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ON A QUASI-LINEAR PARABOLIC EQUATION OCCURRING IN AERODYNAMICS*

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1. Introduction. The equation under discussion in this paper is the following:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u = u(x, t)$ in some domain and ν is a parameter. The occurrence of the first derivative in t and the second in x clearly indicates the equation is parabolic, similar to the heat equation, while the interesting additional feature is the occurrence of the non-linear term $u\ \partial u/\partial x$. The equation thus shows a structure roughly similar to that of the Navier-Stokes equations and has actually appeared in two separate problems in aerodynamics. An equation simply related to (1) appears in the approximate theory of a weak non-stationary shock wave in a real fluid. This is discussed in Ref. 1 (pp. 146-154) where a general solution of (1) is given. The equation is also given in J. Burgers' theory of a model of turbulence (Ref. 2) where he notes the relationship between the model theory and the shock wave. Historically, the equation (1) first appears in a paper by H. Bateman (Ref. 3) in 1915 when he mentioned it as worthy of study and gave a special solution. Eq. (1) is of some mathematical interest in itself and may have applications in the theory of stochastic processes. The aim of this paper is to study the general properties of (1) and relate the various applications. I wish to thank Professor P. A. Lagerstrom and F. K. Chuang for helpful collaboration.

2. Relationship of (1) to Shock Wave Theory. The solutions to Eq. (1) can approximately describe the flow through a shock wave in a viscous fluid. They can be related to the shock wave in several ways. In Ref. 1 an approximation based on the Navier-Stokes equations for one-dimensional non-stationary flow of a compressible viscous fluid gives

$$\frac{\partial w}{\partial t} + \beta w\frac{\partial w}{\partial x} = \frac{4}{3}\nu^*\frac{\partial^2 w}{\partial x^2} \tag{2}$$

for w = excess of flow velocity over a sonic velocity where $\beta = (\gamma + 1)/2$, ν^* = kinematic viscosity at sonic conditions. Eq. (2) is reduced to Eq. (1) by $\beta w = u$; $4/3\ \nu^* = \nu$. In this paper a different discussion, intended to illustrate the production and maintenance of a shock, will be given. In the cases in which we are interested we can say that

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ν is a small parameter, a statement which will be made more precise later. Thus, we study the case $\nu = 0$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \tag{3}$$

in order to see the underlying mechanism of propagation. Eq. (3) is similar to the non-linear equations for propagation of waves of finite amplitude in one dimension (Ref. 4, p. 482)

$$\begin{aligned} \left\{\frac{\partial}{\partial t} + (u + c) \frac{\partial}{\partial x}\right\}(\omega + u) &= 0, \\ \left\{\frac{\partial}{\partial t} + (u - c) \frac{\partial}{\partial x}\right\}(\omega - u) &= 0, \end{aligned} \tag{4}$$

where $\omega = \int_{\rho_0}^{\rho} (dp/d\rho)^{1/2}\, d\rho/\rho$; $c = (dp/d\rho)^{1/2}$ = velocity of sound
p = pressure, ρ = density.

Eq. (4) reduces to (3) exactly if $c = 0$. Thus we have as a model a fluid in which u is transported by the fluid motion itself (i.e. with a velocity u). According to the theory of characteristics for first order equations, the projection of the characteristics of (3) on the (x, t) plane are straight lines whose slope is

$$\frac{dx}{dt} = u. \tag{5}$$

In addition Eq. (3) interpreted geometrically states that u is constant in the characteristic direction. Thus u corresponds to the Riemann invariants $u \pm \omega$ of (4). It follows that each characteristic is straight over its length and carries a definite value u. Therefore knowing the values of u at any particular instant the solution for the future may be found by following the characteristics. However this process may terminate after a finite time when two characteristics intersect. For example this must happen if for $x_1 < x_2$, $u(x_1, t) > u(x_2, t)$. This phenomenon is the steepening of the wave front for waves of finite amplitude known from the study of (4).

