TWO DIMENSIONAL SOURCE FLOW OF A VISCOUS FLUID*

BY
H. C. LEVEY
Aeronautical Research Laboratories, Melbourne

Summary. The steady two-dimensional source-type flow of a viscous heat-conducting perfect gas is investigated. The solutions of physical significance all contain shocks, and bounds are given for the shock-thickness in terms of the shock-strength and the Reynolds number of the flow. It is found that the entropy rises to a maximum within the shock, and this maximum does not disappear even when the viscosity tends to zero.

Introduction. The main part of this investigation is concerned with two-dimensional source flow similar to the flow in a divergent channel with straight walls, for instance, in a nozzle or a diffuser, when boundary layers are neglected. If the fluid is assumed inviscid, no fundamental length exists, but if it is viscous a Reynolds number R characterizing the flow is provided by the ratio of the mass flow (per unit length normal to the plane of the flow) to the viscosity (cf. eq. (1.14)). In ordinary supersonic wind-tunnel nozzles this Reynolds number is of the order of 10^7, but in low-density, hypersonic wind tunnel nozzles (for which indeed the conical shape is being increasingly favoured) it can be of very much smaller order, and deviations of the flow from that obtained in the limit $R \rightarrow \infty$ may be of some interest. The problem has also been considered by Sakurai [1], who derived an equation similar to (1.41), and sketched the solution curves for $R \approx 20$ (see **1.4**).

The problem is, moreover, of theoretical interest since the corresponding problem for an inviscid fluid is one of the few for which an exact solution containing a limit line has been found [2]. This limit line is the sonic circle, and its exterior is doubly covered by the velocity field (**1.2**). The subsonic branch of the solution has a stagnation point at infinity and we aim to find the corresponding solutions for a viscous fluid.

The energy equation is integrated once to give two first order simultaneous differential equations for V and θ as functions of w, where V is essentially the velocity gradient, and w and θ are respectively the speed and temperature in dimensionless notation (**1.1** and **1.3**). All the solutions which have a stagnation point at infinity are shown to have formal asymptotic expansions for V and θ, in this neighbourhood, which agree to the first order with the inviscid solution when $R \rightarrow \infty$ (**1.3**).

The simultaneous differential equations for V and θ are of the singular perturbation type. The highest derivatives are multiplied by a small parameter, namely R^{-1}, and for a first approximation to equations of this type, the small parameter is taken to be zero. It is fairly obvious that as long as the highest derivatives remain 'small', there are solutions of the full equations which differ little from the solutions of the lower-order approximate equations. But it cannot be expected that the full solutions will converge *uniformly* to the approximate solutions over the whole flow field, as the parameter tends to zero. The boundary layer is a case in point, and in general, one may expect regions where the limiting solution is quite different from the inviscid approximation.

For a particular value of the Prandtl number σ, the energy equation can be integrated a second time, and the simultaneous equations are then reduced to one first

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order, non-linear differential equation for V in terms of w, which is still of the perturbed type (1.4 and eq. 6.5). This value of σ is $3/4 + 0(R^{-1})$, and as $\sigma \approx 0.72$ for air, the results obtained below should be significant.

In sec. 2, 3 and 4, the solution curves of this differential equation are discussed in the w-V plane by semi-geometrical methods for fixed, large R. Simpler curves are given which provide bounds enclosing the solution curves. In the limit as $R \rightarrow \infty$, the singularity found is not a limit line, but an *ideal shock* joining parts of inviscid solutions. A limit line does not occur even for the exceptional solutions, physically unrealisable, which do not contain a shock.

A full discussion is given for the case of constant viscosity. For the case of a viscosity proportional to the absolute temperature, the method of solution is outlined, and the results are summarised in sec. 6. In practice the variation of viscosity with temperature lies between these two extremes. There is no qualitative difference between the two cases as regards the shock formation. However, in the case of variable viscosity, the absolute temperature is automatically positive throughout the flow if it is positive at any one point of it, whereas in the case of constant viscosity there are solutions violating this requirement which have to be discarded on physical grounds.

1. The fundamental equations.

1.1. In this section, the equations for the steady flow of a perfect gas are first considered in general, in order to derive a first integral of the energy equation. The equations are then specialised to the case of purely radial, two-dimensional flow.

Let x_1, x_2, x_3 be a right-handed system of Cartesian coordinates, let v_i be the component of velocity in the direction of x_i increasing, and let p, ρ, T, μ, λ, R, C_v, C_p and a, denote respectively the pressure, density, absolute temperature, viscosity, heat-conductivity, gas constant, specific heat at constant volume, specific heat at constant pressure, and local speed of sound $((\partial p/\partial \rho)_S)$. The equations for the conservation of mass, momentum and energy are, with the usual summation convention,

$$\frac{\partial}{\partial x_i}(\rho v_i) = 0, \tag{1.1}$$

$$\rho v_i \frac{\partial v_i}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(p + \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k}\right)\delta_{ij} + 2\frac{\partial}{\partial x_i}(\mu e_{ij}), \tag{1.2}$$

and

$$\rho v_i \frac{\partial}{\partial x_i}(C_v T) + p\frac{\partial v_k}{\partial x_k} = \mu\left\{2e_{ij}e_{ij} - \frac{2}{3}\frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right\} + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i}\right), \tag{1.3}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.4}$$

and δ_{ij} is the Kronecker delta. Multiplication of (1.2) by v_i and addition to (1.3) gives

$$\frac{\partial}{\partial x_i}\left\{\rho v_i C_v T + \frac{1}{2}\rho v_i v_j^2 + p v_i + \frac{2}{3}\mu v_i \frac{\partial v_j}{\partial x_j} - 2\mu v_j e_{ij} - \lambda \frac{\partial T}{\partial x_i}\right\} = 0,$$

and so

$$\rho v_i C_v T + \frac{1}{2}\rho v_i v_j^2 + p v_i + \frac{2}{3}\mu v_i \frac{\partial v_j}{\partial x_j} - 2\mu v_j e_{ij} - \lambda \frac{\partial T}{\partial x_i} = \epsilon_{ijk}\frac{\partial A_k}{\partial x_j}, \tag{1.5}$$

where the A_i are unknown functions of the x_i.

