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ON THE TRANSFORMATION OF THE LINEARIZED EQUATION OF UNSTEADY SUPERSONIC FLOW*

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Summary. The linearized potential equation for unsteady motion in frictionless, supersonic flow is transformed from the classical wave equation to the canonical form $\phi_{xx} - \phi_{yy} - \phi_{zz} = \phi_{\tau\tau}$ with the aid of a modified Lorentz transformation. Possible invariant transformations of the latter, including the classical Lorentz transformation, are discussed. Eleven coordinate systems (each of which has its counterpart in the classical theory of the wave equation) permitting separation of variables are set forth, their derivation being based on the analogy between the hyperbolic metric defined by $(ds)^2 = (dx)^2 - (dy)^2 - (dz)^2$ and the Euclidean (Cartesian) metric. A few practical applications are indicated.

1. Introduction. We consider here the linearized equation for the velocity potential in unsteady, supersonic flow and various coordinate transformations to which it may be usefully subjected in order to effect separation of variables.

In connection with the linearization of the original equations we remark that the assumptions implicit therein impose much stronger restrictions than in such classical fields as electricity and magnetism. Let δ be the fineness ratio, defined as the larger of the maximum thickness of a wing, or the amplitude of transverse motion, divided by the maximum wing chord l, the latter serving as the characteristic length throughout the following analysis. Further let ν be a dimensionless measure of time rate of change, where the characteristic time is lc^{-1}, c being the sonic velocity in the undisturbed medium, and let sl be the average wing span. Then an extension of the two dimensional analysis of Lin, Reissner and Tsien[1] shows that *sufficient* conditions for linearization ($M > 1$) are

$$M\delta \ll 1, \qquad \nu M\delta \ll 1 \tag{1.1}$$

and any *one or more* of

$$|M - 1|, \nu, \qquad \text{or} \qquad s^{-2} \gg \delta^{2/3}, \tag{1.2}$$

where M is the free stream Mach number.

In the case of slender bodies slightly different restrictions obtain.[2]

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[1]C. C. Lin, E. Reissner and H. S. Tsien, J. Math. & Ph. 27, 220 (1948).

[2]J. W. Miles, J. Aero. Sci. 19, 380 (1952).

Because of these restrictions the results obtained herein probably have but limited practical application. Nevertheless, we feel that they are of interest *per se* and, in addition, may lead to useful results in the hands of other investigators. Moreover, they may be of some value in attacking the non-linear equations.

2. The potential equation. The linearized potential equation governing small disturbances with respect to a fixed coordinate system in a perfect fluid is (the wave equation)

$$\phi_{XX} + \phi_{YY} + \phi_{ZZ} = \phi_{TT} , \tag{2.1}$$

where X, Y, Z are dimensionless Cartesian coordinates in a *fixed* reference frame, T is a dimensionless time obtained by multiplying the true time (t) by the sonic velocity (c) and dividing by l, and ϕ is the dimensionless velocity potential (reference quantity: Ul). In the case of a supersonic flight at Mach number M along the negative X axis, the body in question may be brought to rest and (1) reduced to (what may be regarded as) a canonical form by introducing the modified Lorentz transformation

$$\begin{pmatrix} x \\ \tau \end{pmatrix} = (M^2 - 1)^{-1/2}\begin{pmatrix} 1 & M \\ M & 1 \end{pmatrix}\begin{pmatrix} X \\ T \end{pmatrix}, \qquad y = Y, \qquad z = Z \tag{2.2}$$

in which case we have

$$\Box\phi = \phi_{xx} - \phi_{yy} - \phi_{zz} = \phi_{\tau\tau} , \tag{2.3}$$

where $\Box$ may be designated as the "hyperbolic Laplacian" operator.*

Equation (3) may be identified as the wave equation in the coordinates (x, iy, iz, τ), just as Bateman[3] has identified the classical wave equation as Laplace's equation, albeit four dimensional, in (X, Y, Z, iT). Such an identification suggests, by analogy, numerous solutions to (3), which, to be sure, must be appropriately restricted if they are to correspond to physical reality.

If in (3) we pose the harmonic dependence† exp $(-i\kappa\tau)$, with the generality implied by Fourier's theorem, we obtain

$$\Box\phi + \kappa^2\phi = 0 \tag{2.4}$$

which may be appropriately designated as the "hyperbolic Helmholtz equation".

The origin of the moving (x, y, z) coordinates is conveniently chosen at the most upstream point of the body creating the disturbance. Then, in consequence of the supersonic flight velocity, we may assert

$$\phi \equiv 0, \qquad x < 0. \tag{2.5}$$

Accordingly, it is often expedient to introduce the Laplace transformation

$$\Phi = \mathfrak{L}\{\phi\} = \int_0^\infty e^{-sx}\phi\, dx \tag{2.6}$$

*This notation has been used by P. A. Lagerstrom in unpublished lectures at the California Institute of Technology.

