part II.

The numerical results presented in Secs. 7-8 can be deduced from formulas of Réthy. In fact, some similar results were obtained by Valcovici [5] in 1913; but our results are more exact, much more extensive, and based on simpler computations. They suggest methods of estimating wall corrections, entirely analogous to those suggested by Prandtl and Valcovici [5] for ordinary flows with *wake.* However, we have felt it necessary to give a fresh discussion (in Sec. 9), which will be *physically reliable* for cavity flows with small K (say $K < 0.2$). It is notorious that the wake interpretation is entirely inaccurate physically.

The formulas of Réthy have been extensively generalized by Mises [3]. We give, in Secs. 3-5, a further generalization, which permits one to determine *any* flow with free streamlines, whose *hodograph is a circular sector.* In Sec. 6, we discuss a new method for basing effective numerical computations on these formulas.

2. Blocking constant. Consider the idealized cavity flow in a water tunnel with fixed walls, depicted in Fig. 1. We suppose an incompressible, non-viscous liquid, and a stationary liquid-gas interface, with negligible turbulence. Further, we suppose a uniform upstream flow with velocity v_0 , and a uniform free downstream velocity v_1 as the cavity approaches its maximum cross-section A_c , in a tunnel of cross-section A_0 . The rate

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†In Part II, we shall use Q to avoid confusion with elliptic function notation.

[1]Mostly unpublished, *cf.* P. Eisenberg and H. L. Pond, *Water tunnel investigations of steady state cavities*, David Taylor Model Basin Report 668 (1948). Also G. Birkhoff, *Recent progress in free boundary theory*, Proc. 7th Int. Congress Applied Mech., London, 1948, p. 7.

of increase of liquid momentum per unit time is clearly $\rho v_1^2 A_1 - \rho v_0^2 A_0$, where $A_1 = A_0 - A_c$. The total thrust on a long section of liquid is $(p_0 - p_c)A_0 - D$ where p_0 is the upstream pressure in the free stream, and D is the drag. Hence

$$\rho v_1^2 A_1 - \rho v_0^2 A_0 = (p_0 - p_c)A_0 - D.$$

But by Bernoulli's equation, $p_0 + \rho v_0^2/2 = p_c + \rho v_1^2/2$; by conservation of volume, $v_1 A_1 = v_0 A_0$. Substituting,

$$D = \rho A_0\left(\frac{1}{2}(v_1^2 - v_0^2) + (v_0 - v_1)v_0\right) = \frac{1}{2}\rho A_0(v_1 - v_0)^2. \tag{2}$$

By Bernoulli's equation, applied to (1) with $p_f = p_0$,

$$K = (v_1/v_0)^2 - 1, \qquad \text{or} \qquad (1 + K)^{1/2} = v_1/v_0 . \tag{2'}$$

Hence if we define the *drag coefficient*, as is usual, in terms of the *upstream* velocity, so that

$$C_D = 2D/\rho v_0^2 A = (v_1/v_0 - 1)^2 A_0/A, \tag{3}$$

we have the exact relation

$$C_D = [(1 + K)^{1/2} - 1]^2 A_0/A \leqq K^2 A_0/4A. \tag{4}$$

This defines a *blocking constant*, analogous to that occurring in sonic flow with M near unity:

$$K \geqq 2(C_D)^{1/2}(A_0/A)^{1/2}. \tag{4'}$$

Since C_D ordinarily varies between .0625 and 1.00, we have in practice $(A_0/A)^{1/2}/2 \leqq K_{\min} \leqq 2(A_0/A)^{1/2}$. Thus, to achieve $K = .05$, we must have A_0/A about 400, at least.

3. Mathematical assumptions. Henceforth we shall discuss theoretical plane flows with free streamlines, usually (cf. [2], Chap. XI) called flows with "wakes". For greater

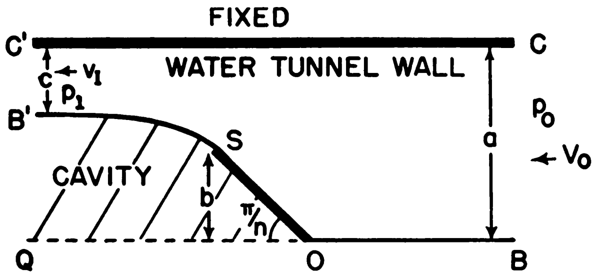

Fig. 1.

physical realism, we shall refer to *cavities* instead of wakes. In Part I, we shall consider mainly infinite cavities behind wedges in the center of tunnels with *fixed walls* (Fig. 1), *free jets* (Fig. 1a), and *bounded jets* issuing from orifices (Fig. 6).

In addition to the drag coefficient defined by (3), for such flows, we shall consider the drag coefficient

$$C_1 = 2D/\rho v_1^2 A = C_D/(1 + K) \tag{3'}$$

based on the *downstream* velocity. Following Valcovici, we show that C_1 gives a wall correction which is much smaller than C_D. In fact, we show that for wedges, the correction is *infinitely* smaller.

Since the drag coefficient is in theory independent of dimensions for a given value of a/b, one need only consider the case in which the downstream velocity is unity;

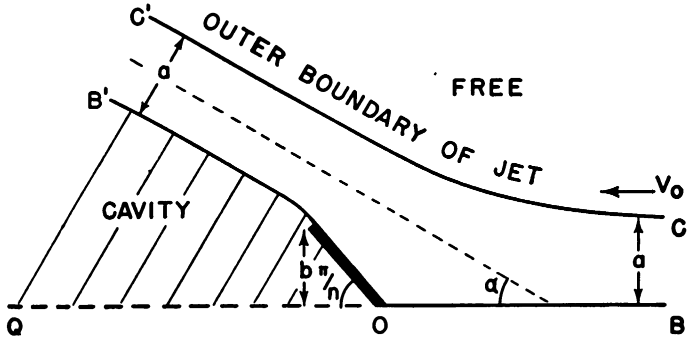

FIG. 1a.

further one may put $a = \pi/v$ where v denotes the upstream velocity. Thus, the stream function is normalized so that it goes from zero on $BOSB'$ to π on CC'. In this notation, since $a = \pi/v$,

$$C_0 = \frac{\pi}{b}\frac{(1 - v)^2}{v^3}; \tag{5}$$

$$C_1 = \frac{\pi}{b}\frac{(1 - v)^2}{v}. \tag{5'}$$

Similarly, one finds from consideration of momentum and continuity for the *open* tunnel, or free jet, case

$$C = 2\frac{a}{b}(1 - \cos\alpha)$$

and, if the previous convention regarding the stream function is adopted, $a = \pi/v$, $v = 1$ so that

$$C = 2\frac{\pi}{b}(1 - \cos\alpha). \tag{6}$$

It is to be noted that the upper half of the flow may be regarded as a jet issuing from an angular orifice. In the fixed wall case, the ratio v of upstream velocity to downstream velocity is simply the "coefficient of contraction". Numerical correlations between v and b for this case, and between α and b for the case of a free jet, have been obtained by Mises [3] who obtained excellent agreement with experimental observations on jets.

4. Technique of conformal transformation. Analytical formulas covering the cases for which we have derived numerical results, in Secs. 6-7, may be found in many places.[2]

[2]M. Réthy, Klausenburger Berichte (1879) and [4]; U. Cisotti, Rendic. Palermo 28 (1909), 307-52 and *Idromeccanica piana*, Milan, Secs. 140, 146; V. Valcovici [5]; R. Von Mises, [3] and [1]; P. Frank and R. von Mises, *Partielle Differentialgleichungen der mathematische Physik*, Ch. XI, Sec. 2.