The flows investigated in the following are those which are independent of x_3, and in the (x_1, x_2)-plane, depend only on the distance, r, from the origin. The only non-vanishing velocity component is that in the direction of r increasing, denoted by u. Referred to polar coordinates, (r, φ) in the (x_1, x_2) plane, (1.5) then yields

$$\rho u\left(C_v T + \frac{p}{\rho} + \frac{1}{2}u^2\right) + \frac{2}{3}\mu u\left(\frac{du}{dr} + \frac{u}{r}\right) - 2\mu u\frac{du}{dr} - \lambda\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial A_3}{\partial \varphi}, \tag{1.6}$$

and

$$0 = \frac{\partial A_3}{\partial r}. \tag{1.7}$$

Hence A_3 is a function of φ only, and $\partial A_3/\partial\varphi$ must be a constant.

From equation (1.1)

$$\rho u r = \kappa \text{ (a constant)}, \tag{1.8}$$

and (1.2) gives

$$\rho u\frac{du}{dr} + \frac{dp}{dr} = \frac{4}{3}\frac{d}{dr}\left(\mu\frac{du}{dr}\right) + \frac{2\mu}{r}\frac{du}{dr} - \frac{2\mu u}{r^2} - \frac{2}{3}\frac{d}{dr}\left(\frac{\mu u}{r}\right). \tag{1.9}$$

With the help of the equation of state of the gas,

$$p = \mathrm{R}\rho T, \tag{1.10}$$

(1.6) may be written

$$\rho u r\left(C_p T + \frac{1}{2}u^2\right) + \frac{2}{3}\mu u\left(r\frac{du}{dr} + u\right) - 2\mu u r\frac{du}{dr} - \lambda r\frac{dT}{dr} = C'. \tag{1.11}$$

Equations (1.8) to (1.11) govern the steady, two-dimensional, purely radial flow of a perfect gas.

These equations are put into non-dimensional form by the substitutions

$$w = \left(\frac{\gamma+1}{2}\right)^{1/2}\frac{u}{a_0} = \left(\frac{\gamma+1}{2}\right)^{1/2}\mathrm{R}^{-1/2}T_0^{-1/2}u, \qquad \theta = \frac{T}{T_0}, \qquad \xi = \log r. \tag{1.12}$$

The point at infinity will be taken to be a stagnation point, and T_0 and a_0 denote the corresponding temperature and speed of sound, respectively; $\gamma = C_p/C_v$. The Prandtl number σ is defined by

$$\sigma = \frac{C_p\mu}{\lambda}, \tag{1.13}$$

and the only remaining dimensionless parameter is

$$R = \frac{3\kappa}{\mu}\frac{\gamma+1}{2\gamma}, \tag{1.14}$$

which may be regarded as the Reynolds number of the flow.

Elimination of p and ρ from equations (1.8) to (1.11), and use of (1.12) to (1.14) leads to the two equations

$$(\beta+1)w^2w' + w\theta' - w\theta - \theta w' = \frac{4}{R}w^2(w'' - w) + 2w^2(2w' - w)\frac{d}{d\xi}\left(\frac{1}{R}\right), \tag{1.15}$$

$$\theta + \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2 - \frac{3}{R(1+\beta)\sigma}\theta' - \frac{8\beta}{R(1+\beta)}ww' = C \text{ (a constant)}, \tag{1.16}$$

where

$$\beta = (\gamma - 1)/(\gamma + 1) \tag{1.17}$$

and a dash denotes differentiation with respect to ξ.

1.2. The "inviscid solution." The solution for source flow in an inviscid perfect gas is obtained by putting $R = \infty$ in the above equations, without any enquiry as to the validity of such a step. It will be seen later that some of the terms which have a factor of order R^{-1}, are themselves large of order R, so that even when R becomes infinite a finite contribution remains. If this latter possibility is ignored, the equations reduce to

$$w\{(1 + \beta)ww' + \theta' - \theta\} - \theta w' = 0 \tag{1.18}$$

and

$$\theta + \beta w^2 = 1, \tag{1.19}$$

since $\theta = 1$ at the stagnation point at infinity by definition.

When θ is eliminated from (1.18) and (1.19) the equation

$$w' = -w \frac{(1 - \beta w^2)}{(1 - w^2)} \tag{1.20}$$

is obtained. Its solution is

$$r = e^{\xi} = \frac{\kappa(1 - \beta)^{-1/2}}{a_0 \rho_0 w} \cdot (1 - \beta w^2)^{(\beta-1)/(2\beta)}, \tag{1.21}$$

where ρ_0 is the stagnation density at $r = \infty$.

The graph of $w(r)$ is given in Fig. 1. The 'sonic speed' (which is the fluid speed at

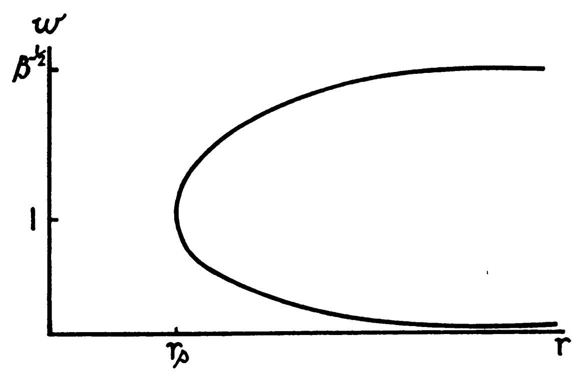

FIG. 1. w versus r for inviscid gas.

which the fluid speed equals the local speed of sound) corresponds to $w = 1$, and the maximum speed attainable, at which the temperature falls to absolute zero, corresponds to $w = \beta^{-1/2}$. If r_s is the value of r for which $w = 1$, (1.20) gives no solution for $r < r_s$, but for $r > r_s$ there are two branches of the $w - r$ curve, one tending to zero, and the other tending to $w = \beta^{-1/2}$ as r tends to infinity. Thus for any r greater than r_s there are two possible values of w, one representing a supersonic speed, and the other a subsonic speed. The streamlines are radial and, in fact, are cusped at the sonic circle, which is a limit line (of a rather special type).