[3]H. Bateman, *Partial differential equations*, Dover Publ., New York, 1944, p. 384.

†The choice $\kappa = (M^2 - 1)^{-1/2}(\omega/c)$ yields the dependence exp $(i\omega t)$ in the original time domain; *cf.* (2.9) *infra*.

which, applied to (4), yields

$$\Phi_{yy} + \Phi_{zz} - \lambda^2\Phi = 0, \tag{2.7}$$

$$\lambda^2 = s^2 + \kappa^2. \tag{2.8}$$

In the application of the end results, it is most convenient to deal with a coordinate x^* measured from the foremost point on the airfoil and the true time t, the corresponding potential being given by

$$\begin{aligned}\phi^*(x^*, y, z, t) &= \phi(x, y, z, \tau)\\ &= \phi[(M^2 - 1)^{-1/2}x^*, y, z, M(M^2 - 1)^{-1/2}x^* - (M^2 - 1)^{1/2}(ct/l)].\end{aligned} \tag{2.9}$$

However, in all of the subsequent discussion we shall deal implicitly with x and τ.

3. Invariant transformations. The general question of transformations under which the classical wave equation remains invariant has been discussed in a series of papers by Bateman.[4] The corresponding transformations of (2.3) and (2.4) follow *via* the analogy suggested above.

It is immediately evident that (2.3) is invariant under independent translations of (x, y, z, τ), a spherical rotation of (y, z, τ) with x fixed in direction (*N.B.*: (2.3) is not invariant under a rotation involving x, as, *e.g.*, in the case of a transformation to Mach coordinates.), a (simultaneous) scale transformation of (x, y, z, τ), and a scale transformation of ϕ.

Rather less obvious are the inversions studied by Bateman. The simplest of these imply that if $\phi(x, y, z, \tau)$ is a solution to (2.3) so also are

$$\psi(x, y, z, \tau) = \mu\phi(\mu x, \mu y, \mu z, \mu\tau), \tag{3.1}$$

$$\mu = (x^2 - y^2 - z^2 - \tau^2)^{-1}, \tag{3.2}$$

$$\chi(x, y, z, \tau) = \nu\phi[\tfrac{1}{2}\nu(y^2 + z^2 + \tau^2 + 1), \tfrac{1}{2}\nu(y^2 + z^2 + \tau^2 - 1), \nu z, \nu\tau], \tag{3.3}$$

$$\nu = (x - y)^{-1}. \tag{3.4}$$

As an example of a more general result, a homogeneous solution of degree -1 is given by

$$\begin{aligned}\phi(x, y, z, \tau) &= (x + y)^{-\alpha}(x - y)^{-\alpha'}(z + i\tau)^{-\beta}(z - i\tau)^{-\beta'}\\ &\quad (x^2 - y^2 - z^2 - \tau^2)^{-\gamma}(-1)^{-\gamma'}P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}\end{aligned} \tag{3.5}$$

from which additional transformations may be obtained *via* the many transformations of the generalized hypergeometric function P (in the Riemann-Papperitz notation).

Perhaps the most interesting (but not necessarily the most important, since many valuable inferences are afforded by the various scale transformations) transformation under which (2.3) remains invariant is that of Lorentz, which is most conveniently written in the normalized form (so that all transformations obtained by assigning

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[4] H. Bateman, Proc. London Math. Soc. (2) **8**, 223 (1909); *ibid* **7**, 70 (1909); **8**, 469 (1910); **10**, **7** (1911).

different values to m are members of a group; *cf.* the work of Lagerstrom[5] and Hayes[6] in steady flow).

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (1 - m^2)^{-1/2} \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad | m | < 1. \tag{3.6}$$

This result has its most direct application in extending a solution for a wing leaving the wing tip $y = 0$ to one making an angle $\tan^{-1} m$ with the direction of flight. (But if m is negative the original tip becomes a trailing edge, and the Kutta condition then must be introduced.) This result has been applied to the problem of a rated wing tip in unsteady flow, starting from the known solution for a rectangular wing and applying the Kutta condition where the edge is trailing (see J. W. Miles, Q. Appl. Math., **11**, 363; 1953).

Additional transformations under which (2.3) remains invariant may be obtained from symmetry considerations among (y, z, τ). Finally, (2.4) is invariant under all of the foregoing transformations not involving τ and to a simultaneous scale transformation of (x, y, z, κ^{-1}).

