II. The question has been raised by Wintner [10] whether the constant 2 occurring as the coefficient in (3) is the least value of α for which

$$\mu \geq -c_0 - \alpha \sum_{n=1}^{\infty} | c_n |^2 \tag{14}$$

holds for an arbitrary periodic function $f(t)$ defined by (2). Although this question will remain unanswered, it can easily be shown, as a consequence of (4), that $\alpha \geqq 1/4\pi^2$. For, suppose (14) holds for all $f(t)$ defined by (2); then, by (4), $-\alpha \sum_{n=1}^{\infty} | c_n |^2 \leqq \pi^2 N^2 + \Re(c_N)$ holds for $N = 1, 2, \cdots$. If c_N is real, it follows that $\pi^2 N^2 + c_N + \alpha c_N^2 \geqq 0$; hence, by a consideration of the discriminant of this last quadratic expression, $1 - 4\alpha\pi^2 N^2 \leqq 0$. For $N = 1$, this implies $\alpha \geqq 1/4\pi^2$, which was to be shown.

References

1. L. Brillouin, *Wave propagation in periodic structures*, New York and London, 1946.
2. P. Hartman and C. R. Putnam, *The least cluster point of the spectrum of boundary value problems*, Am. J. Math. **70**, 849-855 (1948).
3. P. Hartman and A. Wintner, *On the location of spectra of wave equations*, Am. J. Math. **71**, 214-217 (1949).
4. C. R. Putnam, *An oscillation criterion involving a minimum principle*, Duke Math. J. **16**, 633-636 (1949).
5. F. Seitz, *The modern theory of solids*, New York, 1940.
6. M. J. O. Strutt, *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Berlin, 1932.
7. H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. **68**, 222-269 (1910).
8. A. Wintner, *Stability and spectrum in the wave mechanics of lattices*, Phys. Rev. **72**, 81-82 (1947).
9. A. Wintner, *A criterion of oscillatory stability*, Q. Appl. Math. **7**, 115-119 (1949).
10. A. Wintner, *On the non-existence of conjugate points*, to be published.

A MODIFICATION OF SOUTHWELL'S METHOD*

By W. H. INGRAM (*New York*)

J. L. Synge[1] has given a geometrical interpretation of Southwell's method of solution of the problem $Ax = b$ when $A = (a_{ij})$ is symmetric and $\sum \sum a_{ij}x_ix_j$ is a positive definite form. A modification of the method having application to the more general case in which xA_TAx is a positive definite form makes use of the ellipsoids of the Gauss-Seidel process.

For any vector x, there is an error e defined by the equation

$$Ax - b = e, \tag{1}$$

therefore

$$(xA_T - b)W(Ax - b) = eWe; \tag{2}$$

---

*Received Jan. 2, 1951.

[1]J. L. Synge, *A geometrical interpretation of the relaxation method*, Q. Appl. Math., **2**, p. 87 (1944).

W is a diagonal weighting matrix for weighting the relative importance of the equations of the set (1). An orthogonal transformation T exists such that

$$T_T A_T W A T = D \equiv [d_1, d_2, \cdots, d_n)$$

is a diagonal matrix of positive elements d_i. The substitution

$$x = Ty, \qquad x = yT_T,$$

into (2) gives

$$d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2 + \cdots - 2\beta_1 y_1 - 2\beta_2 y_2 - 2\beta_3 y_3 - \cdots = eWe - bWb, \tag{3}$$

an equation in which $\beta = T_T A_T W b$. It is seen that (3) is the equation of a family of hyperellipsoids with the properties

(a) a common center at $(\beta_1/d_1, \beta_2/d_2, \cdots)$
(b) a common orientation,
(c) common principal axes' ratios,
(d) parameter e.

To complete the squares, one adds $\sum \beta_i^2/d_i$ to both sides of (3):

$$\sum \beta_i^2/d_i = \beta D^{-1}\beta = bATD^{-1}T_T A_T b = bWb.$$

It is seen that the ellipsoids converge to a point as $e \to 0$ and that their common center, given by $e = 0$, is the solution of (1).

The solution may be approximated in the same way as for the family of ellipsoids employed by Southwell and by the same process.

THE LAPLACIAN AND MEAN VALUES*

BY R. M. REDHEFFER AND R. STEINBERG (*University of California, Los Angeles*)

Introduction. The Laplacian $\nabla^2 f$ represents *deviation from the average* in the following sense [1]. Let f_0 be the mean value of f over a cube of edge $2t$,

$$8t^3 f_0 = \iiint_{-t}^{t} f(x + u, y + v, z + w)\, du\, dv\, dw. \tag{1}$$

Then

$$\lim_{t \to 0} 6(f_0 - f)/t^2 = \nabla^2 f, \tag{2}$$

provided f is sufficiently smooth. This result is interesting in that it is independent of the co-ordinate system. Also it sheds a certain light on the wave equation. Thus, if the restoring force at a point in a medium is proportional to the deviation from the average, in some sense, then one might expect the equation of motion to be $a \nabla^2 f = bf_{tt}$, where a measures stiffness, b inertia.

*Received April 9, 1951.