It is intended now to use this solution as a guide to a study of the solutions of the equations (1.15) and (1.16), inasmuch as it is expected that source flow, even in a viscous, heat-conducting fluid, should lead to a stagnation point at $r = \infty$, and that viscous effects should become comparatively unimportant there. But a limit line cannot exist in the real fluid, and so, if solutions of (1.15) and (1.16) are sought which behave like the solution of (1.20) for large r and small w, their continuation backward in r should show what phenomena are to be expected in a real fluid when inviscid theory predicts a limit line. To preserve the correspondence between the solutions for large r, the constant C in (1.16) is taken to be unity.

1.3. The solution at large distances from the source, for large, but finite, values of R. In this section, μ is taken to be a constant, which leads to a considerable simplification at this stage. When the complete solution is discussed later, however, the case of μ varying with temperature will also be considered. It should be noticed here that the ratio of the specific heats, γ, is assumed constant throughout, and the Prandtl number, σ, is assumed constant when the viscosity is constant.

It is possible to eliminate any explicit dependence on ξ from the equations (1.15) and (1.16) by the substitution

$$V = -2\frac{dw}{d\xi} = -2r\frac{dw}{dr}, \tag{1.22}$$

which permits us to reduce these equations to two simultaneous first order equations in V, θ and w, namely

$$\frac{1}{R}w^2V\frac{dV}{dw} = -\frac{1}{2}(1+\beta)w^2V - \frac{1}{2}Vw\frac{d\theta}{dw} - w\theta + \frac{1}{2}\theta V + \frac{4w^3}{R}, \tag{1.23}$$

and

$$\theta - 1 + \beta\left\{1 + \frac{4}{R(1+\beta)}\right\}w^2 + \frac{3}{2R\sigma(1+\beta)}wV = 0. \tag{1.24}$$

The variable V is closely related to the fluid acceleration, and satisfies a certain non-linear differential equation which is obtained by eliminating θ from (1.23) and (1.24).

The equations (1.23) and (1.24) respectively may be put into the forms

$$\begin{aligned}\frac{d}{dw}(V-2w) - \frac{R}{2w^2V}(V-2w) &= -\frac{R}{2}\left\{1+\beta-\frac{8\beta}{(1+\beta)R}+\frac{4}{R}\right\} + \frac{R}{2w^2}(\theta-1)\\ &\quad + R\left\{\beta + \frac{1+2\beta}{1+\beta}\frac{4}{R}\right\}\frac{w}{V} - \frac{R}{2w}\left\{1-\frac{3}{(1+\beta)R\sigma}\right\}\frac{d\theta}{dw}\\ &= f\left(V, \theta, \frac{d\theta}{dw}, w\right)\end{aligned} \tag{1.25}$$

and

$$\begin{aligned}\frac{d}{dw}(\theta-1) + \frac{2R\sigma}{3V}(1+\beta)(\theta-1) &= -\frac{2\beta R\sigma}{3V}(1+\beta)\left\{1+\frac{4}{R(1+\beta)}\right\}w^2 - \frac{8\beta\sigma}{3}w\\ &= g(V, w).\end{aligned} \tag{1.26}$$

If $z(w)$, $\zeta(w)$ are defined by

$$z(w) = -\int^{w} \frac{R}{2w^2 V}\, dw, \tag{1.27}$$

$$\zeta(w) = \int^{w} \frac{2R\sigma}{3V} (1 + \beta)\, dw, \tag{1.28}$$

then equations (1.25) and (1.26) have the formal solutions

$$V - 2w = e^{-z} \int e^{z} f \cdot dw + C_1 e^{-z} \tag{1.29}$$

and

$$\theta - 1 = e^{-\zeta} \int e^{\zeta} g\, dw + C_2 e^{-\zeta}. \tag{1.30}$$

These integral equations may be solved iteratively in the neighbourhood of $w = V = 0$, starting with the first approximation

$$V = 2w \tag{1.31}$$

suggested by the inviscid solution. If $(\theta - 1)$ is to be finite near the stagnation point, C_2 must be zero, and the first approximation to (1.30) is

$$\theta - 1 = -\beta\sigma \frac{12 + R(1 + \beta)}{6 + R\sigma(1 + \beta)}\, w^2; \tag{1.32}$$

after substitution of this expression in $f(v, \theta, w)$, (1.29) gives the second approximation

$$V - 2w = 2\left\{\frac{(1 - \beta^2)R\sigma + 6(1 + \beta) - 24\beta\sigma}{(1 + \beta)R\sigma + 6}\right\} w^3 + C_1 e^{-z}. \tag{1.33}$$

This process can be repeated as many times as desired to obtain higher approximations, and gives the asymptotic solution for V and θ near $w = V = 0$. The coefficients of the powers of w in the series so obtained agree with those found for the 'inviscid' solution, except for terms of order R^{-1}. However, the former series are divergent, as the term of order R^{-1} in the coefficient of w^n is unbounded as $n \to \infty$. Also there is an infinite number of solutions having the same sort of behaviour near $w = V = 0$, due to the presence of the term $C_1 e^{-z}$ in (1.33), where C_1 is arbitrary. This is of course due to the fact that the differential equations leading to these expansions are of higher order than the 'inviscid' differential equations. The term $C_1 e^{-z}$ is asymptotically of the form

$$C_1' \exp\left\{-\frac{1}{8} R w^{-2} - \frac{1}{4}(1 + \beta)R \log w\right\},$$

and so is very small if w is small enough, whatever the value of C_1.