4. Coordinate transformations. We shall consider only those coordinate transformations

$$x, y, z = x, y, z(q_1, q_2, q_3) \tag{4.1}$$

for which the hyperbolic line element transforms according to

$$(ds)^2 = (dx)^2 - (dy)^2 - (dz)^2 = h_1^2(dq_1)^2 - h_2^2(dq_2)^2 - h_3^2(dq_3)^2 \tag{4.2}$$

where $h_{1,2,3,}$ are positive, real coefficients.* In the sense that the metric is diagonal, these transformations may be said to be orthogonal, but the absence of cross products like $q_i q_j$ does not necessarily imply the (Euclidean) geometric orthogonality of the parametric family of surfaces q_i = constant with the family q_j = constant. [*E.g.*, x' and y' of (3.6) are not (Euclidean) orthogonal coordinates, but their metric does satisfy (2) above.]

Introducing the transformation defined by (1) and (2) in (2.3), we have by analogy with Lamé's transformation[7] of Laplace's equation

$$\Box\phi = (h_1 h_2 h_3)^{-1} \left[\left(\frac{h_2 h_3}{h_1} \phi_{q_1} \right)_{q_1} - \left(\frac{h_3 h_1}{h_2} \phi_{q_2} \right)_{q_2} - \left(\frac{h_1 h_2}{h_3} \phi_{q_3} \right)_{q_3} \right]. \tag{4.3}$$

It would, of course, be possible to include τ in the transformation, writing

$$(ds)^2 = (dx)^2 - (dy)^2 - (dz)^2 - (d\tau)^2 \tag{4.4}$$

with an obvious extension of (3) for $\Box\phi - \phi_{\tau\tau}$. However, aside from the Lorentz transformation (2.2) we shall include τ only in some rather simple homogeneous transformations, where the introduction of metrical coefficients would appear rather ponderous.

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[5]P. A. Lagerstrom, Jet Propulsion Lab. Rep. 4-36; NACA, T.N. 1685 (1948).

[6]W. D. Hayes, Thesis, Calif. Inst. Tech., Pasadena, Calif., 1947.

*The introduction of the hyperbolic distance in the study of the wave equation is of course not new, having been developed in some generality by Hadamard (ref. 10, *infra;* also ref. 16, 430 ff.). However, the particular problem of separation of variables in hyperbolic space seems to have received little previous consideration.

[7]E. T. Whittaker and G. N. Watson, *Modern analysis*, Macmillan Co., New York, 1948, p. 401.

5. Methods of solution. We shall concern ourselves primarily with obtaining solutions to (2.3), (2.4) and (2.7) by separating variables. It might be thought sufficient to seek solutions of (2.1), on which a considerable literature already exists, but this is not generally the case, due principally to the difficulties associated with moving boundaries. Nevertheless, several interesting results may be so obtained[8], and the approach has the advantage of physical clarity, since T is a direct measure of physical time, whereas τ is not.

An alternative attack based on the existing literature for the wave equation would be to rewrite (2.3) in the form

$$\phi_{\tau\tau} + \phi_{yy} + \phi_{zz} = \phi_{xx} . \tag{5.1}$$

[In view of (1), the remark in ref. 8 that there is no transformation that will fix the coordinate system in the wing and still yield the wave equation seems to require some modification.] This approach has proved quite fruitful in the steady flow case ($\phi_{\tau\tau} = 0$), due both to the physical and mathematical analogies afforded (*e.g.*, von Kármán's acoustic analogy[9]) and, more importantly, the applicability of Hadamard's method[10]. (Nevertheless, in the light of subsequent developments, the most successful of the general methods applicable to the steady flow wing problem appear to have been those of Busemann[11,12,13] and Evvard,[14,15] for which no clearly defined antecedents existed in the classical literature.) While Hadamard's method is not directly applicable to the three dimensional wave equation[16], there exists the even more elegant method of Marcel Riesz[17]. We have not investigated the application of Riesz's method to the unsteady flow problem, but some consideration has been given to this matter by P. A. Lagerstrom[18]. It would appear to be of interest primarily in obtaining a solution to the direct wing problem (ϕ_z specified everywhere on $z = 0$) and in formulating the integral equation for the indirect wing problem (different derivatives of ϕ specified over different parts of $z = 0$).

We turn now to the problem of separation of variables. By analogy with the classical wave equation[19,20] there must exist eleven coordinate systems in which the hyperbolic Helmholtz equation of (2.4) is separable. In general, these systems will differ from their counterparts in Euclidean space, but the cylindrical (with respect to the x axis) systems remain unchanged. Since in each (cylindrical) case the separation of x yields an exponential solution, we consider for these systems the solution of the modified equation (2.7).

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[8] H. Lomax, M. A. Heaslet, and F. B. Fuller, NACA, T.N. 2256 (1950).

[9] Th. von Kármán, J. Aero Sci. **14**, 373 (1947).