However, we believe that our formulas provide a simpler basis for effective numerical computation than any in the literature; certainly, our numerical results are new.

Let the physical plane be the z-plane with $z = x + iy$, and the origin at the stagnation point 0. The complex potential is $W = U + iV$, where U is the velocity potential and V is the stream function. The simplest mathematical representation is obtained by choosing conventions as to sign so that $\zeta = dW/dz$ is the negative complex conjugate

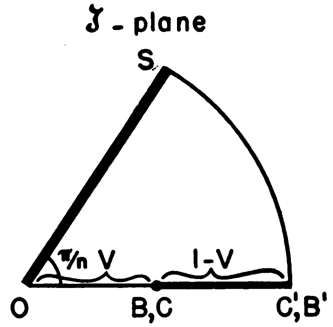

FIG. 2.

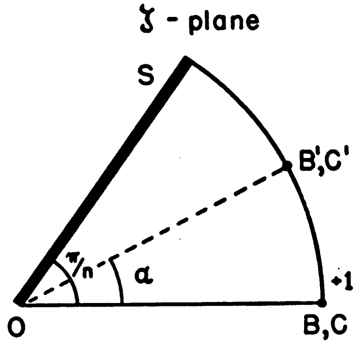

FIG. 2a.

$\xi + i\eta$ of the vector velocity $-\xi + i\eta$. In Figs. 2-2a we show the hodograph or ζ-plane for the fixed wall and free jet cases. *The hodograph is a sector of the unit circle*, of π/n radians.

We shall now present a unified treatment, applicable to all cases in which the preceding hypothesis is satisfied (and also if $1/\zeta$ is a circular sector).

Whenever this is the case, the one-to-one conformal or "schlicht" transformations $\zeta \to \zeta^n$, $\zeta \to (1 + \zeta^n)/(1 - \zeta^n)$, and $\zeta \to \psi$, where

$$\psi = (1 + \zeta^n)^2/(1 - \zeta^n)^2 \tag{7}$$

map the hodograph onto a half-circle, a quadrant, and the upper half-plane, respectively. The most general schlicht transformation doing this is therefore[3] given by

$$\zeta \to \frac{A\psi + B}{C\psi + D} = \frac{A}{C}\,\frac{1 + 2\lambda\zeta^n + \zeta^{2n}}{1 + 2\mu\zeta^n + \zeta^{2n}} = \frac{p}{q} = \tau \tag{8}$$

Here A, B, C, D, λ, μ are real, and $\lambda \neq \mu$, since otherwise we would have $\zeta \to$ const.

[3] C. Caratheodory, *Conformal mapping*, Cambridge, 1932.

Now in the case or Figs. 1-1a, the W-graph, or region of the W-plane corresponding to fluid points, is clearly the infinite strip $0 \leqq V \leqq \pi$. Hence e^W occupies the upper half-plane; this is the case treated by Réthy and Mises (*op. cit. supra*).

We now show that this restriction on the W-graph is unnecessary. The *W-graph can be a plane, half-plane, or infinite strip, with or without cuts.* The reason is simply that in these cases, the upper half τ-plane can be mapped conformally onto the W-graph by a Schwarz-Christoffel[4] transformation of the special form

$$dW = R(\tau)\, d\tau, \qquad (R(\tau) \text{ a } \textit{rational} \text{ function.}) \tag{9}$$

Thus in the case of a symmetric wedge in an infinite stream treated by Bobyleff,[5] the W-graph of half of the flow is the upper half-plane. In any such case, we have

$$dW = d\tau = (q\, dp - p\, dq)/q^2. \tag{9.1}$$

In all the cases of jets (many of which can also be interpreted as flows with cavity or "wake") treated by Réthy and Mises,[6] the W-graph is an infinite strip. Here we can set

$$dW = d\tau/\tau = (q\, dp - p\, dq)/pq = dp/p - dq/q. \tag{9.2}$$

The W-graph is a cut plane in various cases, including flows past oblique plates (Rayleigh) and asymmetric wedges.[7] In any such case, we can normalize to

$$dW = \tau\, d\tau = p(q\, dp - p\, dq)/q^3. \tag{9.3}$$

The W-graph is a cut half-plane in other interesting cases. These include simulations

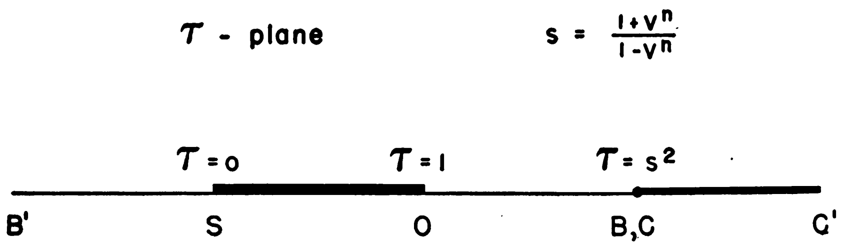

FIG. 3.

of a seaplane float[8] by a plate or wedge under a free surface. In such cases, we can normalize to

$$dW = \tau\, d\tau/(\tau - 1) = p(q\, dp - p\, dq)/q^2(p - q) \tag{9.4}$$

This will be simplified further in Sec. 4, formula (12′).

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[4]See for example [2], Ch. X.

[5]Jour. Russ. phys.-chem. Ges. XIII (1881).

[6]Refs. [1], [3], [4]; also ref. [2], Secs. 11.51, 11.53, and Ex. 5 of Ch. XI. This case also covers the useful case of a bend in a pipe with straight walls, meeting in an angle on the outside, and in a curve on the inside, so as to get constant pressure along the curve. Such a design should minimize the tendency to cavitation and flow separation on the inside of the bends; a cavitation-free expansion can be constructed similarly.

[7]See [2], Chap. XII, Sec. 12.50, and Exs. 3, 8; Rayleigh, *Collected papers*, vol. I, p. 287.

[8]See A. E. Green, Proc. Camb. Phil. Soc. 31 (1935), 589-603; 32 (1936), 67-85 and 248-52; 34 (1938), 167-84. In these papers special cases falling under (9.4) and (9.5) are treated. See also [2], Sec. 12.3.

The W-graph is a cut infinite strip (and the hodograph a circular sector) in various other cases. These include an infinite free jet divided by a flat or wedge,[9] a wedge in a

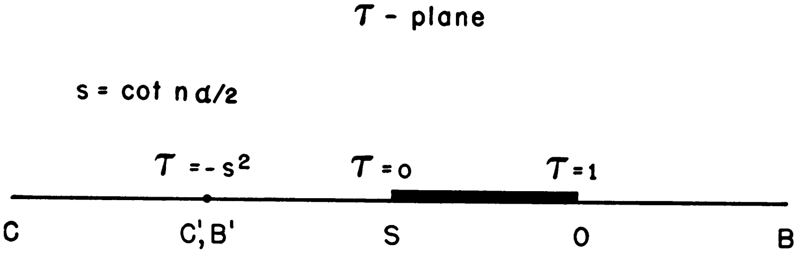


FIG. 3a.

stream bounded by parallel walls, one of which is parallel to a side of the wedge (Fig. 3), and the jet produced by two streams joined in a wedge (Fig. 3a). In any such case, we can normalize to

$$\begin{aligned} dW &= \frac{\tau\, d\tau}{(\tau + \alpha)(\tau - \alpha - 1)} = \frac{1}{\alpha + \alpha^{-1}} \left[\frac{\alpha\, d\tau}{\tau + \alpha} + \frac{\alpha^{-1}\, d\tau}{\tau - \alpha^{-1}} \right] \\ &= \frac{q\, dp - p\, dq}{\alpha + \alpha^{-1}} \left[\frac{\alpha}{q(p + \alpha q)} + \frac{\alpha^{-1}}{q(p - \alpha^{-1} q)} \right] \end{aligned} \tag{9.5}$$

If the rate of flux in the two branches is equal, even though the flow is not symmetric, $\alpha = 1$ and we have by elementary integration $W = (1/2)\ \mathrm{Log}\ (\tau^2 - 1)$.

