where C is an arbitrary constant. When $C = 0$, one finds the solution $\mu_0(\alpha)$. Substituting (13c) into (13a) and (11a) and then (10a)

$$s = (1/2)r \sin 2\theta[1 - \alpha \cos 2\theta]^{-1/2}. \tag{14}$$

Since $s(r, \theta)$ satisfies equations (10a, b), $u(s)$, $w(s)$ satisfy the Blasius Equation (8a, b). Writing s in terms of physical coordinates, we have

$$r = Z[U_\infty(x + y)/\nu xy]^{1/2}, \qquad \sin \theta = (x/x + y)^{1/2}, \qquad \cos \theta = (y/x + y)^{1/2}, \tag{15a}$$

$$s = 2Z(U_\infty/\nu)^{1/2}[(x + y) - a(x - y)]^{-1/2}. \tag{15b}$$

The constant velocity surfaces are the surfaces $s =$ const. since $u = Bl(s)$. It remains to determine the constant $| a |$ in such a way that as $X/Y \to \infty$, s becomes proportional to $Z(u_\infty/\nu y)^{1/2}$ so that one obtains the Sears yawed plate solution. This is the case if $a = -1$, $(x > y)$; $a = 1$ $(x < y)$.

It follows that the present solution extends Sears' result up to the immediate corner of the plate with no change. The constant shear and boundary layer thickness lines are parallel to the axes up to the axis of symmetry. This indicates the existence of a narrow region there where the cross flow terms in the velocity Laplacian are not negligible and the boundary layer equations do not hold.

References

1. H. Blasius, *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Zeits. Math. Phys. **56**, 1 (1908).
2. W. R. Sears, *The boundary layer of yawed cylinders*, Journ. Aero. Sci. **15**, 49 (1948).

NOTE ON MAXIMUM SHOCK DEFLECTION*

By GARRETT BIRKHOFF AND JOHN W. WALSH (*Harvard University*)

The angle $\lambda = \lambda(M)$ of maximum shock deflection, for a given Mach number M of flow, is of interest in various applications. It gives the critical angle for *attached shocks* past a wedge [2, p. 53], and that for *jetless wedge collapse* [3]. We give here a new simple means of determining $\lambda(M)$, for ideal gases, which seem simpler than the usual one [1, §122].

We follow the notation of [2]. By formulas (4.3) and (3.3) of this reference,

$$q_{2n} = \frac{a^{*2}}{q_{1n}} \quad \text{and} \quad \frac{a^{*2}}{q_{1n}^2} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{a^2}{q_{1n}^2} \tag{1}$$

in a polytropic gas, with $p + p_0 = A\rho^\gamma$.

Now suppose we are given the velocity q_1, relative to J, of one impinging stream, and the shock angle β between the impinging stream and the shock front, as in Fig. 1. Of course, β is not known a priori; we shall seek, by variation of β, that value of β which maximizes the deflection angle δ, and thus obtain the desired maximum deflection $\delta_{\max} = \lambda$.

We suppose also that we know p, ρ in the impinging stream. Then the normal shock,

*Received July 15, 1953.

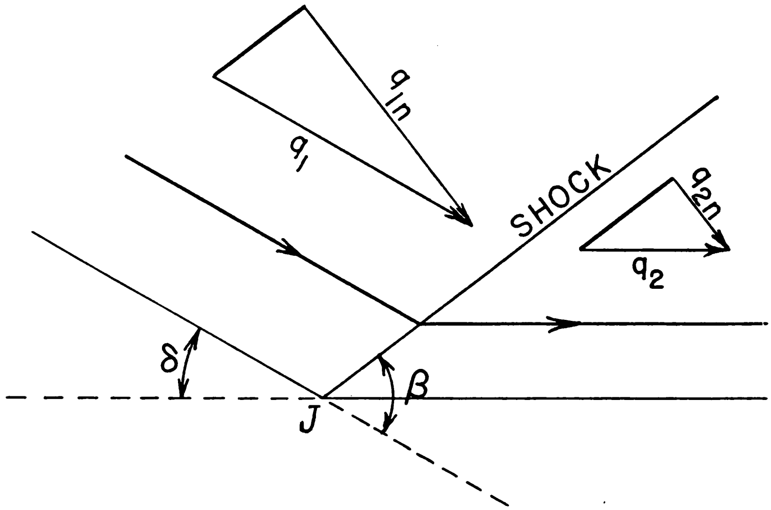

FIG. 1. Stream deflection by a shock.