[10] J. Hadamard, *Lectures on Cauchy's problem,* Yale Univ. Press, 1923.

[11] A. Busemann, Luftfahrtforschung **12**, 210 (1935).

[12] S. Goldstein and G. N. Ward, Aero. Q. **2**, 39 (1940).

[13] Lagerstrom, l. c. ante.

[14] J. C. Evvard, NACA, T.N. 1382 (1947).

[15] G. N. Ward, Q. J. Mech. and Appl. Math. **2**, 136 (1949).

[16] R. Courant & D. Hilbert, *Methoden der mathematische Physik,* J. Springer, Berlin, 1938, vol. 2, p. 443.

[17] M. Riesz, Acta Math. **81**, 1-218 (1949): see also B. Baker and E. T. Copson, *Huygens' principle* Oxford U. Press, 1950, p. 57.

[18] In the unpublished lectures referred to above.

[19] H. P. Robertson, Math. Ann. **98**, 749 (1938).

[20] L. P. Eisenhart, Ph. Rev. **45**, 427 (1934); Ann. Math. **35**, 284 (1934).

6. Cartesian coordinates. Equation (2.7) is separable in the following cylindrical systems: Cartesian, polar, elliptic and parabolic. The separated solutions in Cartesian coordinates are exponential, and the method of generalization is that of Fourier. The great majority of the literature on the supersonic wing problem uses Cartesian coordinates, and no further discussion is warranted here.

7. Cylindrical polars. Let (ρ, φ) be the polar coordinates defined by

$$y + iz = \rho e^{i\varphi}. \tag{7.1}$$

The corresponding transformation of (2.7) yields

$$\rho(\rho\Phi_\rho)_\rho + \Phi_{\varphi\varphi} - (\lambda\rho)^2\Phi = 0. \tag{7.2}$$

The separated solution of (2), which differs from its classical counterpart only in the sign of λ^2, is given by

$$\Phi = K_\mu(\lambda\rho)e^{i\mu\varphi} \tag{7.3}$$

where K_μ is Macdonald's solution to Bessel's equation of order μ and is dictated (in preference to alternative Bessel functions) by the boundary condition at infinity.

In the case of a rectangular wing edge ($\rho = 0$) with upper and lower surfaces $\varphi = 0$ and π, respectively, the solution

$$\left(\frac{\partial}{\partial y}\right)^{m+1}\Phi^{(m)} = K_{m+1/2}(\lambda\rho)\cos\left[\left(m + \frac{1}{2}\right)\varphi\right] \tag{7.4}$$

is appropriate to the boundary condition

$$\Phi_z^{(m)}\big|_{z=0\pm} \sim y^m. \tag{7.5}$$

(More generally, y^m can be replaced by an m-th order polynomial in y.) and may be used to construct a general solution for the oscillating rectangular wing. This approach has been used by Rott[21] for the special case $m = 0$, following Lamb's treatment of the half plane diffraction problem.[22]

8. Elliptic cylinder coordinates. The well known transformation

$$y + iz = b\cosh(\xi + i\eta) \tag{8.1}$$

yields

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} - (\lambda b)^2(\cosh^2\xi - \cos^2\eta)\Phi = 0 \tag{8.2}$$

which separates into Mathieu equations in both ξ and η. A general solution of (2), subject to the appropriate null condition at infinity, is given by

$$\Phi = Gek_n(\xi, -q)[A_n ce_n(\eta, -q) + B_n se_n(\eta, -q)], \tag{8.3}$$

$$q = \tfrac{1}{4}\lambda^2 b^2, \tag{8.4}$$

where the notation is that of McLachlan[23].

[21] N. Rott, J. Aero. Sci. 18, 775 (1951).

[22] H. Lamb, Proc. Lon. Math. Soc. (2), 4, 190 (1906); *ibid* 8, 422 (1910); *Hydrodynamics*, Dover Publ., New York, 1945, p. 538.

[23] N. W. McLachlan, *Theory and application of Mathieu functions*, Oxford U. Press, 1947.

These coordinates have been used to obtain a general solution for an oscillating rectangular wing of (in principle) arbitrary aspect ratio[24], following Sieger's solution of the analogous diffraction problem[25]. While the Laplace inversion of the resulting solution in terms of tabulated functions does not appear to be possible, an expansion in powers of λb leads to an asymptotic (in x) expansion for the potential and to integrals (*e.g.*, lift and moment) that are useful for values of the "effective" aspect ratio less than unity. [The solution for a single edge (*vide supra*) suffices to handle the greater than unity case.]