We can now regard the viscosity as a mechanism for preventing the formation of discontinuities. The viscous stresses depend on changes of rates of strain so that the viscosity ν appears in Eq. (1) multiplying a term of higher order $\partial^2 u/\partial x^2$. This gives (1) the nature of a diffusion equation where velocity (actually momentum) is the quantity diffused. It is, of course, very much different from solid friction as expressed in first order damping terms. The characteristics of (1) are different from those of (3) for any $\nu > 0$ and they are given in the (x, t) plane by the curves $\psi(x, t)$ = const. where

$$\nu \psi_x^2 = 0 \qquad \text{or } t = \text{const.} \tag{6}$$

These characteristics occur as a double set which can be regarded as the limit of a pair of characteristics indicating a high signal speed. Thus here the speed of signals is infinite.

Hence Eq. (1) shows the typical features of shock wave theory: (i) A non-linear term tending to steepen the wave fronts and produce complete dissipation, (ii) A viscous term of higher order which prevents formation of actual discontinuities and which tends to diffuse any differences in velocity.

3. Relationship of (1) to Turbulence Theory. Eq. (1) is related to turbulence theory as a mathematical model. The similarity of the Navier-Stokes equation to Eq. (1) is responsible. Both contain non-linear terms of the type: unknown function times a first derivative; and both contain higher order terms multiplied by a small parameter. The problems in turbulence are not very well defined but in most theories one is interested in some kind of spectrum, the feeding of energy through the spectrum and dissipation of energy. The model equation contains the non-linear terms and viscous terms vital to a study of those topics. Following Burgers, we regard Eq. (1) as a model for decaying free turbulence; he discusses other cases in much detail.

The mathematical (rather than physical) aspect of the model can be emphasized as follows. We study as before the underlying wave propagation for $\nu = 0$. For the usual turbulence theory we are dealing with flows in two or three dimensions so that in addition to equations like (1) a kinematical or continuity restriction must be added. For an incompressible fluid in two dimensions we have

$$u_t + uu_x + vu_y = -\frac{1}{\rho} p_x , \tag{7a}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho} p_y , \tag{7b}$$

$$u_x + v_y = 0. \tag{7c}$$

The continuity equation (7c) is a statement that only transversal waves are present in the flow field. However as discussed in Sec. 2 a typical feature of (1) is the longitudinal steepening effect (i.e. steepening in the flow direction). It does not seem possible to have waves of that type for a system like (7) but the steepening must be transverse to the local flow direction. The underlying structure of (7) is given by the characteristic surfaces $\Psi(x, y, t) = \text{const.}$ which now satisfy the equation

$$\frac{1}{\rho} (\Psi_x^2 + \Psi_y^2)(\Psi_t + u\Psi_x + v\Psi_y) = 0. \tag{8}$$

The vanishing of the first factor gives a double set of surfaces $t = \text{const.}$ corresponding to the characteristic cones which have degenerated into planes, and propagation of pressure signals with infinite speed. The vanishing of the second factor gives stream surfaces. If any steepening effect occurs it will have to be related to these surfaces. In this case an invariant on these stream surfaces is the vorticity $\xi(x, y, t)$ which satisfies the equation

$$\xi_t + u\xi_x + v\xi_y = 0. \tag{9}$$

Thus, as pointed out by Burgers, the situation in the actual case is complicated very much by the kinematical restrictions.

The quantity u in the above model is, of course, a measure of the turbulence. The model is completed by some kind of statistical analysis based on Eq. (1) or else Eq. (1) is considered as a stochastic differential equation subject to random boundary values.

4. General Properties of Eq. (1). It may be expected that Eq. (1) is similar to the heat equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2} \tag{10a}$$

and to the heat equation in a moving medium

$$\frac{\partial \theta}{\partial t} + U \frac{\partial \theta}{\partial x} = \nu \frac{\partial^2 \theta}{\partial x^2} \tag{10b}$$

with respect to the type of boundary value problems which are sensible. In this section some comparisons will be made and other general properties of (1) will be studied.