5. Algebraic simplification. The algebraic simplification in the complex domain given by Mises (refs. of footnote 2) for the case (9.2), is possible in the general case (9), whenever $R(\tau) = R(p/q)$ can be represented explicitly[10] as a linear combination of terms of the form

$$R_i(\tau) = \Pi(\tau - \alpha_{ih})/\Pi(\tau - \alpha'_{ik}).$$

It is not always easy to determine $R(\tau)$ from the physical data of the problem. But there always exist, by the theory of Schwarz-Christoffel transformations, real constants such that

$$R(\tau) = \sum_{i>0} \frac{\beta_i}{\tau - \alpha_i} + \beta_0 + \sum_{j>0} \beta_j(\tau - \alpha_j). \tag{10}$$

Here the α_i represent points at infinity on tubes across which the stream function jumps $\pi\beta_i$, while the α_j correspond to stagnation points.

Once these real constants have been determined (one may in practice have to carry one or two arbitrary constants through all calculations), each $\tau - \alpha_i = \tau_i = p_i/q_i$ can always be expressed as in (8). Moreover, since

[9]See footnote 8 and [2], Chap. XII, Exs. 2, 4, 6; the symmetrical case can be reduced to (9.2). We have sketched figures for the cases not in the literature.

[10]From the point of view of effective computation, the explicitness of the factorization is important. It can be very tedious to determine numerically all the roots of a general polynomial, and to verify that the determination is exact enough for subsequent computations.

$$d\tau_i/\tau_i = d(\mathrm{Ln}(p_i/q_i)) = dp_i/p_i - dq_i/q_i$$

$$d\tau = dp/q - p\,dq/q^2 \tag{11}$$

$$\tau_j\,d\tau_j = dp_j/q_j^2 - p_j\,dq_j/q_j^3$$

dW equals $\zeta^{n-1}\,d\zeta$ times a linear combination of polynomials in ζ^n, divided by denominators of the form $(1 + 2\nu_i\zeta^n + \zeta^{2n})^{m(i)}$, where $m(i) \leqq 3$. Now applying the general formula, $dz = dW/\zeta$, we get after repeated synthetic division by factors q_i, q, and q_j.

THEOREM 1. Let there be given any flow whose hodograph is a circular sector and whose W-graph is a strip, half-plane, or plane, with or without cuts. Then $z(\zeta)$ can be expressed as a sum of integrals of the form

$$\int \zeta^{n-2}(a_i + b_i\zeta^n)\,d\zeta/(1 + 2\nu_i\zeta^n + \zeta^{2n})^{m(i)}. \tag{12}$$

Here the $m(i)$ are positive integers and the *real* coefficients a_i, b_i, ν_i can be explicitly evaluated in terms of the λ, μ of (8), by rational operations.

In the case (9.4), the reduction is especially simple since $p - q = (\lambda - \mu)\zeta^n$ and $p = (p - q) + q$. It yields

$$dz = 2n\zeta^{n-2}\,d\zeta\left\{\frac{i + \mu + \zeta^n}{q} + \frac{(\lambda - \mu)(1 + \zeta^n)}{q^2} + \frac{\lambda + \mu}{(\lambda - \mu)\zeta^n} + \frac{2}{(\lambda - \mu)}\right\} \tag{12'}$$

In any case, the quadratic functions $1 + 2\nu_i\zeta^n + \zeta^{2n}$ which occur in the denominator of (12) can be factored, in the real or complex domain. Thus

$$1 + 2\nu + \zeta^{2n} = \begin{cases} (\zeta^n \pm v^n)(\zeta^n \pm v^{-n}) & \text{if} \quad |\nu| > 1 \\ (\zeta^n \pm 1)^2 & \text{if} \quad |\nu| = 1 \\ (\zeta^n + e^{in\alpha})(\zeta^n - e^{-in\alpha}) & \text{if} \quad |\nu| < 1 \end{cases} \tag{13}$$

Here $v = \nu \pm (\nu^2 - 1)^{1/2}$ and $\alpha = (1/n)\,\mathrm{Cos}^{-1}\,\nu$ respectively, so that the factorization is explicit. We conclude, using partial fractions again.

THEOREM 2. In the complex domain, we can represent $z(\zeta)$ as a sum of constant multiples of integrals of the form

$$\int \zeta^{n-2}\,d\zeta/(\zeta^n - \beta_i)^m, \qquad (m = \text{positive integer}). \tag{14}$$

In the case (9.2) treated by Réthy-Mises, we always have $m = 1$. A slightly weaker result holds in general. Since

$$d\left[\frac{\zeta^h}{(\zeta^n - \beta_i)^{m-1}}\right] = -\frac{n(m-1)\zeta^{h+n-1}}{(\zeta^n - \beta_i)^m}\,d\xi + \frac{h\zeta^{h-1}\,d\zeta}{(\zeta^n - \beta_i)^{m-1}},$$

we get through repeated integration by parts, the following result.

THEOREM 3. Under the hypothesis of Theorem 1, $z(\zeta)$ can be expressed as a sum of constant multiples of terms of the form $\zeta^i/(\zeta^n - \beta_i)^m$, and of integrals of the form

$$\int \zeta^i\,d\zeta/(\zeta^n - \beta_i). \tag{14'}$$

6. Effective numerical computation. In the general case, the effective computation of $z(\zeta)$ involves various questions, which we shall only touch on. By introducing new variables $-\omega_i = \zeta^n/\beta_i$, we see that the evaluation of (14′) can be effectively accomplished by constructing a table of the *incomplete beta functions*

$$\int_0^t \omega^\beta \, d\omega/(1 + \omega) \tag{15}$$

in the complex domain. After considerable study, we are impelled to the conclusion that this is a more effective way to compute $z(\zeta)$, in the general case when the hodograph is a circular sector than the method found in the literature, for a general n. We can certainly state

THEOREM 4. Under the hypothesis of Theorem 1, $z(\zeta)$ can be explicitly expressed in terms of elementary functions and incomplete beta functions.

If $n = h/k$ is rational, then the substitution $\zeta = \beta_i^{1/h}\sigma^k$ reduces the complex integrals (14′) to the form $\int \sigma^\nu \, d\sigma/(\sigma^h - 1)$, where y, h are integers. Moreover

$$-h \int \frac{\sigma^\nu \, d\sigma}{1 - \sigma^h} = \sum_{k=1}^{h} - \int \frac{\omega^{-k\nu} \, d\sigma}{1 - \omega^k \sigma} = \sum_{k=1}^{h} \omega^{-k(\nu+1)} \operatorname{Log}(\sigma - \omega^{-k}). \tag{16}$$

Hence we conclude (cf. Mises [1], pp. 821-2, for the case (9.2)).