9. Parabolic cylinder coordinates. In this case, we have

$$y + iz = \tfrac{1}{2}(\xi + i\eta)^2, \tag{9.1}$$

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} - \lambda^2(\xi^2 + \eta^2)\Phi = 0 \tag{9.2}$$

and the resulting solution appears in the form

$$\Phi = D_\mu[(2\lambda)^{1/2}\xi]D_{-\mu-1}[(2\lambda)^{1/2}\eta] \tag{9.3}$$

where D_μ is Weber's parabolic cylinder function of order μ in the notation of Whittaker[26].

In practice the manipulation of the D_μ is involved, but, having introduced (ξ, η), it may be possible to write down solutions in terms of more elementary functions. Thus, following Lamb (ref. 22), we find the solution

$$\Phi = e^{\lambda\xi\eta}\left[A + B\int_0^{(\lambda/2)^{1/2}(\xi+\eta)} e^{-\zeta^2}\,d\zeta\right] + e^{-\lambda\xi\eta}\left[C + D\int_0^{(\lambda/2)^{1/2}(\xi-\eta)} e^{-\zeta^2}\,d\zeta\right]. \tag{9.4}$$

Evaluating A, B, C, D, by the imposition of the boundary conditions

$$\Phi_z|_{z=0\pm} = -W(s), \qquad y > 0 \tag{9.5a}$$

$$\Phi = 0, \qquad y \le 0 \tag{9.5b}$$

together with the null requirement at infinity, we find for the Laplace transform of the potential on the upper surface ($\eta = 0$) of a rectangular wing

$$\Phi|_{z=0+} = \lambda^{-1}W(s)\ \mathrm{erf}\ [(\lambda y)^{1/2}] \tag{9.6}$$

in agreement with the result obtained by the method suggested in section 7 for the case where the boundary condition on the wing is independent of y.

10. Hyperbolic coordinates. Let r denote the hyperbolic radius, *viz.*

$$r = (x^2 - y^2 - z^2)^{1/2}. \tag{10.1}$$

Then the Euclidean transformation to spherical polar coordinates suggests the analogous transformation

$$x = r\cosh\xi, \qquad y = r\sinh\xi\cos\varphi, \qquad z = r\sinh\xi\sin\varphi. \tag{10.2}$$

If we restrict both r and ξ to be positive and real, only points within the downstream Mach cone $[x > (y^2 + z^2)^{1/2}]$ of the origin are included in the transformation, and the surfaces obtained by holding r, ξ or φ constant are circular hyperboloids (of two sheets,

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[24] J. W. Miles, Aero. Q. 4, 231 (1953).
[25] B. Sieger, Ann. d. Physik 27, 626 (1908).
[26] ref. 7, sec. §16.5.

although only the sheet directed along $+x$ is included), circular cones and planes, respectively. The Mach cone itself, being a characteristic surface of $\Box\phi$, has the degenerate specification $r = 0$ *and* $\xi = \infty$. We remark that the tangent planes to the surfaces $r =$ constant and $\xi =$ constant have complementary (rather than perpendicular) slopes.

Introducing the transformation (2) in (4.3) we obtain

$$\Box\phi = (r \sinh \xi)^{-2}[\sinh^2 \xi(r^2\phi_r)_r - \sinh \xi(\sinh \xi\phi_\xi)_\xi - \phi_{\varphi\varphi}] \tag{10.3}$$

and, separating variables, a general solution to (2.4) is given by

$$\phi = r^{-1/2}Z_{\nu+1/2}(\pi r)B_\nu^\mu(\cosh \xi)e^{i\mu\varphi} \tag{10.4}$$

where $Z_{\nu+\frac{1}{2}}$ is a solution to Bessel's equation of order $\nu + \frac{1}{2}$, and B_ν^μ is a solution to the generalized Legendre equation

$$[(1 - z^2)B_z]_z + [\nu(\nu + 1) - \mu^2(1 - z^2)^{-1}]B = 0. \tag{10.5}$$

Solutions of type (4) have been studied by Hayes (ref. 6) for the case of steady flow ($\eta = 0$), where the r dependence reduces to r^ν. The resulting solutions are homogeneous of order ν, or, in the terminology of the day, "generalized conical flows".

An alternative attack on (2.3), after posing the dependence on ξ and φ already found, is to assume the solution to be homogeneous in (τ/r). Thus, it is found that a homogeneous solution of order k is given by

$$\phi = r^k \mid 1 - (\tau/r)^2 \mid^{1/2(k+1)}B_\nu^{k+1}(\tau/r)B_\nu^\mu(\cosh \xi)e^{i\mu\varphi} \tag{10.6}$$

a result that is reminiscent of a homogeneous solution to the wave equation proposed by Bateman[27] and has possible application to transient loading problems.