Thus there is an infinite number of solutions for the flow of the viscous heat-conducting gas, which, in the neighbourhood of the stagnation point at infinity, are asymptotically equal to the solution for the inviscid gas when the viscosity and the heat-conductivity tend to zero. It is very difficult to investigate the solutions of (1.23) and (1.24) in other parts of the flow field, but it is found that there is a particular value of σ which enables the problem to be reduced to that of the solution of a first order non-linear differential equation. This equation is still of the singular perturbation type, but its solutions can be discussed, after some trouble, and it is with this problem that we shall be concerned hereafter.

1.4. The equation (1.24) may be written

$$\theta - 1 + \beta\left\{1 + \frac{4}{R(1 + \beta)}\right\}w^2 - \frac{3}{(1 + \beta)R\sigma}\frac{d}{d\xi}(\theta - 1) - \frac{4\beta}{R(1 + \beta)}\frac{d}{d\xi}w^2 = 0, \quad (1.34)$$

and this equation, as it stands, can be integrated when

$$\sigma = \sigma_0 = \frac{3}{4} + \frac{3}{R(1 + \beta)}\,. \quad (1.35)$$

For, when E is defined by

$$E = \theta - 1 + \beta\left\{1 + \frac{4}{R(1 + \beta)}\right\}w^2, \quad (1.36)$$

the equation takes the form

$$E - \frac{4}{R(1 + \beta) + 4}\frac{dE}{d\xi} = 0, \quad (1.37)$$

which has the general solution

$$E = A \exp\left\{\left[\frac{1}{4}(1 + \beta)R + 1\right]\xi\right\} \quad (1.38)$$

where A is an arbitrary constant. Now, E must tend to zero at the stagnation point at infinity, and so $A = 0$ and $E \equiv 0$. In fact, E is effectively the difference between the total energy, per unit mass, of the fluid and the value of this total energy at the stagnation point, and (1.38) shows that $E \to \pm\infty$ at this stagnation point unless $E \equiv 0$. Therefore, when σ has the value given by (1.35),

$$\theta = 1 - \beta\left\{1 + \frac{4}{R(1 + \beta)}\right\}w^2. \quad (1.39)$$

This permits us to reduce the problem to a first order differential equation for V as a function of w.

The value chosen for σ is actually not far from the truth, as experimental values for air are in the neighbourhood of 0.72, and when R is large, the value chosen will be close to 0.75. From the form of (1.15) and (1.16), (with constant R) it follows from [5] that the solutions are continuous in σ, and in fact have continuous derivatives of any order with respect to σ in $w > 0$, $-\infty < \sigma < \infty$. In the neighbourhood of $w = 0$ the variation with σ can be seen from (1.33). Thus the only effect of choosing the above particular value of σ is to simplify the equations rather than materially affect their solutions, and we expect that the solutions investigated hereafter will be representative of—and, in fact, very close to—the actual flows which would occur in air under the same boundary conditions. In [1], Sakurai fits values of σ_0 to experimental values for various gases by choosing appropriate values of R, (negative for air), and sketches the solution curves for $\sigma_0 = 0.88$ ($R \approx 20$), Meyer's theoretical value.

With the help of equation (1.39), θ and its derivatives are eliminated from equation (1.23) to give

$$\frac{w^2 V}{R}\frac{dV}{dw} = \frac{1}{2}V\left\{1 - \left(1 - \frac{4\beta}{R(1 + \beta)}\right)w^2\right\} - w\left\{1 - \left(\beta + \frac{4}{R}\frac{1 + 2\beta}{1 + \beta}\right)w^2\right\}. \quad (1.40)$$

For convenience in the subsequent algebra, terms which are genuinely of order R^{-1} compared with terms of order 1, are now neglected on the right-hand side of (1.40). That is, in each of the curly brackets, the coefficients of w^2 are replaced by 1 and β respectively. Thus, the equation to be investigated is

$$\frac{w^2 V}{R}\frac{dV}{dw} = \frac{1}{2} V(1 - w^2) - w(1 - \beta w^2). \tag{1.41}$$

The methods hereafter applied to (1.41) are quite applicable to (1.40), but the algebra becomes much more involved, merely because of the unwieldy forms of the two coefficients of w^2. The difference between the two equations is very small for large R, and physically the difference is that in equation (1.40) the sonic speed and maximum speed vary slightly with R, while in equation (1.41) they are fixed at their inviscid values. The variation in w between the two is only of order R^{-1}, while in the subsequent discussion the most significant variations in w are of order $R^{-1/2}$.

2. Properties of the solution curves.

Although equation (1.41) is of a simple appearance, it cannot be integrated to find closed solutions, so that the discussion requires indirect methods. In the first place, in order to investigate the form of the solutions we use the standard techniques of curve sketching to discuss the behaviour of the solution curves in the w-V plane, and to this end the behaviour of the curves of zero slope and zero curvature is established in **2.1**, while **2.2** gives some of the more elementary properties of the solution curves.

2.1. Let C_1 be the curve on which dV/dw is zero in equation (1.41), that is, the curve given by

$$V = f(w) = 2w\,\frac{1 - \beta w^2}{1 - w^2}, \tag{2.1}$$

which is also the inviscid solution curve (1.20).

Let C_2 be the curve on which d^2V/dw^2 is zero in equation (1.41), that is, the curve given by

$$V^3 - w(1 + \beta w^2)V^2 - \tfrac{1}{2}R(1 - w^2)(1 - \beta w^2)V + Rw(1 - \beta w^2)^2 = 0. \tag{2.2}$$

It has the following properties.

(i) When $w = 0$ (stagnation point), $V = 0$ or $\pm(R/2)^{1/2}$.

(ii) When $w = 1$ (sonic speed),

$$V = -(1 - \beta)^{2/3}R^{1/3}\{1 + O(R^{-1/3})\}. \tag{2.3}$$

(iii) When $w = \beta^{-1/2}$ (maximum speed), $V^2 = 0$ and $V = 2\beta^{-1/2}$. At $V^2 = 0$, there is a double point, the curve having there the slopes $-\frac{1}{2}(1 - \beta)R\{1 + O(R^{-1})\}$ and $4\beta/(1 - \beta)\{1 + O(R^{-1})\}$.