THEOREM 5. If $n = h/k$ is rational in Theorem 1, then we can express $z(\zeta)$ explicitly in terms of complex logarithms and rational functions of $\zeta^{1/k}$.

However, unless k is small, the effective computation of a flow field by this method will be extremely small. Hence Theorem 4 however elegant, does not adequately describe the real problems of obtaining numerical results of physical interest. Thus physically, in the problems of Sec. 2, one is given n, and wishes to correlate the asymptotic velocity with the ratio b/a = (plate width)/(channel width); not λ or μ. It is most convenient to correlate these as follows.

In the *fixed wall* case, $\zeta = 1$ when $W = -\infty$, $e^W = 0$; $\zeta = v$ when $W\, e^W = \infty$. Comparing with $e^W = p/q$, we get $q = (1 - \zeta^n)^2$ and $p = (\zeta^n - n^n)\,(\zeta^n - v^{-n})$. This gives, after *reducing to partial fractions,*

$$dz = -\frac{2n\zeta^{n-2}\, d\zeta}{1 - \zeta^n} + \frac{nv^n\zeta^{n-2}\, d\zeta}{(1 - v^n\zeta^n)} + \frac{n\zeta^{n-2}\, d\zeta}{v^n - \zeta^n}\,. \tag{17}$$

In the *free jet* case, $\zeta = 1$ when $e^W = \infty$ and $\zeta = e^{i\alpha} = \beta$ is complex when $e^W = 0$. This gives similarly

$$dz = \frac{2n\zeta^{n-2}\, d\zeta}{1 - \zeta^n} - \frac{n\beta^n\zeta^{n-2}\, d\zeta}{1 - \beta^n\zeta^n} - \frac{n\zeta^{n-2}\, d\zeta}{\beta^n - \zeta^n} \tag{17a}$$

The application of the incomplete beta function to this "fixed wall" case is relatively simple. Thus if we write $v\zeta = e^{\pi i/n}\tau$, etc., in (17), we see that we could compute $b(v)$, corresponding in the z-plane to $\zeta = e^{\pi i/n}$, from a table of the real incomplete beta function

$$\Phi_n(y) = \int_0^v \tau^{n-2}\, d\tau/(1 + \tau^n) = \frac{1}{n}\int_0^{n\nu} \omega^{-1/n}\, d\omega/(1 + \omega). \tag{18}$$

We could also obtain the velocity (hence pressure) distribution along the wedge, the tunnel wall, and the dividing streamline from the same table. The case of the free jet is less simple, and will be discussed in Sec. 7.

In this connection, we note that the real *definite* integral $\Phi_n(1)$ can be expressed[11] in terms of the logarithmic gamma function $\psi(n) = \Gamma'(n)/\Gamma(n)$. Specifically,

$$\Phi_n(1) = \frac{1}{2n}\left[\psi\left(1 - \frac{1}{2n}\right) - \frac{1}{2n}\,\psi\left(\frac{1}{2} - \frac{1}{2n}\right)\right]. \tag{19}$$

Hence, if v is near one (case of large channel), we need only compute $\Phi_n(v) - \Phi_n(1)$ and $\Phi_n(1/v) - \Phi_n(1)$, which are presumably easy, in terms of series (cf. Sec. 8).

7. Numerical results for plate. In case the wedge is a plate, $n = 2$ and the preceding remarks are unnecessary. We can integrate (17) easily, getting

$$z = -4\,\mathrm{Tanh}^{-1}\,\zeta + 2v\,\mathrm{Tanh}^{-1}\,v\zeta + \frac{2}{v}\,\mathrm{Tanh}^{-1}\,(\zeta/v) \tag{20}$$

$$b = (\pi/v) - \pi + 2(v - 1/v)\,\mathrm{Tan}^{-1}\,v, \tag{21}$$

in the *fixed wall* case. The drag coefficient C_0 and C_1 of formulas (5), (5′) may now be correlated with values of b/a, using v as a parameter. Numerical results are given in Table 1, and the results are shown graphically in Fig. 4. The "wall correction" for a closed water tunnel may thus be estimated.

The numerical results demonstrate clearly that *basing the drag coefficient on the downstream velocity leads to a smaller correction.* This is the result of Prandtl and Valcovici [5], but based now on much more extensive evidence. This conclusion will be discussed further in Sec. 8.

In the *free jet*[12] case, we get similarly from (17a) with $n = 2$,

$$dz = \frac{2\,d\zeta}{1-\zeta} + \frac{2\,d\zeta}{1+\zeta} - \beta^2\left(\frac{d\zeta}{1-\beta\zeta} + \frac{d\zeta}{1+\beta\zeta}\right) - \frac{1}{\beta^2}\left(\frac{d\zeta}{1-\zeta/\beta} + \frac{d\zeta}{1+\zeta/\beta}\right), \tag{20a}$$

$$b = \pi(1 - \mathrm{Cos}\,\alpha) - 2\,\mathrm{Sin}\,\alpha\,\mathrm{Log}\,\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right). \tag{21a}$$

This expression together with the drag formula (6) gives the values of Table 2. These values are also shown graphically in Fig. 4.

In the "fixed wall" case of the closed tunnel, the pressure at the tunnel wall is a quantity which may be readily measured; thereby, a further comparison with theory is made possible. If p_0 is the static pressure and v is the stream velocity as a great distance upstream from the lamina, then the static pressure p, where the stream velocity is u, is determined by the relation

$$\frac{p_0 - p}{(\rho v^2)/2} = \frac{u^2}{v^2} - 1. \tag{22}$$

[11]See [2], p. 315; in [1], p. 110, a different notation is used. The function $\Psi(x)$ is tabulated in vol. I of The British Association Tables, and many other places.—Using this result, we can correlate C_D with the case of a symmetrical (Bobyleff) or assymetrical wedge in an infinite stream, in the case that the stagnation point it at the leading edge. Existing tables of the incomplete beta function do not cover the range needed for the present problem.

[12]In the free jet case, if $n = 2h$ is an even integer, we can also slightly simplify the computations since

$$(1 + \alpha\zeta^h + \zeta^{2h})(1 - \alpha\zeta^h + \zeta^{2h}) = 1 + (2 - \alpha^2)\zeta^n + \zeta^{2n}.$$