associated as usual with the oblique shock by considering those velocity components that are normal to the shock front, has on the supersonic (approaching) side a velocity of

$$q_{1n} = q_1 \sin \beta. \tag{2}$$

Substituting from (2) and (1), we obtain the important formula

$$\frac{q_{2n}}{q_1} = \left[\frac{a^*}{q_{1n}}\right]^2 \frac{q_{1n}}{q_1} = \sin \beta \left[\frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1} \frac{a^2}{q_1^2} \csc^2 \beta\right]. \tag{3}$$

Equivalently, since $M_1^2 = q_1^2/a^2$,

$$\frac{q_{2n}}{q_1} = A \sin \beta + K \csc \beta, \tag{4}$$

where A and K are given by

$$A = (\gamma - 1)/(\gamma + 1), \qquad K = 2/M_1^2 (\gamma + 1), \tag{5}$$

for an ideal gas.

Using trigonometry, we now find that the component of q_2 *parallel* to q_1 is (cf. Fig. 1)

$$q_1 \cos^2 \beta + q_{2n} \sin \beta. \tag{6}$$

The component of q_2 *perpendicular* to q_1 is similarly

$$q_1 \cos \beta \sin \beta - q_{2n} \cos \beta. \tag{6'}$$

Taking the ratio, after dividing through the numerator and denominator by q_1, and using (4), we get

$$\tan \delta = \frac{\cos \beta \sin \beta - \cot \beta \,(A \sin^2 \beta + K)}{\cos^2 \beta + A \sin^2 \beta + K}. \tag{7}$$

We next rationalize the right side of (7), by the substitution $\tau = \cot \beta$. This gives $\csc^2 \beta = \tau^2 + 1$. Substituting in (6′) after dividing through the numerator and denominator by $\sin^2 \beta$, we get

$$\tan \delta = \tau\left[\frac{1 - A - K(\tau^2 + 1)}{\tau^2 + A + K(\tau^2 + 1)}\right] = \tau\left[\frac{E}{F + G\tau^2} - B\right], \tag{8}$$

where

$$B = K/(K + 1), \qquad E = (1 - A)/(K + 1), \qquad F = A + K,$$
$$G = K + 1. \tag{8'}$$

This is equivalent to [2, formula (4.26)]. This is, incidentally, a very convenient formula to use in computing shock polars.