An energy equation for (1) can be found by multiplying (1) by u and integrating over a spatial domain, for example $(x_1 \leq x \leq x_2)$. This gives

$$\begin{aligned} &\frac{1}{2} \int_{x_1}^{x_2} \frac{\partial}{\partial t} (u^2)\, dx + \frac{1}{3} \{u^3(x_2\,,\, t) - u^3(x_1\,,\, t)\} \\ &\quad = \nu \left\{ u(x_2\,,\, t) \frac{\partial u}{\partial x}(x_2\,,\, t) - u(x_1\,,\, t) \frac{\partial u}{\partial x}(x_1\,,\, t) \right\} - \nu \int_{x_1}^{x_2} \left(\frac{\partial u}{\partial x}\right)^2 dx. \end{aligned} \tag{11}$$

The various terms in Eq. (11) have the following meaning:

$\frac{1}{2} \int_{x_1}^{x_2} \frac{\partial u^2}{\partial t}\, dx$ = total rate of change of kinetic energy in system,

$\frac{1}{3} \{u^3(x_2\,,\, t) - u^3(x_1\,,\, t)\}$ = net flux of kinetic energy out of system across boundaries,

$\left\{ \left(u \frac{\partial u}{\partial x}\right)_{x_2} - \left(u \frac{\partial u}{\partial x}\right)_{x_1} \right\}$ = rate of work done on system at boundaries,

$\nu \int_{x_1}^{x_2} \left(\frac{\partial u}{\partial x}\right)^2 dx$ = total dissipation of energy by viscosity in system.

The non-linear term in (1) provides a means of feeding energy into the system across the the boundaries. We can have a steady-state solution to (1) in an infinite domain $(-\infty < x < \infty)$ with an energy balance.

$$\frac{1}{3} (u_1^3 - u_2^3) = \nu \int_{-\infty}^{+\infty} \left(\frac{\partial u}{\partial x}\right)^2 dx, \tag{12}$$

where $u_1 = u(-\infty, t)$, $u_2 = u(+\infty, t)$ and $u_1 > u_2$. A linear equation like (10a) or (10b) can have no such (bounded, non-zero) solution in an infinite domain. For such a steady state the total dissipation, as given by (12), is independent of the value of ν. The steady solution is actually

$$u = -u_1 \tanh \frac{u_1(x - x_1)}{2\nu}, \tag{13}$$

where u_1 and x_1 are constants. Thus $u_2 = -u_1$. This gives the steady flow through a shock wave and shows how the non-linear terms are responsible for a change from $u > 0$ (supersonic) to $u < 0$ (subsonic). As $\nu \to 0$ the steep front of (13) at $x = x_1$ approaches a discontinuity which corresponds to the shock wave in a fluid where $\nu = 0$. This shows one reason why the conditions on continuity of solutions for $\nu = 0$ have to be relaxed if the solutions are to correspond to reality. Using (13) it is also possible to give steady state solutions for finite domains $(0 < x < l)$ which have regions of rapid transition either in the interior or adjacent to the boundary.

A translation property of (1) is also of interest. If we consider (1) in a coordinate system moving in the positive x-direction with a constant velocity U, defined by

$$\begin{aligned} \bar{x} &= x - Ut \\ \bar{t} &= t \end{aligned} \tag{14}$$

we obtain

$$\frac{\partial u}{\partial \bar{t}} + (u - U) \frac{\partial u}{\partial \bar{x}} = \nu \frac{\partial^2 u}{\partial \bar{x}^2}, \tag{15}$$

so that $w = u - U$ satisfies the same equation in $(\bar{x}, \bar{t})$ as u in (x, t):

$$\frac{\partial w}{\partial \bar{t}} + w \frac{\partial w}{\partial \bar{x}} = \nu \frac{\partial^2 w}{\partial \bar{x}^2}.$$

In this sense (1) is invariant under a Galilean transformation. Bateman used the steady-state solution (13) as $w(\bar{x}, \bar{t})$ in order to show a shock $u(x, t)$ progressing with a velocity U

$$u = U - u_1 \tanh \frac{u_1(x - x_1 - Ut)}{2\nu}.$$

Laws of similarity are also of importance in understanding the joint effects of non-linearity and viscosity in (1). For clarity, consider a solution u of (1) depending on the following parameters:

l = significant length; e.g. size of domain,

u_0 = significant initial amplitude,

ν = viscosity.