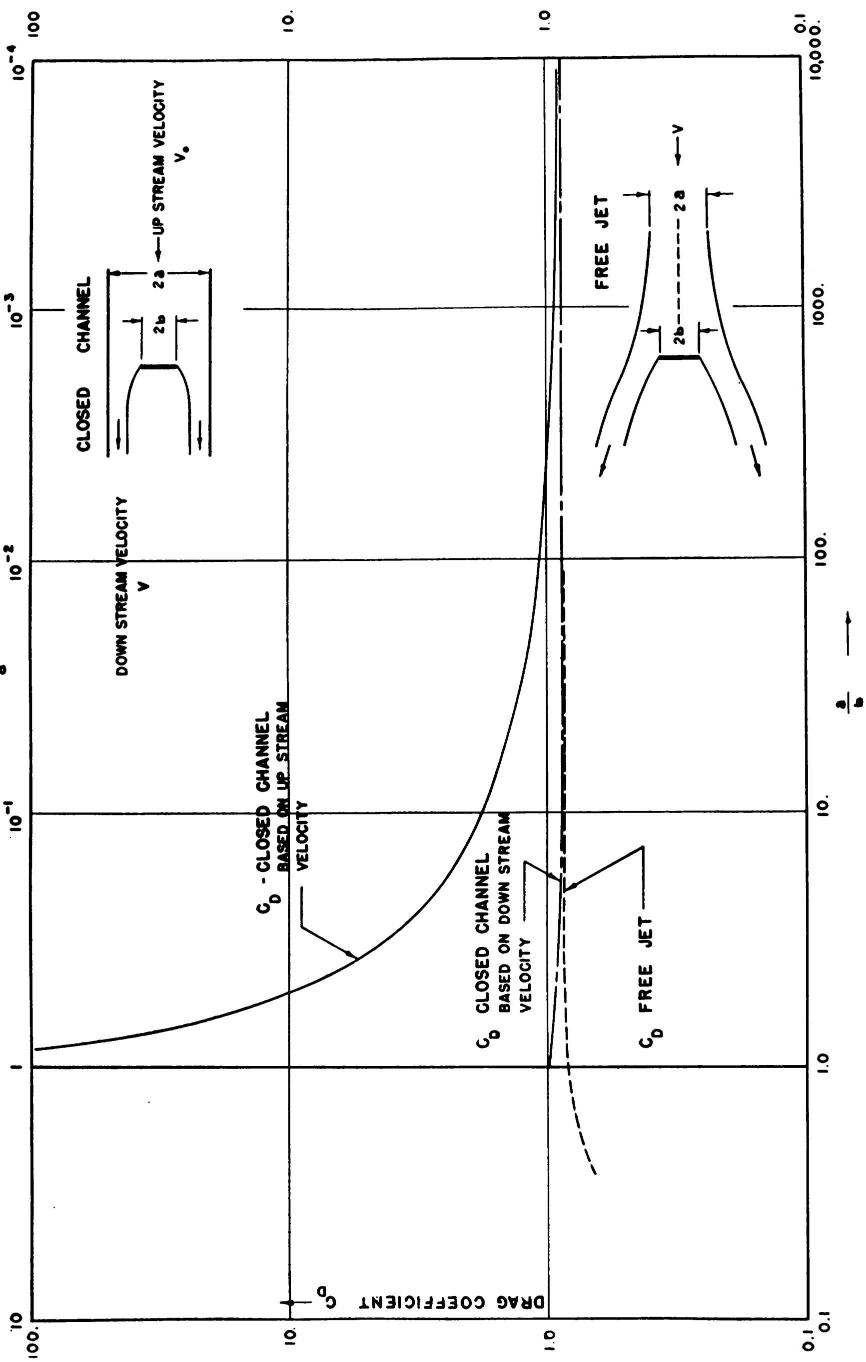

CLOSED CHANNEL
UP STREAM VELOCITY
DOWN STREAM VELOCITY
FREE JET
C_D - CLOSED CHANNEL BASED ON UP STREAM VELOCITY
C_D CLOSED CHANNEL BASED ON DOWN STREAM VELOCITY
C_D FREE JET
DRAG COEFFICIENT C_D

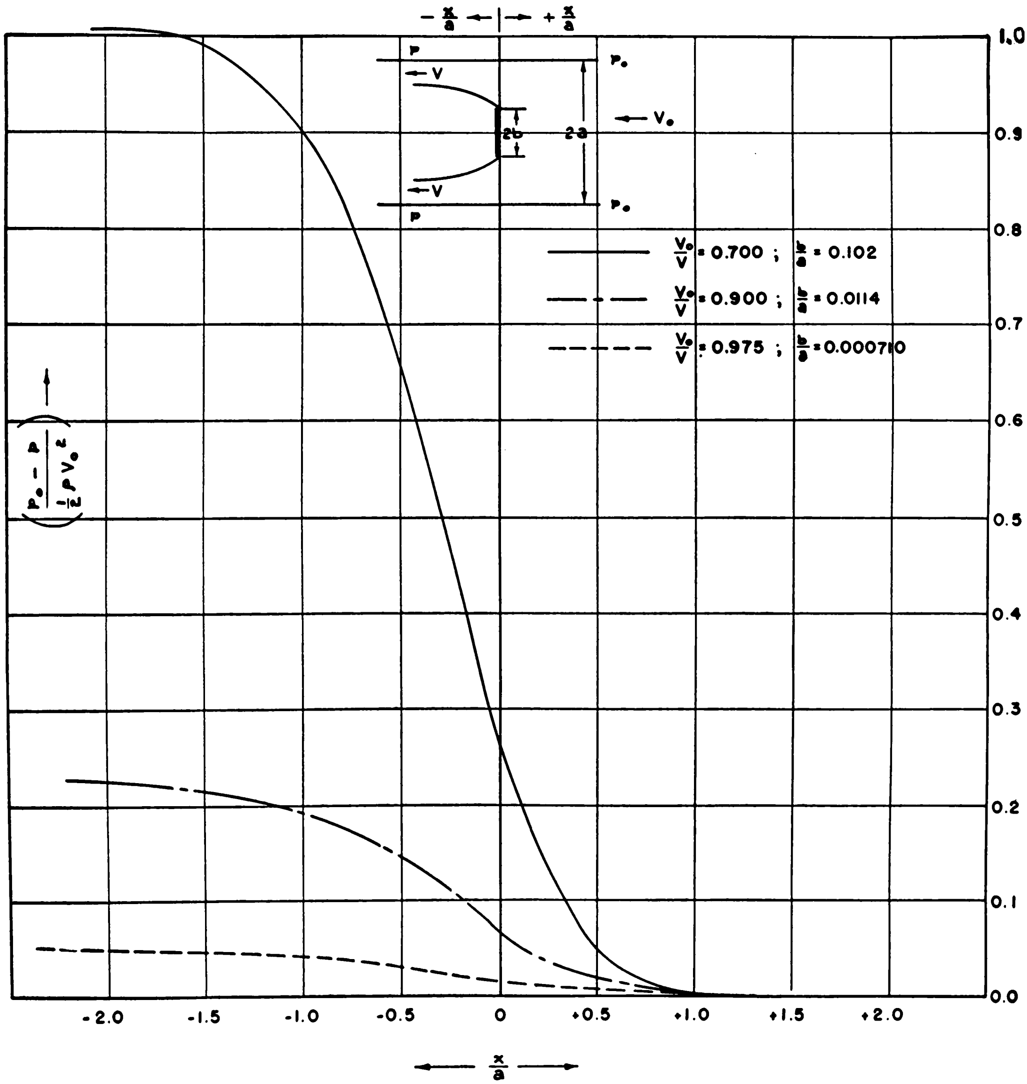
FIG. 4.

Positions on the tunnel wall are $z = x + i\pi/v$ and one gets at once from (20), with $v \leqq u \leqq 1$

$$x = -2 \log \frac{1+u}{1-u} + v \log \frac{1+vu}{1-vu} + \frac{1}{v} \log \frac{u+v}{u-v},$$

$$= -4 \tanh^{-1} u + 2v \tanh^{-1} vu + \frac{2}{v} \tanh^{-1} v \frac{v}{u},$$

where the origin of z is at the stagnation point 0. The pressure coefficient (22) has been evaluated as a function of x for three values of v; the results are shown in Fig. 5, and are tabulated in Tables 3a, 3b, 3c.