(iv) For $w \gg R^{1/2}$ there are three branches of the curve, given asymptotically by

$$V \sim 2\beta w,$$

$$V \sim -\tfrac{1}{2}Rw,$$

$$V \sim \beta w^3.$$

(v) The curve C_2 has infinite slope at (w_m, V_m) where

$$w_m \sim 1 - \frac{3}{2^{2/3}}(1-\beta)^{1/3}R^{-1/3},$$

$$V_m \sim 2^{-1/6}(1-\beta)^{2/3}R^{1/3}. \tag{2.4}$$

(vi) Provided that

$$|1 - w^2|^{-1}R^{-1/3} = o(1) \tag{2.5}$$

there is a branch of C_2 given by

$$V = g_1(w) = f(w)\{1 + O(R^{-1}(1-w^2)^{-3})\} \tag{2.6}$$

which lies close to C_1 in the range $0 \le w < \infty$. The condition (2.5) means that the difference between w and 1 is of greater order than $R^{-1/3}$ as $R \to \infty$; that is, it excludes a small range of speeds near the sonic speed. In the range $0 \le w < w_m$, with the above small neighbourhood of $w = 1$ again excluded, there are two other branches given by

$$V = g_2(w) = \left\{\frac{R}{2}(1-w^2)(1-\beta w^2)\right\}^{1/2}\{1 + o(1)\}, \tag{2.7}$$

and

$$V = g_3(w) = -\left\{\frac{R}{2}(1-w^2)(1-\beta w^2)\right\}^{1/2}\{1 + o(1)\}.$$

In the range $1 < w < \beta^{-1/2}$, there is only one branch $V = g_1(w)$ which is actually the continuation of $V = g_3(w)$.

(vii) The curves C_1 and C_2 intersect only at $w = 0$ and $w = \beta^{-1/2}$. For $w \ll 1$,

$$f(w) \sim 2w + 2(1-\beta)w^3 + 2(1-\beta)w^5 + 2(1-\beta)w^7 + \cdots \tag{2.8}$$

and

$$\begin{aligned} g_1(w) \sim 2w &+ \{2(1-\beta) + 8R^{-1}\}w^3 \\ &+ \{2(1-\beta) + 8(5-4\beta)R^{-1} + 128R^{-2}\}w^5 + \{2(1-\beta) \\ &+ 8(14 - 16\beta + 3\beta^2)R^{-1} + 64(17-12\beta)R^{-2} + 2688R^{-3}\}w^7 + \cdots. \end{aligned} \tag{2.9}$$

Let C_3 be the curve on which $d(V - f(w))/dw = 0$ in (1.41); that is, the curve given by

$$V = \frac{2w(1-w^2)(1-\beta w^2)^2}{(1-\beta w^2)^3 - 4w^2R^{-1}\{1 + (1-3\beta)w^2 + \beta w^4\}}. \tag{2.10}$$

Then in the range $0 \le w < \beta^{-1/2}$, under the condition (2.5), C_3 lies between C_1 and $V = g_1(w)$. This fact is of use in connection with the approach to the limiting form **3.1.**

2.2. The solution curves may be shown to have the following properties:—

(i) The point $(\beta^{-1/2}, 0)$ is a saddle point, the solution curves passing through it being tangent to the two branches of C_2 passing through this point, see **2.1**, (iii).

(ii) For large w there is a family of curves asymptotic to C_1, and another family asymptotic to $V = -\frac{1}{2}Rw$.

(iii) There is an infinite number of solution curves approaching $w = 0$ for large negative V. These may be shown to have the form

$$V = -\tfrac{1}{2}Rw^{-1} + A - (\tfrac{1}{2}R - 2)w + O(w^2), \tag{2.11}$$

where A is an arbitrary constant.

(iv) By a method similar to that used in (**1.3**), (or by substitution of a power series), the asymptotic form of the solution curves through (0, 0), in the neighbourhood of this point, is found to be

$$\begin{aligned} V \sim 2w + \{2(1-\beta) + 8R^{-1}\}w^3 + \{2(1-\beta) + 8(5-4\beta)R^{-1} \\ + 128R^{-2}\}w^5 + \{2(1-\beta) + 8(14 - 16\beta + 3\beta^2)R^{-1} \\ + 64(20 - 15\beta)R^{-2} + 3456R^{-3}\}w^7 + \cdots + C'e^{-z} \end{aligned} \tag{2.12}$$

so that by comparison with (2.8) and (2.9), all these solution curves lie above $f(w)$ and $g_1(w)$ for w small enough, as the coefficient of w^7 in (2.12) is greater than the coefficient of w^7 in (2.9) by an amount $192(1-\beta)R^{-2} + 768R^{-3}$.

The above properties, and a knowledge of the regions of positive and negative curvature and positive and negative slope, enable the solution curves to be sketched, as in