11. Prolate hyperboloidal coördinates. This system is derived by analogy to the conventional prolate spheroidal set. Modifying the transformation to the latter, we arrive at the new transformation

$$\begin{aligned} x &= \xi\eta - 1 \\ y &= (\xi^2 - 1)^{1/2}(\eta^2 - 1)^{1/2} \cos \varphi \\ z &= (\xi^2 - 1)^{1/2}(\eta^2 - 1)^{1/2} \sin \varphi. \end{aligned} \tag{11.1}$$

As in §10, the entire manifold (ξ, η, φ) is mapped in the downstream Mach cone from the (x, y, z) origin, but to avoid ambiguity we impose the restrictions

$$1 \leq \eta < \xi. \tag{11.2}$$

The surface of revolution $\xi =$ constant, as defined by

$$\frac{(x + 1)^2}{\xi^2} - \frac{(y^2 + z^2)}{(\xi^2 - 1)} = 1 \tag{11.3}$$

is evidently a circular hyperboloid of two sheets (*cf.* §10) directed along the x axis. The same result holds for $\eta =$ constant, it being necessary only to replace ξ by η in (3). Again, the respective families of surfaces are not orthogonal in the Euclidean sense.

Introducing the transformation (1) in (4.3), we obtain

$$\Box\phi = (\xi^2 - \eta^2)^{-1}\{[(\xi^2 - 1)\phi_\xi]_\xi - [(\eta^2 - 1)\phi_\eta]_\eta - (\xi^2 - 1)^{-1}(\eta^2 - 1)^{-1}\phi_{\varphi\varphi}\}. \tag{11.4}$$

[27] ref. 3, p. 384.

Posing the solution

$$\phi = f(\xi)g(\eta)e^{i\mu\varphi} \tag{11.5}$$

in (2.4), separation of variables yields the Lamé equation

$$[(\xi^2 - 1)f_\xi]_\xi + [\kappa^2\xi^2 - \mu^2(\xi^2 - 1)^{-1} + \nu]f = 0 \tag{11.6}$$

and an identical equation for $g(n)$.

We remark that $f(\xi)$ and $g(\eta)$ need not be the same solution to (6); *i.e.*, they may be Lamé functions of the first and second kinds, respectively, or *vice versa.*

12. Oblate hyperboloidal coordinates. Modifying the conventional transformation to oblate spheroidal coordinates, we write

$$\begin{aligned} x &= \xi\eta, \\ y &= (\xi^2 + 1)^{1/2}(\eta^2 - 1)^{1/2}\cos\varphi, \\ \xi &= (\xi^2 + 1)^{1/2}(\eta^2 - 1)^{1/2}\sin\varphi. \end{aligned} \tag{12.1}$$

In this case, symmetry exists between ξ and $i\eta$, and the surfaces ξ = constant and η = constant, *viz.*

$$\frac{x^2}{\xi^2} - \frac{(y^2 + z^2)}{(\xi^2 + 1)} = 1, \tag{12.2}$$

$$\frac{(y^2 + z^2)}{(\eta^2 - 1)} - \frac{x^2}{\eta^2} = 1 \tag{12.3}$$

are circular hyperboloids of two and one sheets, respectively, directed along the x axis. The restricted range

$$1 \leq \eta < +(\xi^2 + 1)^{1/2} \tag{12.4}$$

includes all points inside the downstream Mach cone, while $\eta > (\xi^2 + 1)^{1/2}$ gives points outside of this cone.

The transformation of the hyperbolic Laplacian yields

$$\Box\phi = (\xi^2 + \eta^2)^{-1}\{[(\xi^2 + 1)\phi_\xi]_\xi - [(\eta^2 - 1)\phi_\eta]_\eta - (\xi^2 + 1)^{-1}(\eta^2 - 1)^{-1}\phi_{\varphi\varphi}\} \tag{12.5}$$

and the resulting separation of (2.4) again leads to Lamé functions.

13. Hyperboloidal coordinates. Modifying the conventional transformation to ellipsoidal coordinates (in the notation of Hobson[28]), we write

$$\begin{aligned} x &= h^{-1}k^{-1}\xi\eta\zeta, \\ y &= h^{-1}(k^2 - h^2)^{-1/2}(\xi^2 - h^2)^{1/2}(\eta^2 - h^2)^{1/2}(\zeta^2 - h^2)^{1/2}, \\ z &= k^{-1}(k^2 - h^2)^{-1/2}(\xi^2 - k^2)^{1/2}(k^2 - \eta^2)^{1/2}(\zeta^2 - k^2)^{1/2}, \end{aligned} \tag{13.1}$$

$$0 < h \leq \eta \leq k \leq \xi < \zeta. \tag{13.2}$$

The restricted range of (2) includes all points within the Mach cone.