Then it is possible to express any solution of (1) in terms of these non-dimensional variables:

$$R = \frac{ul}{\nu}, \qquad R_0 = \frac{u_0 l}{\nu}, \qquad \tau = \frac{t\nu}{l^2}, \qquad \xi = \frac{x}{l}$$

as

$$R = F(R_0, \tau, \xi). \tag{16}$$

This relationship is, of course, general. However it should be compared with the corresponding linear case where it is possible to express the solution as

$$\frac{u}{u_0} = F(\tau, \xi). \tag{17}$$

These results are easily derived by introducing the dimensionless variables τ and ξ in the corresponding equations and seeing what is required of u to make the equations dimensionless. As one example, compare two solutions with the same l but different values of viscosity ν_1, ν_2. In the linear case we can say that the ratio $[u_2/u_{0_2} = u_1/u_{0_1}]_\xi$ when $t_2 = t_1\nu_1/\nu_2$. In the non-linear case however we can only say that u_{0_2} must be adjusted so that $u_{0_2} = u_{0_1}\nu_2/\nu_1$ if the F is to have the same value, and in that case $u_2 = u_1\nu_1/\nu_2$. As might be expacted the non-linear equation cannot give much information under linear transformations.

Various writers (Refs. 5 and 6) have studied the existence and uniqueness of solution of different types of problems for quasi-linear parabolic equations. For initial value problems it is clear that only one condition is needed at $t = 0$, $u(x, 0) = u_0(x)$. Using this and treating Eq. (1) as an inhomogeneous heat equation, they reduce the problem to an integral equation. Picard iteration procedures can be used to prove existence and uniqueness of solution in the neighborhood of the initial line. The boundary conditions always involve constant values of u(or u_x) on lines $x = \text{const}$. The problem of radiation where conditions like $u(x, t) = f(t)$ have to be considered, has not been treated. It is much harder to prove existence and uniqueness, or to discover necessary and sufficient conditions for this, in the radiation case. Some further remarks about uniqueness will be made in Sec. 5.

5. General Solution of Initial Value Problem. The general solution developed here applies directly to the case when the initial values are known in some domain

$$u(x, 0) = u_0(x), \tag{18}$$

and the boundary conditions are of a simple type. For example, we may have

$$u(x_1, t) = 0, \qquad u(x_2, t) = 0. \tag{19}$$

The solution is supposed to be bounded function having the necessary derivatives.

The general result is: If $\theta(x, t)$ is any solution to the heat equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2}, \tag{20}$$

then

$$u(x, t) = -2\nu \frac{\theta_x}{\theta} \tag{21}$$

is a solution to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

The result can be checked directly by differentiation. However it was derived as follows. Let

$$u = \frac{\partial \phi}{\partial x}, \qquad \phi = \phi(x, t) \tag{22}$$

and substitute in (1). Integrating with respect to x we obtain

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \nu \frac{\partial^2 \phi}{\partial x^2}, \tag{23}$$

where the function of integration is omitted. Like the heat equation (23) is invariant under the transformation $x \longrightarrow ax$, $t \longrightarrow a^2 t$ (a = constant). This suggests finding solutions of the form

$$\phi(x, t) = F\{\theta(x, t)\} \tag{24}$$

where θ satisfies (20). Introducing (24) into (23) we have

$$F' \cdot \theta_t + \frac{1}{2} F'^2 \cdot \theta_x^2 = \nu\{F'' \theta_x^2 + F' \theta_{xx}\}. \tag{25}$$

Hence we obtain the ordinary differential equation for $F(\theta)$

$$\frac{1}{2}\left(\frac{dF}{d\theta}\right)^2 = \nu \frac{d^2F}{d\theta^2}, \tag{26}$$

which has the solution

$$F(\theta) = -2\nu \log(\theta - c_1) + c_2 . \tag{27}$$

Therefore, F is the log of a solution to the heat equation and $u(x, t)$ can be expressed as in (21).