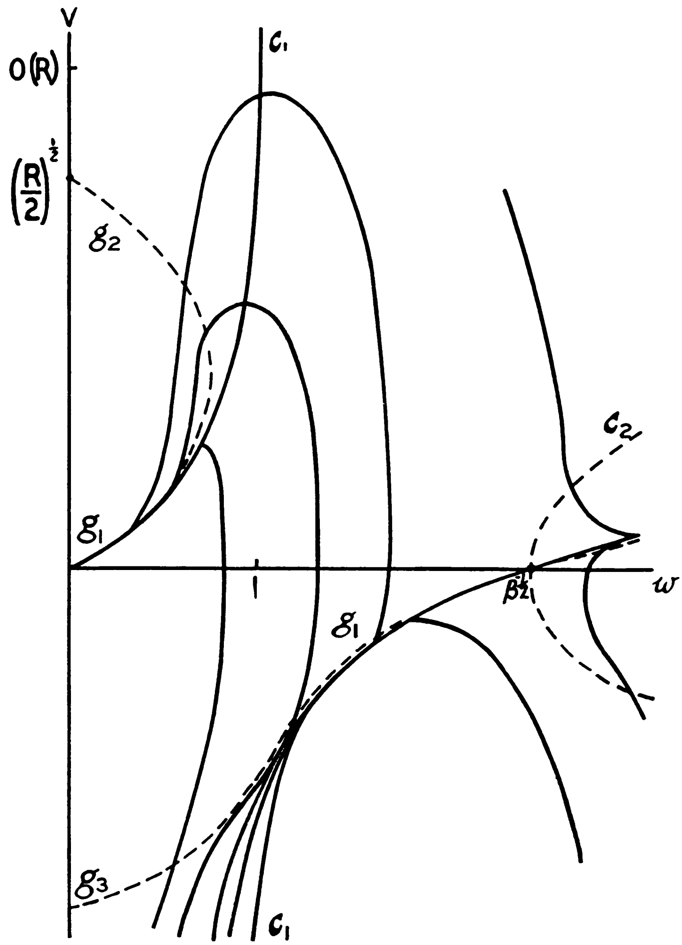

FIG. 2. The w-V plane for constant μ.

C_1 is curve of zero slope.

C_2, g_1, g_2, g_3 are the curves of inflexion.

Fig. 2. It is seen that the solution curves passing through the origin may be divided into three main classes:—

(I) those which pass through $V = g_2(w)$, cross the line $w = \beta^{-1/2}$ and are ultimately asymptotic to C_1, for large w;

(II) Those which pass through $V = g_2(w)$, bend round to cross the w-axis, and are ultimately asymptotic to the negative V-axis (see (2.11));

(III) Those which pass through $V = g_1(w)$, and then cross the w-axis and become asymptotic to the negative V-axis, as for class II. The solutions of the first class, whose curves penetrate into the region of negative temperature are discarded as physically impossible just as in inviscid theory. However, it will be seen later that all possible flows are automatically confined to $0 \leq w < \beta^{-1/2}$ when μ is taken proportional to T.

3. The approach to the limit.

Now that the general shape of the solution curves in the w-V plane has been established, we proceed to investigate whether any limiting curves are approached as $R \to \infty$, and if so, how closely these limiting curves are approximated when R is large. A clue is given by considering the differential equation (1.41) in the form

$$\frac{dV}{dw} = \frac{R}{w^2 V}\left\{\frac{1}{2} V(1 - w^2) - w(1 - \beta w^2)\right\}, \tag{3.1}$$

which shows that, at a fixed point, $dV/dw \to \infty$ as $R \to \infty$. So that considerable portions of the solution curves may be expected to have very large slope. In fact, it will be shown that these portions are very nearly vertical straight lines. The technique used is to find bounding curves which lie on either side of the solution curve considered, and which approach a limiting curve as $R \to \infty$.

3.1. In Fig. 3 consider the solution curve passing through the point $P(b, c)$ on $V = g_2(w)$ so that b and c satisfy equation (2.2) and

$$c = \left\{\frac{R}{2}(1 - b^2)(1 - \beta b^2)\right\}^{1/2}\{1 + o(1)\}, \tag{3.2}$$

but assume that

$$(1 - b^2)^{-1} R^{-1/3} = o(1). \tag{3.3}$$

After a considerable amount of calculation based on the value of the solution derivative on the bounding curves*, the following results may be established.

For $w > b$, the solution curve through P lies above the segment PQ' of the line

$$V = h_\epsilon(w) = c + \frac{R}{b^2}\left\{\frac{1}{2}(1 - b^2) - \frac{b}{c}(1 - \beta b^2)\right\}(1 - \epsilon)(w - b), \qquad 0 < \epsilon < 1, \tag{3.4}$$

where at Q'

$$w = b + \tfrac{1}{2}\epsilon b(1 - b^2), \tag{3.5}$$

and below the tangent

$$V = h(w) = c + \frac{R}{b^2}\left\{\frac{1}{2}(1 - b^2) - \frac{b}{c}(1 - \beta b^2)\right\}(w - b), \tag{3.6}$$

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*For example, on PQ' the solution derivative is greater than $h'_\epsilon(w)$ so the solution curve through P cannot cross PQ'.

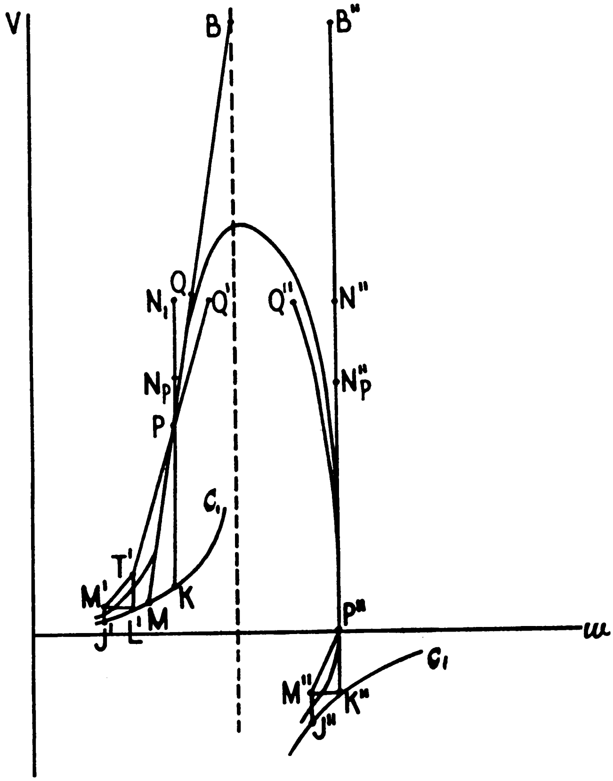

FIG. 3. A typical shock solution in the w-V plane.