The coordinate surfaces are defined by

$$\frac{x^2}{\xi^2} - \frac{y^2}{\xi^2 - h^2} - \frac{z^2}{\xi^2 - k^2} = 1, \tag{13.3}$$

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[28]E. W. Hobson, *Spherical and ellipsoidal harmonics*, Cambr. U. Press, 1931, Ch. XI.

where ξ may be replaced by either η or ζ. The ξ and ζ surfaces are elliptic hyperboloids of two sheets directed along the x axis (*i.e.*, the section x = constant yields an ellipse, while either y or z = constant yields a hyperbola), while the η surfaces are elliptic hyperboloids of one sheet directed along the y axis.

The hyperbolic Laplacian is given by

$$\Box\phi = (\zeta^2 - \xi^2)^{-1}(\zeta^2 - \eta^2)^{-1}L_\zeta\{\phi\} - (\zeta^2 - \xi^2)^{-1}(\xi^2 - \eta^2)^{-1}L_\xi\{\phi\} - (\zeta^2 - \eta^2)^{-1}(\xi^2 - \eta^2)^{-1}L_\eta\{\phi\}, \tag{13.4}$$

$$L_\xi\{\phi\} = (\xi^2 - h^2)^{1/2}(\xi^2 - k^2)^{1/2}[(\xi^2 - h^2)^{1/2}(\xi^2 - k^2)^{1/2}\phi_\xi]_\xi\ , \tag{13.5}$$

and the resulting separation of (2.4) yields Lamé functions in each of the three variables. L_ζ is identical with L_ξ , but $(\xi^2 - k^2)^{1/2}$ must be replaced by $(k^2 - \eta^2)^{1/2}$ in L_η .

14. Hyperboloido-conal coordinates. If, in (13.1) *et. seq.* we assume ζ to be very large, so that ζ, $(\zeta^2 - h^2)^{1/2}$ and $(\zeta - k^2)^{1/2}$ each may be replaced by r, we have, in analogy to the spheroconal coordinates of classical potential theory (see ref. 26),

$$\begin{aligned} x &= h^{-1}k^{-1}r\xi\eta, \\ y &= h^{-1}(k^2 - h^2)^{-1/2}r(\xi^2 - h^2)^{1/2}(\eta^2 - h^2)^{1/2}, \\ z &= k^{-1}(k^2 - h^2)^{-1/2}r(\xi^2 - k^2)^{1/2}(k^2 - \eta^2)^{1/2}. \end{aligned} \tag{14.1}$$

These are the hyperboloido-conal coordinates first introduced by Robinson[29] and applied by him to delta wings in both steady and unsteady flow. (Robinson uses various notations in the papers cited and also introduces Jacobian elliptic functions). The surfaces obtained by setting r, ξ or η constant are circular hyperboloids of two sheets directed along the x axis, elliptic cones directed along the x axis and elliptic cones directed along the y axis, respectively. In particular, the surface $\xi = k$ degenerates to the triangular lamina bounded by $y = \pm k^{-1}(k^2 - h^2)^{1/2}x$ and $z = 0\pm$ and lying entirely inside the Mach cone, thereby furnishing the desired separation of variables for the delta wing with subsonic leading edges.

The hyperbolic Laplacian in these coordinates is given by

$$\Box\phi = r^{-2}[(r^2\phi_r)_r - (\xi^2 - \eta^2)^{-1}(L_\xi\{\phi\} + L_\eta\{\phi\})], \tag{14.2}$$

where L_ξ and L_η are given by (13.5). A solution to (2.3) that vanishes on the Mach cone and is regular on the x axis is given by

$$\phi = \psi_n(r,\ \tau)F_n^m(\xi)E_n^m(\eta), \tag{14.3}$$

where E and F denote Lamé functions of the first and second kind in the notation of Hobson[28], and

$$(r^2\psi_r)_r - n(n + 1)\psi = r^2\psi_{\tau\tau}\ . \tag{14.4}$$

Solutions to (4) may be obtained by comparison with the $(r,\ \tau)$ portions of (10.4) and (10.6), *viz.*

$$\psi_n(\tau,\ r) = r^{-1/2}Z_{n+1/2}(\kappa r)e^{i\kappa\tau}, \tag{14.5}$$

$$\psi_n(\tau,\ r) = r^k\ |\ 1 - (\tau/r)^2\ |^{1/2(k+1)}\ B_n^{k+1}(\tau/r). \tag{14.6}$$

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[29]A. Robinson, RAE 2151, ARC 10222 (1946); J. Roy. Aero. Soc. 52, 735 (1948); Rep. 16, Cranfield Coll. Aero. (1948); Proc. 7th. Inter. Congr. Appl. Mech. 2, 500 (1948); *cf.* also Haskind and Falkovich, Akad. Nauk. SSSR Prikl. Mat. Mech. 11, 371 (1947) and Germain and Bader, Recherche Aero. 1949, 3 (1949).