RATIO DISTANCE ALONG CHANNEL WALL TO CHANNEL HALF WIDTH

FIG. 5.

8. Case of bounded jet. A case of particular physical interest is that of a semi-infinite closed channel terminating in a free jet. The physical plane is depicted in Fig. 6, where

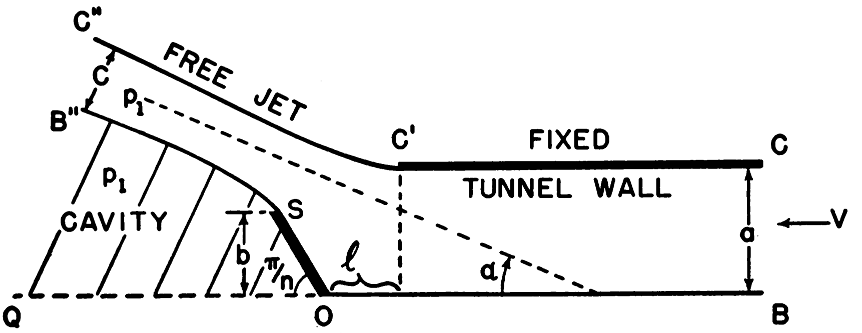


FIG. 6.

the upper half of the flow pattern is shown. As in Sec. 1, momentum considerations give the drag coefficient. If we normalize as before so that $v_1 = 1$, $a = \pi/v$, the drag coefficient based on the *upstream* velocity becomes

$$C_0 = \frac{\pi}{b}(1 - 2v \operatorname{Cos} \alpha + v^2)/v^3 \tag{23}$$

$$C_1 = \frac{\pi}{b}(1 - 2v \operatorname{Cos} \alpha + v^2)/v. \tag{23'}$$

Using (9.2), we get almost immediately

$$dz = dz_1 + dz_2 , \tag{17b}$$

where dz_1 is given by (17) and dz_2 by (17a). We therefore have $b = b_1 + b_2$, where b_1 is given by (21) and b_2 is given by (21a). Moreover in general

$$z = v \log \frac{1 + v\zeta}{1 - v\zeta} + \frac{1}{v} \log \frac{v + \zeta}{v - \zeta} - \beta \log \frac{1 + \beta\zeta}{1 - \beta\zeta} - \frac{1}{\beta} \log \frac{1 + \zeta/\beta}{1 - \zeta/\beta},$$

where the origin of z is at the stagnation point on the lamina. Consider the channel wall CC'; along this wall ζ is real and may be replaced by u, $v \leqq u \leqq 1$, and the complex coordinate along CC' is

$$z = x + i\pi/v = v \log \frac{1 + vu}{1 - vu} + \frac{1}{v} \log \frac{v + u}{u - v} + \frac{i\pi}{v} - (A + A^*),$$

where

$$A = e^{i\alpha} \log \frac{1 + e^{i\alpha}u}{1 - e^{i\alpha}u},$$

and A^* is the conjugate of A. In particular at C', $u = 1$ so that the distance from the plate to the channel opening is

$$x_{c'} = 1 = v \log \frac{1 + v}{1 - v} + \frac{1}{v} \log \frac{1 + v}{1 - v} - (A + A^*).$$

When $u = 1$, one finds that

$$A + A^* = 2 \cos \alpha \log \cot \alpha/2 - \pi \sin \alpha$$

and

$$l = \left(v + \frac{1}{v}\right) \log \frac{1 + v}{1 - v} + \pi \sin \alpha - 2 \cos \alpha \log \cot \alpha/2,$$

$$= 2\left(v + \frac{1}{v}\right) \tan h^{-1} v + \pi \sin \alpha - 2 \cos \alpha \log \cot \alpha/2. \tag{23''}$$

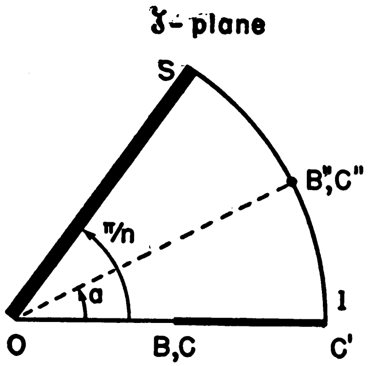


Fig. 7.

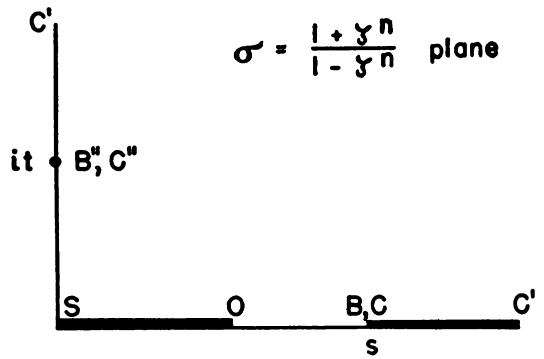


Fig. 8.

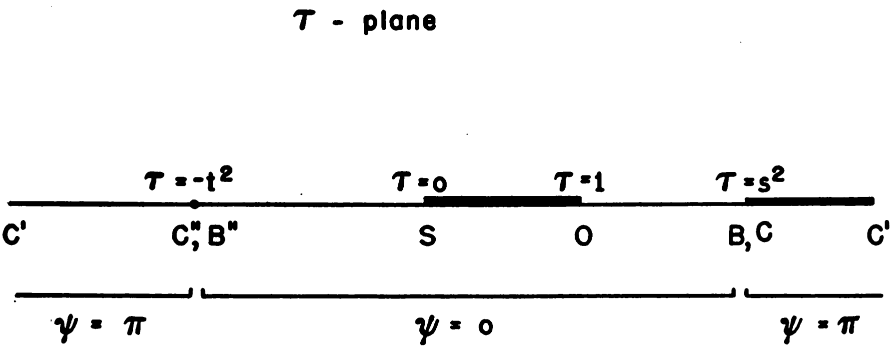


Fig. 9.

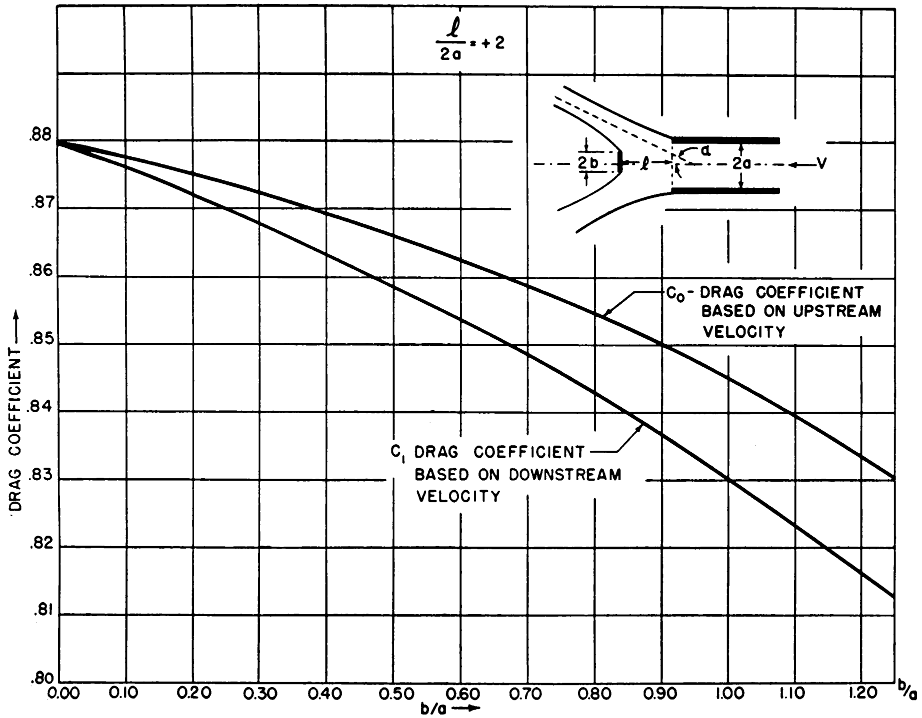


FIG. 10a.