Integrating (21) with respect to x we obtain the equivalent relation

$$\theta(x, t) = C(t) \exp\left(-\frac{1}{2\nu}\int_b^x u(\xi, t)\, d\xi\right), \tag{28}$$

where b is an arbitrary constant and $C = \theta(b, t)$. Without loss of generality b may be normalized to the value zero. The initial values are simply related. If

$$u(x, 0) = u_0(x), \tag{29}$$

then

$$\theta(x, 0) = \theta_0(x) = C_0 \exp\left(-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right). \tag{30}$$

Various representations of the solution to the heat equation may be used. A representation suitable for an infinite domain $(-\infty < x < \infty)$ is given by

$$\theta(x, t) = \frac{1}{2(\pi\nu t)^{1/2}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] \theta_0(\xi)\, d\xi. \tag{31}$$

Thus given θ_0 from (30), θ is found from (31) and $u(x, t)$ from (21). If an integration by parts is carried out in the expression for θ_x (21) becomes

$$u(x, t) = \frac{\int_{-\infty}^{+\infty} \exp\left[-(x-\xi)^2/4\nu t\right] \exp\left[-(2\nu)^{-1}\int_0^\xi u_0(\eta)\, d\eta\right] u_0(\xi)\, d\xi}{\int_{-\infty}^{+\infty} \exp\left[-(x-\xi)^2/4\nu t\right] \exp\left[-(2\nu)^{-1}\int_0^\xi u_0(\eta)\, d\eta\right] d\xi}, \tag{32}$$

an expression for $u(x, t)$ in terms of its initial values.

The uniqueness of the solution to (1) in the domain $(-\infty < x < \infty)$ under the initial conditions (29) can be discussed as follows. Any solution $u(x, t)$ of (1) defines a function $\theta(x, t)$ according to (28), where $C(t)$ can be found such that $\theta(x, t)$ satisfies the heat equation (20). For each u, $C(t)$ is uniquely determined within a multiplicative constant. The initial values of $\theta(x, t)$ depend only on the initial values of $u(x, t)$ according to (30). Now assume there are two solutions u, v of (1) having the same initial values. If $u_0(x)$ is suitably restricted $\theta(x, t)$ is uniquely determined by its initial values. But u, v as solutions of (1) are computed from their corresponding θ according to (21). In this formula the factor $C(t)$ cancels out. Since θ is uniquely determined, u and v are identical for all (x, t) and the solution to (1) is unique.

6. Examples of Solutions. The first example is that of a shock wave approaching a steady state. We choose a step-function as the initial condition where the strength of jump is chosen so that the center of the jump will remain fixed. Then we see how viscosity smoothes even a sharp discontinuity.

The initial conditions are

$$u_0(x) = u_1 \quad x < 0,$$
$$= -u_1 \quad x > 0. \tag{33}$$

From (28), putting $b = 0$

$$\theta_0(x) = C_0 \exp(u_1x/2\nu) \quad x > 0,$$
$$= C_0 \exp(-u_1x/2\nu) \quad x < 0. \tag{34}$$

Upon substituting (34) in (31) we obtain

$$\theta(x, t) = \frac{C_0}{2} \exp\left(\frac{u_1^2 t}{4\nu}\right)\left\{2ch\,\frac{u_1x}{2\nu} + \exp\left(\frac{u_1x}{2\nu}\right) \operatorname{erf}\frac{x + u_1t}{2(\nu t)^{1/2}} - \exp\left(-\frac{u_1x}{2\nu}\right) \operatorname{erf}\frac{x - u_1t}{2(\nu t)^{1/2}}\right\} \tag{35}$$

Hence from (21) the solution is

$u(x, t)$

$$= -u_1 \frac{2sh(u_1x/2\nu) + \{\exp[u_1x/2\nu]\operatorname{erf}[(x+u_1t)/2(\nu t)^{1/2}] + \exp[-u_1x/2\nu]\operatorname{erf}[(x-u_1t)/2(\nu t)^{1/2}]\}}{2ch(u_1x/2\nu) + \{\exp[u_1x/2\nu]\operatorname{erf}[(x+u_1t)/2(\nu t)^{1/2}] - \exp[-u_1x/2\nu]\operatorname{erf}[(x-u_1t)/2(\nu t)^{1/2}]\}} \tag{36}$$

It is easily verified that as $t \to 0$ the initial conditions are satisfied if we use

$$\operatorname{erf}(\infty) = 1, \qquad \operatorname{erf}(-\infty) = -1. \tag{37}$$

For large values of t/ν (and $|x| \neq ut$) we can substitute in (36) the asymptotic formulas for erf,

$$\operatorname{erf} z \cong 1 - \frac{1}{\pi^{1/2} z} \exp(-z^2) + \exp(-z^2) O\left(\frac{1}{z^3}\right), \qquad z > 0. \tag{38}$$

This shows that the approach to the steady state given by Eq. (13) is very rapid. The deviations from the steady state die out like $\exp(-u_1^2 t/4\nu)$.