The homogeneous solutions may be applied to the solutions of the gust loading of a delta wing, but the practical difficulties entailed by the introduction of the Lamé functions are considerable.

A more detailed discussion of the properties of these very useful coordinates is given in the papers cited[29].

15. Parabolic coordinates. A set of coordinates closely related to the classical parabolic coordinates is defined by the analogous transformation [*cf.* (9.1)]

$$x = \tfrac{1}{2}(\xi^2 + \eta^2), \qquad y = \xi\eta \cos\varphi, \qquad z = \xi\eta \sin\varphi. \tag{15.1}$$

While the entire (ξ, η, φ) manifold is mapped inside the downstream Mach cone, we avoid ambiguity by imposing the restriction

$$0 \leq \eta < \xi. \tag{15.2}$$

However, the roles of ξ and η could equally well be reversed by virtue of their symmetry in (1). The surfaces ξ = const. and η = const. are non-orthogonal (in the Euclidean sense) families of paraboloids.

The transformation of the hyperbolic Laplacian yields

$$\Box\phi = (\xi^2 - \eta^2)^{-1}[\xi^{-1}(\xi\phi_\xi)_\xi - n^{-1}(\eta\phi_\eta)_\eta] - (\xi\eta)^{-2}\phi_{\varphi\varphi} \tag{15.3}$$

and a general solution to (2.4) is given by

$$\phi = \xi^{-1}\eta^{-1}W_{\nu,1/2\mu}(i\kappa\xi^2)W_{\nu,1/2\mu}(i\kappa\eta^2)e^{i\mu\varphi}, \tag{15.4}$$

where W denotes a Whittaker function[30].

16. Paraboloidal coordinates. A set of coordinates that bears the same relation to the coordinates of §13 as the (relatively little used) paraboloidal to the ellipsoidal coordinates of classical potential theory[31] is given by

$$\begin{aligned} x &= 2^{-1/2}(\xi^2 + \eta^2 + \zeta^2 - h^2 - k^2), \\ y &= (k^2 - h^2)^{-1/2}(\xi^2 - h^2)^{1/2}(\eta^2 - h^2)^{1/2}(\zeta^2 - h^2)^{1/2}, \\ z &= (k^2 - h^2)^{-1/2}(\xi^2 - k^2)^{1/2}(k^2 - \eta^2)^{1/2}(\zeta^2 - k^2)^{1/2}. \end{aligned} \tag{16.1}$$

The ranges of (ξ, η, ζ) are specified by (13.2) for points inside the Mach cone. The coordinate surfaces are given by

$$2x - \frac{y^2}{(\xi^2 - h^2)} - \frac{z^2}{(\xi^2 - k^2)} = \xi^2, \tag{16.2}$$

where ξ may be replaced by either η or ζ in (2). The surfaces ξ, η or ζ = constant are respectively elliptic, hyperbolic and elliptic paraboloids directed along the x axis.

17. Other possibilities. There exist further coordinate systems having orthogonal metrics, as assumed in (4.2), but, in view of the classic investigations of Schroedinger's equation (refs. 19, 20), it does not appear that separation of variables could be achieved in other than the eleven systems enumerated in §6-16.

It should perhaps be pointed out that there exist coordinate systems that do not satisfy (4.2) but that nevertheless may be extremely useful in practice. Thus, the rotation

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[30] ref. 7, ch. XVI

[31] J. C. Maxwell, *Treatise on electricity and magnetism*, Oxford Univ. Press, 1904, p. 240.

to "Mach coordinates", (which, of course, are also Cartesian and therefore lead to solutions by separation), furnished by

$$x = 2^{-1/2}(\xi + \eta), \qquad y = 2^{-1/2}(-\xi + \eta), \qquad z = z \tag{17.1}$$

yields

$$(ds)^2 = 2d\xi \, d\eta - (dz)^2, \tag{17.2}$$

$$\Box\phi = 2\phi_{\xi\eta} - \phi_{zz} . \tag{17.3}$$

These coordinates have been used to advantage by Evvard[32] in attacking the unsteady flow problem, although it should be remarked that his end results are of questionable validity for time dependences other than linear.*

In the case of steady flow ($\kappa = 0$) there are many more coordinate systems in which (4.2) is a valid representation and $\Box\phi = 0$ can be separated. Bipolar coordinates furnish a cylindrical example, while toroidal coordinates (in which Laplace's equation is separable) can be appropriately modified.

[32] J. C. Evvard, NACA, T.N. 1699 (1948).

*In NACA, T.N. 951 (1950) it is stated that the results of T.N. 1699 are only "approximate."