for $b \leq w \leq 1$; in particular, it lies below the segment PQ of the tangent, where at Q

$$w = b + \tfrac{1}{2}\,\epsilon(1 - \epsilon)b(1 - b^2). \tag{3.7}$$

For $w < b$, the solution curve through P lies above the segment, PM, of the tangent through P, and below the segment PT' of the line $V = h_\epsilon(w)$, where, at T',

$$w = b_1 = b\left\{1 - \frac{2bc}{R(1 - b^2)(1 - \epsilon)}\,\frac{\epsilon c - 2f(b)}{\epsilon c - \epsilon f(b)}\right\}. \tag{3.8}$$

This solution curve also lies below the straight-line segment $T'M'$, where M' has the coordinates

$$w = b_2 = b_1\{1 - 2b_1(1 - \beta b_1^2)(\lambda - 1)^{1/2}(1 - b_1^2)^{-1/2}R^{-1/2}\}, \qquad V = f(b_1), \tag{3.9}$$

with $\lambda = 2\epsilon^{-1}f(b)/f(b_1)$.

We denote the points on C_1 where $w = b$, $w = b_1$, and $w = b_2$ by K, L' and J' respectively, and as $(b - b_1)$, $(b_1 - b_2)$ and $(b - b_2)$ are all $O(R^{-1/2}(1 - b^2)^{-1/2})$ from (3.8) and (3.9), it follows readily from (2.1) that

$$M'J' = O(R^{-1/2}(1 - b^2)^{-5/2}). \tag{3.10}$$

Also, if $w = b_3$ at the point M, then

$$b - b_3 = O(R^{-1/2}(1 - b^2)^{-1/2}), \tag{3.11}$$

so that when the solution curve through P has approached C_1 to within a distance of $O(R^{-1/2}(1 - b^2)^{-5/2})$ the distance between the solution curve and the vertical line through P is only $O(R^{-1/2}(1 - b^2)^{-1/2})$.

Furthermore, as the solution curve through P cannot cross $V = g_1(w)$ it also cannot cross C_3 , (by the remarks following (2.10)), and so, for this solution curve, $v - f(w)$ decreases monotonically to zero as w decreases from b_2 to 0. Thus, for the solution curve through P, the vertical difference between this curve and C_1 is less than $M'J'$ for w between 0 and b_2 .

To sum up the progress so far:— if we keep b fixed and let $R \to \infty$, the solution curve through P approaches C_1 for $0 \leq w < b$ (since $M' \to K$), and the vertical line KPN_p , where N_p has the coordinates

$$w = b, \qquad V = c + \frac{R^p \epsilon}{4bc}(1 - b^2)^2(1 - \epsilon)\{c - f(b)\}, \qquad 0 < p < 1. \tag{3.12}$$

When $p = 1$, the distance of the curve from the vertical through P lies between N_1Q and N_1Q', which are both $O(1 - b^2)$, by (3.5) and (3.7).

3.2. Let the solution curve through $w = \beta^{-1/2}$ cross the curve $V = g_2(w)$ at $w = w_i$, then with reference to **2.5**, the curves for which $b > w_i > 0$ belong to class II and those for which $0 < b < w_i$ belong to class I, and by using appropriate bounding curves it may be shown that $1 - w_i = O(1)$. (Actually, it will be shown in **4.2** that $w_i \to \beta^{1/2}$ as $R \to \infty$.) Accordingly, if we suppose that $b > w_i$ for the solution curve already considered, it will cross the w-axis at $w = d$ say, where $0 < d < \beta^{-1/2}$. Furthermore, it may be shown that, if $(1 - b^2)^{-1}R^{-1/3} = o(1)$, then $(d - 1)$ is positive and at least of order $(1 - b)$, and hence $(d - 1)^{-1}R^{-1/3} = o(1)$. This means that b and d differ from unity to at least the same order in R, and lie on opposite sides of unity.

It is seen from (3.1) that a solution curve crosses the w-axis with a vertical tangent, and from Fig. 2, the solution curve always lies to the left of this tangent. For the solution curve through the point P'', $w = d$, $V = 0$, first consider the part with $1 < w \leq d$, $V \geq 0$. This lies to the left of the line $w = d$, and a bounding curve is required which lies to the left of the solution curve. It is possible to find an inclined straight line which has this property, but this is not sufficient, as we need later to take an integral of V^{-1} over the bounding curve, and the result should be finite. We desire a bounding curve which has the same singularity at $w = d$, $V = 0$ as the solution curve, that is, which behaves like $(d - w)^{1/2}$ near this point. The first two terms of the Taylor expansion of V near $w = d$, $V = 0$ give a satisfactory bounding curve, $P''Q''$, given by

$$V = h_1(w) = \{2(1 - \beta d^2)Rd^{-1}\}^{1/2}(d - w)^{1/2} + \frac{d^2 - 1}{3d^2}R(d - w), \tag{3.13}$$

where at Q''

$$w = d\left\{1 - \frac{1}{6}(d^2 - 1)\right\}. \tag{3.14}$$

For the part of the curve $1 < w \leq d$, $V \leq 0$, the solution curve lies to the left of the vertical line segment $P''K''$, and to the right of the line segment $P''M''$, where M'' has the coordinates

$$w = d_1 = d\{1 - 2(1 - \beta d^2)(d^2 - 1)^{-1/2}R^{-1/2}\}, \qquad V = f(d). \tag{3.15}$$