The passage to the limit $\nu \to 0$ in the solution of (1) is important for determining the behavior of discontinuities in the solution to the equation (3) with $\nu = 0$ from the start. Putting $\nu = 0$ in (36) we have, if $u_1 > 0$

$$u(x, t) = -(\text{sign } x)u_1,$$

so that the initial conditions are preserved and we have a stationary shock wave. The invariant quantity across this shock wave is the kinetic energy u_1^2. This is to be taken as the rule for treating discontinuities in the solution to (3), from the viewpoint of an observer at rest relative to the discontinuity. In addition the flow velocity must decrease in passing through the shock. In the case the initial velocities are not chosen as in (33) the situation is the same in a moving coordinate system. The solution for a shock tends to a quasi-steady state progressing with a certain velocity U, as indicated in Sec. 4. For example if

$$u(x, 0) = u_0(x) = u_3, \qquad x < 0$$
$$= u_4, \qquad x > 0 \qquad \text{where } u_3 > u_4, \tag{39}$$

we may define a moving coordinate system so that the solution following (33) applies. Let

$$u_1 = (u_3 - U) = -(u_4 - U) \tag{40}$$

or

$$U = \frac{1}{2}(u_3 + u_4). \tag{41}$$

Then according to (15), the solution (36) applies to (39) if x is replaced by $x - Ut$, and u by $u - U$. The speed of propagation of the shock U is the average of the velocities on both sides.

The second example shows the decay of an arbitrary periodic initial disturbance. This corresponds in the turbulence model theory to the decay of free turbulence in a box. The initial and boundary conditions are:

$$u(x, 0) = u_0(x), \qquad 0 \le x \le l \tag{41a}$$

$$u(0, t) = u(l, t) = 0, \qquad t > 0. \tag{41b}$$

From (21) and (30) these conditions induce the following conditions on $\theta(x, t)$

$$\theta(x, 0) = \theta_0(x) = C_0 \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right], \tag{43a}$$

$$\theta_x(0, t) = \theta_x(l, t) = 0 \tag{43b}$$

The problem for the heat equation specified by conditions (43) has a unique (bounded) solution. We can represent the solution to the heat equation in a standard way by a Fourier series in x whose coefficients are exponentials in t

$$\theta(x, t) = A_0 + \sum_{n=1}^{\infty} \exp\left[-\nu \frac{n^2\pi^2}{l^2}\, t\right] A_n \cos\frac{n\pi x}{l}, \tag{44}$$

so that (43b) is satisfied. The coefficients A_0, A_n are determined at $t = 0$ as

$$A_0 = \frac{1}{l}\int_0^l \theta_0(x)\, dx = \frac{C_0}{l}\int_0^l \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right] dx \tag{45a}$$

$$A_n = \frac{2}{l}\int_0^l \theta_0(x)\cos\frac{n\pi x}{l}\, dx = \frac{2C_0}{l}\int_0^l \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right] dx \tag{45b}$$

Hence from (21) the solution is

$$u(x, t) = \frac{2\nu\pi}{l}\,\frac{\sum_{n=1}^{\infty} \exp\left[-\nu n^2\pi^2 t/l^2\right] n A_n \sin(n\pi x/l)}{A_0 + \sum_{n=1}^{\infty} \exp\left[-\nu n^2\pi^2 t/l^2\right] A_n \cos(n\pi x/l)} \tag{46}$$

For large values of time t only the first term in the numerator remains so that

$$u(x, t) \cong \frac{2\nu\pi}{l} \frac{A_1}{A_0} \exp\left[-\nu \frac{\pi^2}{l^2} t\right] \sin \frac{\pi x}{l} \tag{47}$$