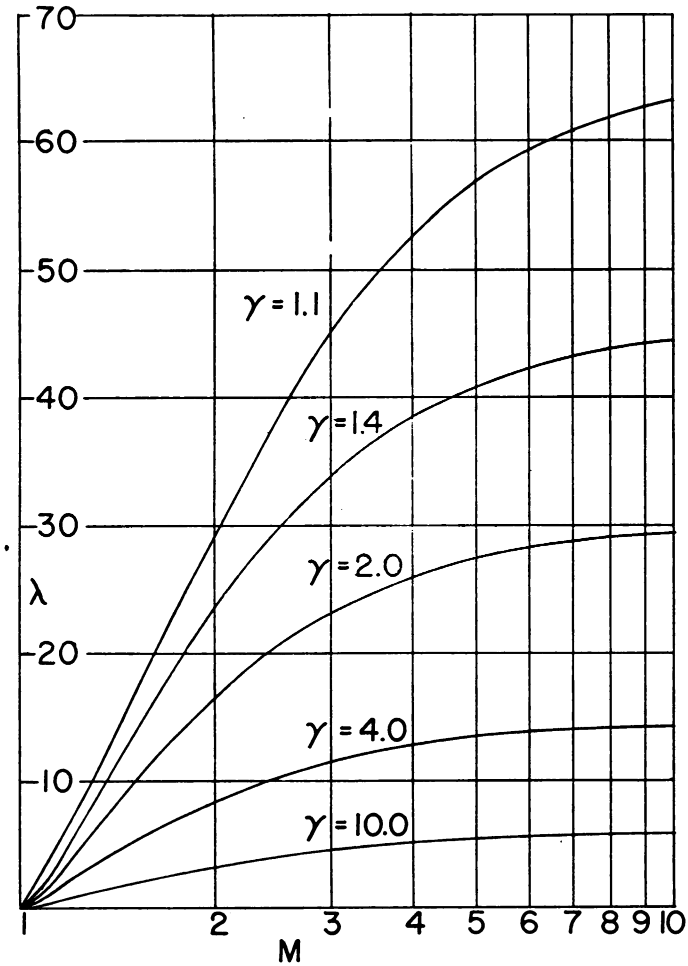

FIG. 2. Maximum stream deflection λ, as a function of γ and the Mach number M of the entry flow.

It is especially convenient in computing the *limiting angle* $\delta_{\max} = \lambda$. Since $\tan \delta$ is an increasing function, this is equivalent to maximizing the right-hand side of (8). But this maximum may easily be found by setting the derivative equal to zero. The derivative is

$$-B + \frac{E}{F + G\tau^2} - \frac{2EG\tau^2}{(F + G\tau^2)^2}.$$

Multiplying through by the denominator, we see that this derivative vanishes if and only if τ^2 satisfies

$$BG^2\tau^4 + (2BFG + EG)\tau^2 + (BF^2 - EF) = 0. \tag{9}$$

Our interest is confined to $M > 1, \gamma > 1$. Then it may easily be shown, using (8′), that the two roots of the equation are real; further, one of these roots is negative and has no physical significance since it leads to an imaginary value of τ.

A graph of $\lambda(M, \gamma)$, computed by using formula (9), is shown as Fig. 2.

BIBLIOGRAPHY

[1] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience, 1948.

[2] H. W. Liepmann and A. E. Puckett, *Introduction to aerodynamics of a compressible fluid*, John Wiley and Sons, 1947, 51-60.

[3] J. M. Walsh, R. G. Shreffler and F. J. Willig, *Limiting Conditions for jet formation in high velocity collisions*, J. Appl. Phys. 24 (1953), 349-359.

ON THE THEORY OF THE BULGE TEST[1]

BY E. W. ROSS, JR. AND W. PRAGER (*Brown University*)

Summary. It is shown that the use of Tresca's yield condition and the associated low rule leads to a simple theory for the bulge test for perfectly plastic or strain-hardening naterials. The basic equations can be integrated in closed form even for finite deflections.

1. Introduction. The ductility of sheet metal under balanced biaxial tension is letermined by the bulge test: a circular sheet of uniform thickness is clamped round he periphery and subjected to unilateral fluid pressure which causes the sheet to bulge lastically. The strain at the pole of the bulge is measured by means of a grid inscribed n the originally flat sheet, and the stress at the pole is computed from the applied ressure and the measured curvature and thickness of the deformed sheet. The dimenions of the sheet are chosen so that its flexural stiffness is negligible[2]; on the other hand, he strains cannot be treated as infinitesimal.

The first consistent theory of the bulge test was given by Hill.[3] This theory is based n the yield condition and flow rule of v. Mises.[4] It is assumed that the relevant quantities an be represented as power series in the ratio between the maximum deflection and the adius of the die aperture through which the sheet is made to bulge. Powers higher than

[1]Received Nov. 13, 1953. The results presented in the paper were obtained in the course of research onsored by Watertown Arsenal Laboratory under Contract DA-19-020-ORD-2598.

[2]Even for a very thin sheet neglecting the flexural stiffness may not be justified in the neighborhood the edge. Such edge effects are known to be highly localized, however, and may therefore be neglected the discussion of the states of stress and strain in the neighborhood of the pole of the bulge. For a scussion of plastic edge effects the reader is referred to a paper by F. K. G. Odqvist (Reissner Annivery Volume, J. W. Edwards, Ann Arbor, Mich., 1949, p. 449).

[3]R. Hill, Phil. Mag. (7) 41, 1133-1142 (1950).

[4]R. v. Mises, Goettinger Nachrichten 1913, 582-592 (1913).