$$\begin{aligned}
 \alpha^* &= \begin{bmatrix} 36.02 & 16 & 4 \\ 15 & 9 & 2.99 \\ 6 & 4 & 2 \end{bmatrix}; \quad b = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \div 100; \\
 m = \alpha b &= \begin{bmatrix} 3 & 0 & 4 \\ -6 & 0 & -12 \\ 3 & 0 & -12 \end{bmatrix} \div 1200 = \begin{bmatrix} .002500 & 0 & .003333 \\ -.005000 & 0 & -.010000 \\ .002500 & 0 & -.010000 \end{bmatrix}; \\
 m^2 &= \begin{bmatrix} 21 & 0 & 60 \\ -54 & 0 & -168 \\ 45 & 0 & 156 \end{bmatrix} \div (1200)^2 = \begin{bmatrix} .000015 & 0 & .000042 \\ -.000037 & 0 & -.000117 \\ .000031 & 0 & .000108 \end{bmatrix}; \\
 \gamma \approx I - m + m^2 &= \begin{bmatrix} .997515 & 0 & -.003291 \\ .004963 & 1 & .009883 \\ -.002469 & 0 & .990108 \end{bmatrix}; \\
 x^* = \gamma x &\approx \begin{bmatrix} 2.096271 \\ -6.169037 \\ 7.049273 \end{bmatrix}.
 \end{aligned}$$

The new inverse matrix α^* may be calculated from Equation (24). A complete new solution gives the values

$$x^* = \begin{bmatrix} 2.096277 \\ -6.169052 \\ 7.049277 \end{bmatrix}$$

and it is seen that good accuracy is obtained with our method using only the first three terms of the series given in Equation (37).

ON A SOLUTION OF THE ENERGY EQUATION FOR A ROTATING PLATE STARTED IMPULSIVELY FROM REST*

By RONALD F. PROBSTEIN (*Princeton University*)

1. Introduction. The problem of the steady, laminar incompressible flow of a fluid over an infinite plate rotating at a constant velocity was first solved by von Kármán [1]. Recently the associated heat transfer problem for the von Kármán example was

*Received Nov. 3, 1952. This work has been supported by the United States Air Force, Air Research and Development Command, under Contract AF 33(038)-250.

calculated by Millsaps and Pohlhausen [2]. Neglecting heat conductivity Nigam [3] has obtained the corresponding non-steady velocity distributions valid for small times for the rotating plate started impulsively from rest with a constant angular velocity. It is the purpose of the present paper to give the temperature distributions associated with the velocity fields found by Nigam. Apart from an academic viewpoint this solution offers an interesting mathematical analogy to the steady hypersonic laminar flow over a semi-infinite flat plate with a "self-induced"¹ pressure gradient.

Under the assumption of an incompressible fluid a solution of the energy equation will be obtained for a thermally insulated surface and for a non-insulated surface with a wall temperature distribution which is a parabolic function of the radius. By analogy with Pohlhausen's steady, laminar boundary layer analysis [5], the assumption of constant density requires that the azimuthal plate velocity must be small compared with the sound speed. Furthermore, in the non-insulated case as the viscous dissipation is to be included, it is implied that the energy dissipation associated with viscosity is of the same order as the energy associated with the imposed temperature difference. The maximum permissible difference between the wall and ambient temperature must therefore be small, that is, of the order of the square of the azimuthal plate velocity. Both of these conditions place a limit on the radius beyond which the solutions to be obtained are no longer valid.

Like Nigam's solution of the equations of motion, the present solution of the energy equation will neglect all time dependent terms of order t^2 or higher; that is, only the early stages of the motion when t is small will be considered. Such a procedure is tantamount to an asymptotic expansion in which the functions are expressed in ascending power series in t and the first coefficients is obtained. This of course has the limitation that no information is given regarding the time to reach the steady state. (See Ref. 6 for an analogous procedure in the case of the impulsively started cylinder.)

2. Basic equations. The basic partial differential equations for the time dependent motion ($\partial/\partial t \neq 0$) of an incompressible fluid (density $\rho = \text{constant}$) are most suitably expressed for this problem in cylindrical coordinates r, θ, z with radial, azimuthal and axial velocity components u, v, w . Since the angular velocity of the plate Ω will be assumed constant for all time $t > 0$ the problem is axially-symmetric and hence the partial derivatives with respect to θ vanish. Therefore the energy equation is written as

$$c_p \left\{ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right\} = \frac{\lambda}{\rho} \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right\} + \nu \left\{ 2 \left(\frac{\partial u}{\partial r} \right)^2 + 2 \left(\frac{u}{r} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 \right\}, \quad (1)$$

where, T is the temperature of the fluid, c_p the specific heat expressed in mechanical units, λ the thermal conductivity, and ν the kinematic viscosity.