This may be contrasted with the solution to corresponding linear problems at large times

$$\theta(x, t) = B_1 \exp\left[-\nu \frac{\pi^2}{l^2} t\right] \sin \frac{\pi x}{l}, \tag{48}$$

where

$$B_1 = \frac{2}{l} \int_0^l u_0(x) \sin \frac{\pi x}{l} \, dx.$$

The solutions (47) and (48), are seen to have the same form in dependence on (x, t), but different amplitudes. The similarity of the solutions is an expression of the fact that when the amplitudes are small the non-linear equation behaves like the linear one. However the decay process over intermediate ranges of time is considerably different. We can find out something about this process by considering special cases. For example, consider a simple sine wave

$$u_0(x) = u_0 \sin \frac{\pi x}{l} \cdot \tag{49}$$

The coefficients are explicitly evaluated in this case as

$$A_0 = \frac{C_0}{l} \int_0^l \exp\left[-\frac{u_0 l}{2\pi\nu}\left(1 - \cos\frac{\pi x}{l}\right)\right] dx = C_0 \exp\left[-\frac{u_0 l}{2\pi\nu}\right] I_0\left(\frac{u_0 l}{2\pi\nu}\right), \tag{50}$$

$$A_n = \frac{2C_0}{l} \int_0^l \exp\left[-\frac{u_0 l}{2\pi\nu}\left(1 - \cos\frac{\pi x}{l}\right)\right] \cos\frac{n\pi x}{l} \, dx = 2C_0 \exp\left[-\frac{u_0 l}{2\pi\nu}\right] I_n\left(\frac{u_0 l}{2\pi\nu}\right), \tag{51}$$

so that (46) becomes

$$u(x, t) = \frac{4\nu\pi}{l} \frac{\sum_{n=1}^{\infty} \exp\left(-\nu n^2\pi^2 t/l^2\right) n I_n(u_0 l/2\pi\nu) \sin(n\pi x/l)}{I_0(u_0 l/2\pi\nu) + 2\sum_{n=1}^{\infty} \exp\left(-\nu^2 n^2 \pi^2 t/l^2\right) I_n(n_0 l/2\pi\nu) \cos(n\pi x/l)} \cdot \tag{52}$$

The conditions at $t = 0$ are satisfied by (52) for

$$I_0(z) + 2 \sum_{n=1}^{\infty} I_n(z) \cos nx = \exp(z \cos x). \tag{53}$$

The significant parameter occurring in the solution is the Reynolds number R_0 based on the size of the box

$$R_0 = \frac{u_0 l}{\nu} \cdot \tag{54}$$

The complexity of (52) is due to excitation of higher frequencies because of the non-linearity of (1). The same excitation of higher frequencies was noted by Burgers by directly splitting the solution of (1) into the form

$$u(x, t) = \sum_{n=1}^{\infty} E_n(t) \sin nx$$

and observing the coupling between the E_n . This can be contrasted with solutions to the heat equation under the same initial conditions

$$\theta(x, t) = u_0 \exp\left[-\nu \frac{\pi^2}{l^2} t\right] \sin \frac{\pi x}{l} \tag{55}$$

which, for all time, shows only the fundamental frequency. This is typical of linear equations as is the linear dependence on the amplitude of the initial disturbance. The solution (52) emphasizes the non-linear dependence on the initial conditions. The initial amplitude enters through R_0 . As $R_0 \to 0$, $u(x, t) = \theta(x, t) + 0(R_0)$. This estimate shows to what approximation the non-linear terms can be neglected and emphasizes that the dimensionless parameter R_0 (not merely u_0) should be small ($R_0 \ll 1$). For large values of R_0 all the $I_n(R_0/2\pi)$ are almost equal so that large changes in R_0 (increases in u_0) produce relatively little effect on the solution. Asymptotically we have

$$I_n\left(\frac{R_0}{2\pi}\right) \cong \frac{\exp(R_0/2\pi)}{R_0^{1/2}}\left\{1 - \frac{4n^2 - 1}{8R_0/2\pi} + O\left(\frac{1}{R_0^2}\right)\right\} \tag{56}$$