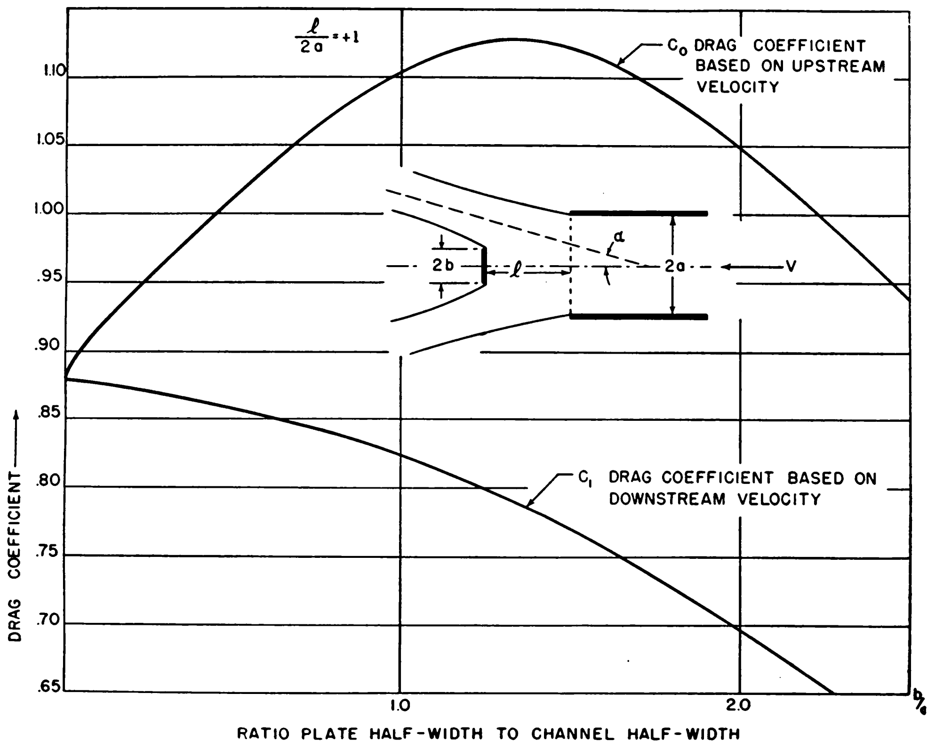


FIG. 10b.

A positive value of l corresponds to a plate position in the region of the free jet, a negative value to a plate position in the region of the closed channel.

Graphical evaluation of the relation between α and v as given by (23''), has been carried out for $1/(2\pi/v) = 0, \pm 1, +2$; and the correspondence of C_0 and C_1 on $b/(\pi/v)$ for each of those values of 1 is shown in Figs. 10a to 10d. The dependence of the angle, α,

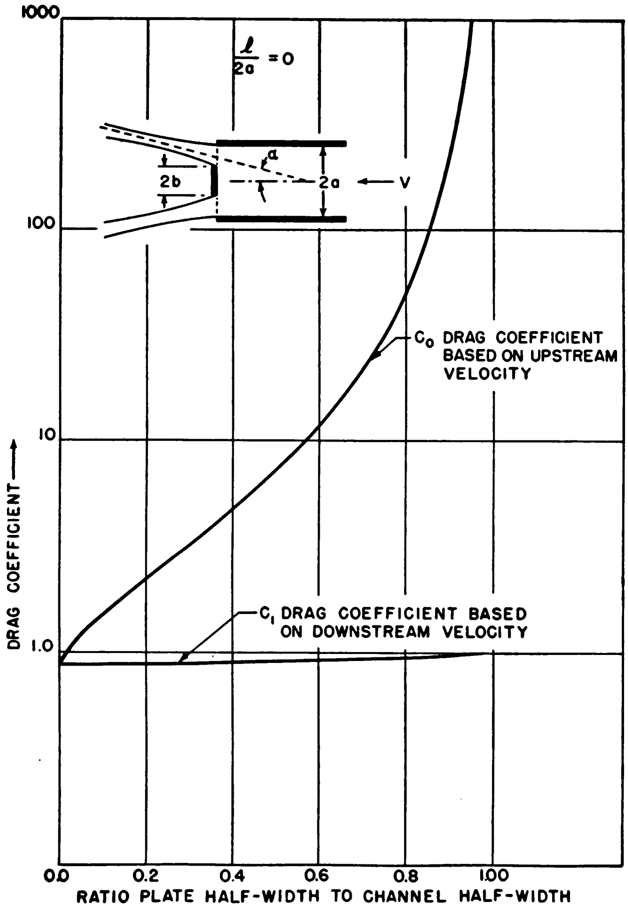


FIG. 10c.

of the jet deflection on v is shown in Fig. 11; the dependence of the ratio, b/a, of plate half width to channel half width on v is shown in Fig. 12. Table 4 summarizes the numerical results.

It is interesting to see what could be determined from tables of the incomplete beta function, in the cases of a wedge in free and bounded jets. By (20a), in a free jet, one could obtain the dependence of the wedge size b on the jet angle α, since $|\beta| = 1$ and $|\zeta| = 1$ at the points where the free streamline begins, knowing (18) only along a contour consisting of the real axis and the unit circle. By (17b), the same procedure is applicable to wedges in bounded jets, and one can locate the orifice as well.

However, it seems impossible to obtain the pressure distribution along the tunnel

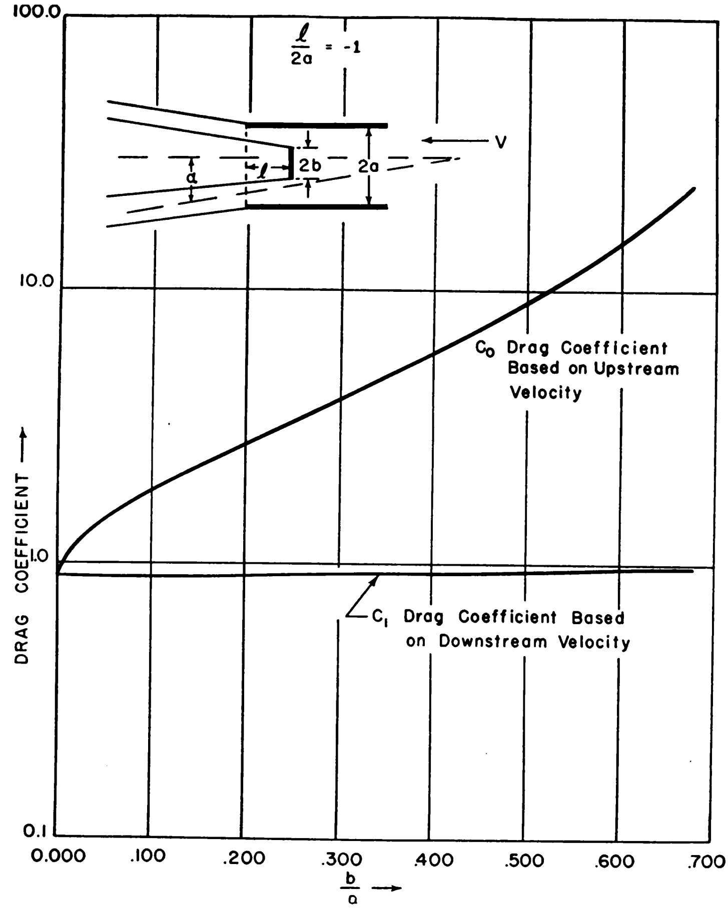


Fig. 10d.

wall and dividing streamlines in these cases, without tabulating (18) over the entire complex plane. Hence we conclude that, with general n (not if $n = 2$), the fixed wall case is substantially easier than the free jet case, which is about as hard as the bounded jet case.