Nigam has shown that if the infinite plate at time $t = 0$ is suddenly made to rotate with a constant angular spin Ω about the axis $r = 0$, that the velocity components may

¹The term "self-induced" refers to the pressure gradient generated as a result of the curvature of the boundary layer over a flat plate; this pressure gradient in turn affects the boundary layer growth [4].

be expressed as follows:

$$u = \Omega^2 r t f(\eta), \quad (2)$$

$$v = \Omega r g(\eta), \quad (3)$$

$$w = -4\nu^{1/2} \Omega^2 t^{3/2} h(\eta), \quad (4)$$

where, $\eta = z/2 (\nu t)^{1/2}$ and the functions $g(\eta)$ and $f(\eta)$ satisfy the second order ordinary linear differential equations

$$g'' + 2\eta g' = 0, \quad (5)$$

$$f'' + 2\eta f' - 4f = -4g^2. \quad (6)$$

Note of course, that as the isothermal Blasius velocity distribution remained unaffected by heat transfer in the Pohlhausen analysis [5], so here, the solutions for the velocity components are unchanged by the heat transfer.

It is now assumed that the energy equation is satisfied by a relation of the form

$$c_r T = c_r T_\infty + (\Omega r)^2 k(\eta) + \Omega^2 \nu t m(\eta), \quad (7)$$

where T_∞ is the ambient temperature. On substituting Eq. (7) along with Eqs. (2)-(4) into the energy equation, neglecting all terms of order t^2 or higher, and equating coefficients of t and r^2 , the following two ordinary inhomogenous linear differential equations are obtained for the functions $k(\eta)$ and $m(\eta)$:

$$k'' + 2\eta \sigma k' = -\sigma g'^2, \quad (8)$$

$$m'' + 2\eta \sigma m' - 4\sigma m = -16k, \quad (9)$$

where $\sigma = c_r \nu \rho / \lambda$.² The problem will now be particularized to the case of $\sigma = 1$ since an analytic rather than numerical solution is desired for the comparison of any results. The close relation between the above equations with $\sigma = 1$ and Eqs. (5) and (6) satisfied by g and f , is then immediately evident.

3. Solutions of the equations. The solutions for m and k may now be expressed as

$$m = 2[(C + 2)g + (D - 1) - f], \quad (10)$$

$$k = \frac{1}{2}[Cg + D - (g - 1)^2], \quad (11)$$

where C and D are constants of integration, the additional two required integration constants A and B being contained in the general solution for f which is given by Nigam's Eq. (10).³ It is of course to be noted from Eq. (11) that the quantity involving k in the expression for the temperature distribution is a function only of the azimuthal velocity v . If g is required to satisfy the boundary conditions that either at time zero or infinitely far from the plate surface ($\eta = \infty$) the azimuthal velocity $v = 0$, and at the plate surface ($\eta = 0$) for all times greater than zero $v = v_{wall} = \Omega r$, then g is given by

$$g = 1 - 2\pi^{-1/2} \int_0^\eta e^{-\eta^2} d\eta = 1 - \operatorname{erf} \eta = \operatorname{erfc} \eta. \quad (12)$$

²The Prandtl number $c_p \nu \rho / \lambda$ is equal to $\sigma c_p / c_r$. In the case where $c_p = c_r$, the Prandtl number and σ are of course identical.

³The sign in front of the particular solution in Eq. (10) of Ref. 3 should be negative.

The particular solution of the energy equation for the thermally insulated plate is obtained by using the boundary conditions that the temperature gradient normal to the surface must vanish, and infinitely far from the plate surface and at time $t = 0$ the temperature has its ambient value T_∞ . The temperature distribution in this case is therefore

$$c_v T = c_v T_\infty + v(v_{wall} - \frac{1}{2}v) + \Omega^2 \nu t \{ (\pi^{-1/2} e^{-\eta^2} - \eta \operatorname{erfc} \eta)^2 - 2\eta^2 \operatorname{erfc} \eta + (4\pi)^{-1/2} \eta e^{-\eta^2} \} \quad (13)$$