As the first approximation for large R_0 the $I_n(R_0/2\pi)$ may be cancelled in (52) to give

$$u(x, t) \doteq \frac{4\nu\pi}{l} \frac{\sum_{n=1}^{\infty} \exp(-\nu n^2\pi^2 t/l^2) n \sin(n\pi x/l)}{1 + 2\sum_{n=1}^{\infty} \exp(-\nu n^2\pi^2 t/l^2) \cos(n\pi x/l)} \,. \tag{57}$$

Equation (57) is an approximation which should describe what happens for t greater than some $t_1 > 0$. Equation (57) can be put in a simpler form by using an identity for theta functions (Ref. 7, p. 489)

$$u(x, t) = -\frac{2\nu\pi}{l} \sum_{n=1}^{\infty} \frac{(-)^n \sin(n\pi x/l)}{\sinh \nu(n\pi^2 t/l^2)} \tag{58}$$

For large R_0 , Eq. (58) gives the spectrum, involving all higher frequencies, explicitly. The solution is independent of the initial amplitude and the spectrum damps exponentially with the first power of the wave number for large n. For small n and $\nu n^2 t/l^2 \ll 1$ the coefficients depend on $1/n$. The dissipation is proportional to u_x^2 and is thus independent of n for the first few n. The exponential cut off for large n assures a finite total dissipation. For $0 \leq x < l$ (58) can be written in another form,

$$u(x, t) \doteq \frac{x}{t} + 2\frac{l}{t} \sum_{m=1}^{\infty} (-)^m \frac{sh(mx/\nu t)}{sh(ml/\nu t)} \,. \tag{59}$$

For $x/\nu t$ large, the series can be approximately summed as

$$u(x, t) \doteq \frac{l}{t}\left\{\tanh\left(\frac{l - x}{2\nu t}\right) - \left(1 - \frac{x}{l}\right)\right\}, \tag{60}$$

a form which shows a steep front near $x = l$. The general picture presented by the above considerations is the following. The initial sine wave (49) shows after the first instant a tendency to develop a steep front near $x = l$, if R_0 is sufficiently large. After a while this steep front broadens and dies out until at the end only a sine wave remains. This sine wave has an amplitude which is smaller than that of the corresponding linear problem because of the increased dissipation over the intermediate ranges of t. It is clear that similar considerations apply to any initial distribution of the same general form as a sine wave.

7. Concluding remarks. The simple examples which have been worked out are intended to illustrate some general features of the interaction of non-linearity and viscosity. The main effects are always the same, namely, steepening of the velocity profiles by non-linearity and prevention of discontinuities, diffusion of momentum and dissipation of energy by viscosity. Under different interpretations these effects are considered responsible for the formation of steep but continuous shock wave fronts and for the finite dissipation and feeding of energy through the spectrum. As an example of different interpretations consider a velocity distribution with a steep front. The discontinuous front of non-viscous flow has infinite dissipation when considered in viscous flow and it contributes terms like $1/n$ to the spectrum. When viscosity is considered from the outset the front is steep but continuous, the dissipation is finite and independent of ν; while the front is steep there are some terms like $1/n$ in the spectrum but the spectrum dies out like exp $(-n)$ for large n. Another important general feature is the non-linear dependence of the solution on a characteristic Reynolds number $R_0 = u_0 l/\nu$, $u_0 x/\nu$. For low $R_0 (R_0 \ll 1)$ the non-linearity is not important and the solution behaves like the solution to the corresponding heat equation but as R_0 increases the solution changes very much. For large R_0 it is typical that there are ranges of (x, t) for which the solution depends very little on the variations in R_0. Part of the problem for the future is a more precise determination of the ranges in which the various approximations are valid.

The same type of result applies to some special solutions in higher dimensions. For the equation in two or three dimensions which is analogous to (1)

$$\mathbf{q}_t + \mathbf{q} \cdot \nabla \mathbf{q} = \nu \nabla^2 \mathbf{q} \tag{61}$$

it can be verified that the method of Sec. 5 applies. If $\theta(x, t)$ is any solution of

$$\theta_t = \nu \nabla^2 \theta, \tag{62}$$

then

$$\mathbf{q} = -2\nu \nabla (\log \theta) \tag{63}$$

is a solution to (61). It should be noted that (63) gives an irrotational flow field.

For future work it seems worthwhile to investigate simple solutions further and study the radiation problems. Then the three-dimensional cases may also be studied.

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