9. Stability of the pressure coefficient. We can sharpen the conclusion of Sec. 7. In the fixed wall case, for any n, *the theoretical pressure coefficient is approximately independent of v, if based on the velocity at the separation point,* (downstream velocity). This we call the principle of stability of the pressure coefficient.

Theorem 5. In the fixed wall case, the pressure coefficient[13] along the wedge, has a zero derivative with respect to v, at $v = 1$, for given wedge dimensions.

Proof. By (7) and (9.1), in the case of a wedge in an infinite stream, the complex potential τ satisfies

$$\tau = (1 + \zeta^n)^2/(1 - \zeta^n)^2. \tag{24}$$

[13]The pressure coefficient is $2(p - p_c)/\rho v^2$, where p_c is an ambient (cavity) pressure, and v an ambient velocity. Our proposal that one should *use the free streamline velocity for v, instead of the free stream velocity.*

Hence if we write $v = 1 - \epsilon$, we have

$$\tau(v) = (1 + v^n)^2/(1 - v^n)^2 = 4/n^2\epsilon^2(1 + \cdots) = 0(1/\epsilon^2). \qquad (25)$$

Now consider the functional relation between τ and the complex potential W for the fixed wall case, for the same ζ. (They have the same hodograph; hence this is schlicht.) We normalize so that (in Fig. 1) $W(S) = 0$ and $W(0) = 1$—i.e., so as to keep the total

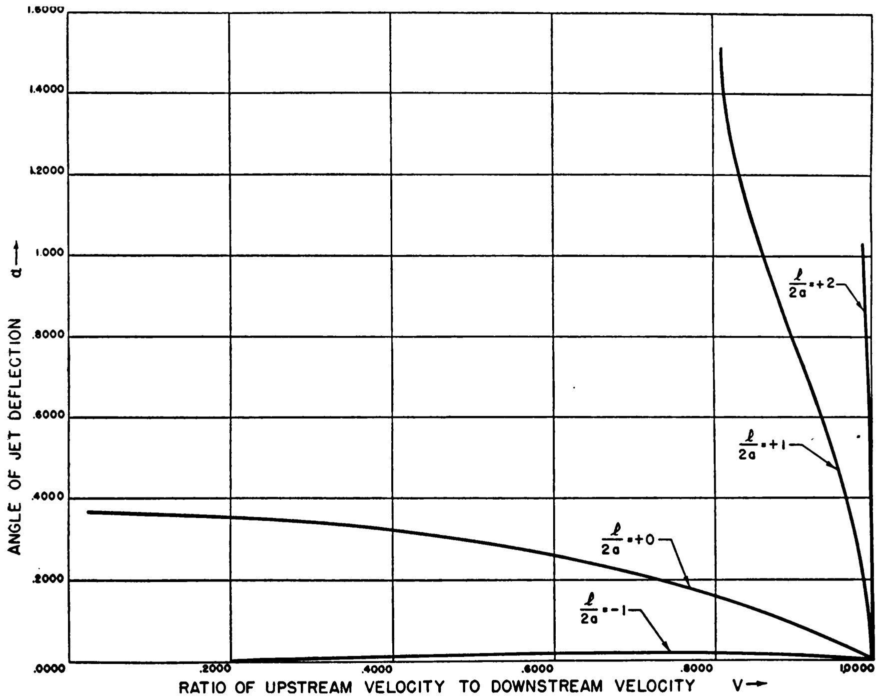


FIG. 11.

change in W along the wedge constant. This makes the W-graph a strip of unknown width $\alpha\pi$, whence by (8)

$$W = \alpha \operatorname{Ln} (A\tau + B)/(C\tau + D). \qquad (26)$$

We solve for α, A, B, C, D by matching W and τ at $\zeta = 0, 1, v, e^{i\pi/n}$. This gives explicitly, writing $\tau(v) = x$,

$$W = \operatorname{Ln}\left(\frac{x}{x - \tau}\right) \Big/ \operatorname{Ln}\left(\frac{x}{x - 1}\right)$$

$$= \tau\left\{\frac{1 + (\tau/2x) + (\tau^2/3x^2) + \cdots}{1 + (1/2x) + (1/3x^2) + \cdots}\right\} = \tau\{1 + 0(\epsilon^2)\}. \qquad (27)$$

From (27) we conclude that changes in $z = \int dw/\zeta$ are $0(\epsilon^2) = 0(1 - v)^2$ for corresponding values of ζ and hence p, completing the proof.

This suggests that in practice, cavity C_D should be computed on velocity at the separation point, and not on upstream velocity. In estimating velocity at the separation point, we can either (i) use the Bernoulli equation and measure the cavity pressure, or (ii) assume uniform velocity in the section of maximum cavity diameter, and estimate the latter's cross-section.—This discussion is only applicable physically to the case of a clear cavity.

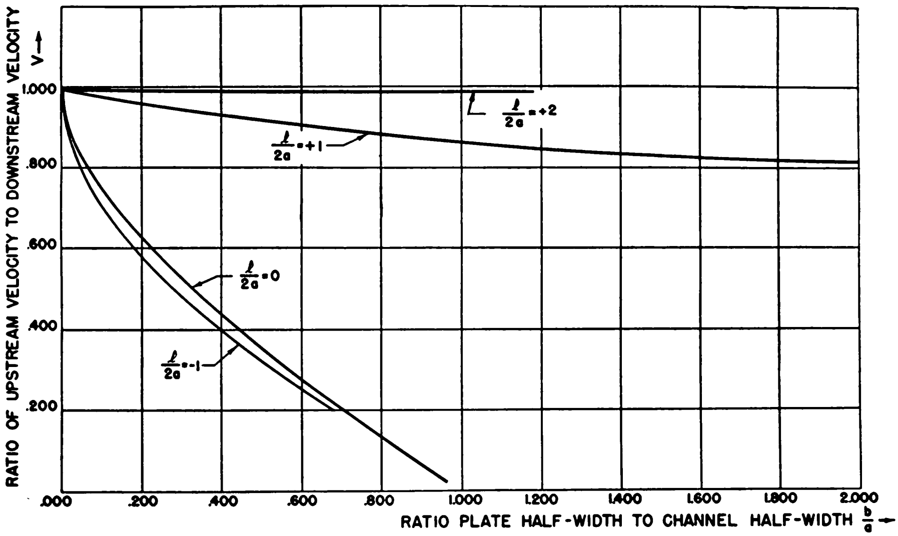


Fig. 12.

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