W is a diagonal weighting matrix for weighting the relative importance of the equations of the set (1). An orthogonal transformation T exists such that

$$T_T A_T W A T = D \equiv [d_1, d_2, \cdots, d_n)$$

is a diagonal matrix of positive elements d_i. The substitution

$$x = Ty, \qquad x = yT_T,$$

into (2) gives

$$d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2 + \cdots - 2\beta_1 y_1 - 2\beta_2 y_2 - 2\beta_3 y_3 - \cdots = eWe - bWb, \tag{3}$$

an equation in which $\beta = T_T A_T W b$. It is seen that (3) is the equation of a family of hyperellipsoids with the properties

(a) a common center at $(\beta_1/d_1, \beta_2/d_2, \cdots)$
(b) a common orientation,
(c) common principal axes' ratios,
(d) parameter e.

To complete the squares, one adds $\sum \beta_i^2/d_i$ to both sides of (3):

$$\sum \beta_i^2/d_i = \beta D^{-1}\beta = bATD^{-1}T_T A_T b = bWb.$$

It is seen that the ellipsoids converge to a point as $e \to 0$ and that their common center, given by $e = 0$, is the solution of (1).

The solution may be approximated in the same way as for the family of ellipsoids employed by Southwell and by the same process.

THE LAPLACIAN AND MEAN VALUES*

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Introduction. The Laplacian $\nabla^2 f$ represents *deviation from the average* in the following sense [1]. Let f_0 be the mean value of f over a cube of edge $2t$,

$$8t^3 f_0 = \iiint_{-t}^{t} f(x + u, y + v, z + w)\, du\, dv\, dw. \tag{1}$$

Then

$$\lim_{t \to 0} 6(f_0 - f)/t^2 = \nabla^2 f, \tag{2}$$

provided f is sufficiently smooth. This result is interesting in that it is independent of the co-ordinate system. Also it sheds a certain light on the wave equation. Thus, if the restoring force at a point in a medium is proportional to the deviation from the average, in some sense, then one might expect the equation of motion to be $a \nabla^2 f = bf_{tt}$, where a measures stiffness, b inertia.

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A harmonic function could be characterized as one such that the limit (2) is zero, in other words, such that its value at a point is close to its average over a suitable neighborhood. This seems interesting too, since a harmonic function can be characterized by an exact equality of that sort over the surface of a sphere.

A result of the type (2) is true in one or two dimensions, and also when the average is taken over the surface, edges or vertices of the cube, rather than over its volume. From (2) for the boundary of a square in two dimensions one can deduce the fact that the line integral of a harmonic function in two dimensions is independent of path. Again, (2) over the vertices of a square or cube is related to methods of numerical integration [2], where one wishes to find $f_0$by computing f or its derivatives at a point.

The Problem. What we want to do here is to characterize the point sets for which a result like (2) holds. It is seen from [1] that symmetry in the coordinate planes through (x, y, z) is sufficient, but it is not necessary. For a solid set, f is taken as a triple integral, divided by the volume of the set; for an area one uses a double integral divided by the area; for a space curve one uses a line integral divided by the length; and for a finite point set one takes the sum of the values, divided by the number of the points. Conciseness requires that the result be given here in a form that applies to all these cases at once, and to any number of dimensions. Accordingly we denote the number, length, surface, or volume by a general measure m:

I. Let S be a bounded point set on which is defined a measure m such that $0 < m(S) = M < \infty$. If the vector P represents a fixed point of S, and the vector Q a variable point of S, the following statements are equivalent:

(a) The set S has mass M, its center of mass is at P, and its moment of inertia about every axis through P is I.

(b) $\int f[P + t(Q - P)]\, dm(Q) - Mf(P) \sim (t/2)^2 I\, \nabla^2 f(P)$ as $t \to 0$ for an arbitrary function f having continuous third derivatives near P.

We do not dwell on the proof, which is rather trivial. There is no restriction in taking $P = 0$. Then, Taylor's theorem with remainder about $P = 0$ (in Cartesian co-ordinates with $Q = (x, y, z)$) shows (b) to be equivalent to the mass and center of mass results of (a) and

$$\int x^2 = \int y^2 = \int z^2 = I/2, \tag{3}$$

$$\int yz = \int zx = \int xz = 0. \tag{4}$$

(In each integration Q ranges over S.)

The identity $x^2 = [(z^2 + x^2) + (x^2 + y^2) - (y^2 + z^2)]/2$ shows that (3) are equivalent to

$$\int (y^2 + z^2) = \int (z^2 + x^2) = \int (x^2 + y^2) = I. \tag{5}$$

Equations (4) and (5) are equivalent to the identity in l, m, n:

$$\int [(mz - ny)^2 + (nx - lz)^2 + (ly - mx)^2] \equiv (l^2 + m^2 + n^2)I;$$

and this is the moment of inertia condition of (a).

As for the speed with which the limit in (b) is approached, we get, again using Taylor's expansion, the following:

II. In I(b), the error is $O(t^3)$. If f has continuous fourth derivatives near P, the error can be made $O(t^4)$ by a further restriction on S. Whatever set S is chosen, the error cannot be made $o(t^4)$ even if the class of functions is restricted to polynomials.

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A NOTE ON ASYMPTOTIC STABILITY*

By H. A. ANTOSIEWICZ (*Montana State College*)

1. In this note we shall develop a stability criterion for a vector differential equation of the form

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where the elements of the matrix $A(t) = (a_{ij}(t))$, $i, j = 1, 2, \cdots n$, are real continuous and uniformly bounded functions for all positive $t \geqq t_0$.

A. Wintner** recently established the following criterion: Let $\lambda_1(t)$ be the greatest, and $\lambda_2(t)$ the least characteristic value of the matrix $\frac{1}{2}[A(t) + A'(t)]$, and let $\| x(t) \|$ denote the Euclidean length of the vector $x(t)$. If $\int^\infty \lambda_1(t)\, dt < \infty$, $\int^\infty \lambda_2(t)\, dt < \infty$, then $\| x(t) \| \to \kappa \neq 0$ as $t \to \infty$ for every non-trivial solution $x(t)$ of (1).

It is to be noted that the condition of integrability of $\lambda_1(t)$, $\lambda_2(t)$ over (t_0, ∞) implies $\int^\infty [\text{trace } A(t)]\, dt < \infty$. Furthermore, this condition automatically excludes the important case $A(t) =$ const. unless $A(t) =$ const. is skew-symmetric.

In the following we shall establish a stability criterion which is free of the above objection, i.e. which will also apply to the general case $A(t) =$ const. We shall consider a condition to be satisfied by the matrix $A(t)$ which will suffice to insure that $\| x(t) \|$ of every non-trivial solution $x(t)$ of (1) tends to zero as $t \to \infty$. According to Liapounoff†, the trivial solution $x(t) \equiv 0$ is then said to be asymptotically stable.

2. Consider a function $V(x, t)$ which is defined and continuous for all x and t in R: $| x_i | \leqq c$, $t \geqq T$ $(i = 1, 2, \cdots n)$. If for equation (1) there exists in R a function $V(x, t)$ which is of fixed sign and admits of an infinitely small upper bound, and for which dV/dt by virtue of (1) is opposite in sign to $V(x, t)$ in R, then the trivial solution $x(t) \equiv 0$ of (1) is asymptotically stable. Liapounoff proved that the existence of such a function

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