For the case where the wall temperature is a parabolic function of the radius (i.e. $T_{wall} = \text{constant } r^2 + T_\infty$) the boundary conditions for zero time and infinite distance from the plate are the same as in the insulated problem. On the other hand, the boundary conditions at the plate surface become $m(0) = 0$ and $k(0) = \alpha$, where the dimensionless constant α controls the rate of change of the temperature gradient along the radius. Hence, the solution for the temperature is

$$c_v T = c_v T_\infty + v \left\{ \left(\frac{1+2\alpha}{2} \right) v_{wall} - \frac{v}{2} \right\} + \Omega^2 \nu t \{ (\pi^{-1/2} e^{-\eta^2} - \eta \operatorname{erfc} \eta)^2 - (1+2\alpha)\eta^2 \operatorname{erfc} \eta - \pi^{-1/2}(1+2\eta^2) \operatorname{erfc} \eta + \pi^{-1/2} \eta e^{-\eta^2} [2\pi^{-1} + (1+2\alpha)] \}. \quad (14)$$

The present approach can be extended to wall temperature distributions different from parabolic by use of a method similar to that adopted by Chapman and Rubesin [7] for solving the steady, compressible laminar flow over a semi-infinite flat plate with an arbitrary distribution of surface temperature.

4. Discussion of results. The main point of interest in the solutions of the energy equation represented by Eqs. (13) and (14) is their mathematically analogous nature to the solutions for the flow over a semi-infinite flat plate with a self-induced pressure gradient. (See footnote 1 and Ref. 4). Examination shows that both of the solutions obtained can be considered to be partially composed of a Crocco type integral of the steady flow energy equation for the flow over a flat plate with a Prandtl number equal to unity and a constant surface temperature under the boundary layer approximations (i.e. $T = a + bv + cv^2$). In addition there is what might be called an "induced time" effect represented by the $\Omega^2 \nu t$ terms which correspond formally to the self-induced pressure gradient already mentioned.

The portion of the solutions involving only the azimuthal velocity arises from the dissipation term $\nu(\partial v/\partial z)^2$ and the analogue of the constant surface temperature in the steady flow flat-plate case is the parabolic distribution chosen for the impulsively started rotating plate.

By analogy to the work in Ref. 4 the terms in v and $\Omega^2 \nu t$ are but the first of an asymptotic expansion in ascending powers of t . Thus the present work for small times bears a closer analogy to the treatment in which the steady hypersonic flow over a semi-infinite flat plate for the region close to the leading edge is considered [8]. In this case solutions are obtained by expansion in powers of $x^{1/2}$, rather than in powers of $x^{-1/2}$ as in Ref. 4 (x is the distance along the plate surface with the leading edge as the origin). The latter solution, which is valid for the region toward the back end of the plate, would in effect correspond to the large-time asymptotic expansion for the present problem.

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ON A CERTAIN ASYMPTOTIC EXPANSION

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1. Introduction. This little note is concerned with the asymptotic expansion of a certain function, defined by a series resembling that of the modified Bessel function $I_\nu(z)$, viz.

$$f(z) = \sum_{k=1}^{\infty} \frac{(z/2)^{2k}}{k!k!k^{1/2}}. \quad (1,1)$$

This function plays a role in a certain physical problem, and values of $f(z)$ for large positive values of the argument had to be determined. In itself the problem is of little general interest but it gives an opportunity to stress again the importance of the use of factorial series and to show the simple means they provide to derive an asymptotic expansion in such a case.

2. The factorial series. The modified Bessel function $I_\nu(z)$ has the expansion

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k!(k+\nu)!}. \quad (2,1)$$

As $k!k^{1/2} \sim (k+1/2)!$ this series for $p = 1, 2$ is, apart from some trivial modifications as multiplication by $(z/2)^{1/2}$ and the addition of a term with $k = 0$, closely related to the series (1, 1).

This relationship is used in the following way. The ratio $(k+1/2)!/(k!k^{1/2})$ can be expanded into a factorial series (cf. Nörlund, Differenzengleichungen):

$$\frac{(k+1/2)!}{k!k^{1/2}} = \sum_{s=0}^{\infty} \frac{(-1)^s \binom{s-1/2}{s} B_s^{(s+1/2)}}{(k+3/2)(k+5/2) \cdots (k+s+1/2)}, \quad (k > 0), \quad (2,2)$$

where $B_s^{(s+1/2)}$ are generalised Bernoulli numbers. As these alternate in sign with increasing values of s , the terms of the series (2, 2) are all positive. Or,

$$\frac{1}{k!k^{1/2}} = \sum_{s=0}^{\infty} (-1)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \frac{1}{(k+s+1/2)!} \quad (k > 0), \quad (2,3)$$