Let the points K'', J'' lie on C_1 with $w = d$ and $w = d_1$ respectively, and let the point N_p'' lie on the line $w = d$, with

$$V = \frac{R^p(d^2 - 1)^2}{18d}\left\{1 + \left[\frac{108d^2(1 - \beta d^2)}{R(d^2 - 1)^3}\right]^{1/2}\right\}, \qquad 0 < p \leq 1, \tag{3.16}$$

so that N_1'' has the same value of V as Q''. Then

$$Q''N_1'' = \frac{1}{6}\, d(d^2 - 1), \tag{3.17}$$

$$K''M'' = \frac{2d(1 - \beta d^2)}{(d^2 - 1)^{1/2}R^{1/2}}, \tag{3.18}$$

and it follows that

$$M''J'' = O(R^{-1/2}(d^2 - 1)^{-5/2}). \tag{3.19}$$

The solution curve, after crossing $M''J''$, then crosses C_2 at $w = d_2$ say, where $1 < d_2 < d_1$, and thereafter lies below it. Thus in the range $1 < w \leq d_2$ the solution curve lies between C_1 and $V = g_1(w)$, and as the difference in height between $V = f(w)$ and $V = g_1(w)$ is $O(R^{-1} \mid w^2 - 1 \mid^{-4}$, (from (2.6)), the solution curves lies within $O(R^{-1/2}(d^2 - 1)^{-5/2})$ of $V = f(w)$ from $w = 1 + O(R^{-1/8}(d^2 - 1)^{5/8})$ to $w = d_1$. Continued backward from this range, the curve crosses the line $w = 1$ with $V < -(1 - \beta)^{2/3}R^{1/3}$ (from 2.3) and is ultimately asymptotic to $w = 0$, as given by (2.11). At the point N_p'', the difference in w between the solution curve and the line $w = d$ is of order $R^{p-1}(d^2 - 1)$, so that as $R \to \infty$, the solution curve approaches C_1, between $1 < w \leq d$, and the straight line $K''P''N_p''$.

In the range $b < w < d$, define the points B, B'' to have respectively the coordinates

$$w = 1, \qquad V = c + \frac{R}{2b^2c}(1 - b^2)(1 - b)\{c - f(b)\}, \tag{3.20}$$

and

$$w = d, \qquad V = c + \frac{R}{2b^2c}(1 - b^2)(1 - b)\{c - f(b)\}. \tag{3.21}$$

Then the solution curve obviously lies above the line $Q'Q''$, and below the lines QB (the continuation of PQ), and BB''. Thus the maximum V attained is of order $(1 - b)^2R$.

The required bounding curves have now been found for the only important class of solution curves, and in the next section, this part of the discussion is completed by finding a relation between b and d, which shows the shock-character of the solution.

4. The "diffuse" shock and the main solution.

It has been shown in the previous section that over some part of the range of w, a typical solution curve of class II approaches the inviscid solution curve in the w-V plane, as $R \to \infty$. It will be shown in this section that the steeply 'humped' part of a typical solution curve of this class approaches a solution appropriate to an ideal shock.

4.1. We require first an estimate of the actual physical distance between the points P and P''. It is sufficient to consider the difference in ξ between these points, and since

$$V = -2dw/d\xi,$$

then

$$\xi_P - \xi_{P''} = 2\int_b^d \frac{dw}{V}\,. \tag{4.1}$$

From (3.4), (3.5), (3.12), (3.13), and the rest of Sec. 3,

$$\xi_P - \xi_{P''} < 2\int_b^{w_{Q'}} \frac{dw}{h_\epsilon(w)} + 2\int_{w_{Q''}}^{d} \frac{dw}{h_1(w)} + 2(d-b)\min(V_{Q''}^{-1}, V_{Q'}^{-1}), \tag{4.2}$$

and

$$\xi_P - \xi_{P''} > 2\int_b^1 \frac{dw}{h(w)} + 2(d-1)[h(1)]^{-1}. \tag{4.3}$$

The integrals which appear in (4.2) and (4.3) are respectively

$$\frac{2b^2c}{R(1-b^2)(1-\epsilon)(c-f(b))}\log\left\{1+\frac{1}{4}b^{-1}R\epsilon(1-\epsilon)(1-b^2)^2(c-f(b))\right\}, \tag{4.4}$$

$$\frac{6d^2}{R(d^2-1)}\log\left\{1+\frac{1}{6}d^{-1}R^{1/2}(d^2-1)^{3/2}[3(1-\beta d^2)]^{-1/2}\right\}, \tag{4.5}$$

and

$$\frac{2b^2c}{R(1-b^2)(c-f(b))}\log\left\{1+\frac{1}{2}b^{-2}R(1-b)(1-b^2)(c-f(b))\right\}, \tag{4.6}$$

so that, for large R, (by what has gone before) they are all

$$O\left\{\frac{\log[R(1-b)^3]}{R(1-b)}\right\}. \tag{4.7}$$

Also, by their definitions, $V_{Q''}$, $V_{Q'}$ and $h(1)$ are all $O\{R(1-b)^2\}$, so that the terms not involving integrals in (4.2) and (4.3) are all

$$O\{R^{-1}(1-b)^{-1}\}. \tag{4.8}$$

Hence

$$\xi_P - \xi_{P''} = O\left\{\frac{\log[R(1-b)^3]}{R(1-b)}\right\} \tag{4.9}$$

for large R.

4.2. With the aid of the above result we are now able to find the asymptotic relation between b and d, which in the limit $R = \infty$, will be seen to be equivalent to Prandtl's relation between the speeds on opposite sides of an ideal shock.

Equation (1.15), (with constant μ), may be put into the form

$$\frac{d}{d\xi}\left\{(1+\beta)w+\frac{\theta}{w}\right\} = \frac{\theta}{w}+\frac{4}{R}\left\{\frac{d^2w}{d\xi^2}-w\right\}, \tag{4.10}$$

and so

$$\int_{\xi_P}^{\xi_{P''}} \frac{d}{d\xi}\left\{(1+\beta)w+\frac{\theta}{w}\right\}d\xi = -\frac{2}{R}(V_{P''}-V_P)+\int_{\xi_P}^{\xi_{P''}}\left(\frac{\theta}{w}-\frac{4w}{R}\right)d\xi$$

$$= \frac{2c}{R} - O(\xi_P - \xi_{P''})$$

as θ/w is $O(1)$ between ξ_P and ξ_P''. From (3.2), the first term on the right-hand side is asymptotically

$$\{2R^{-1}(1-b^2)(1-\beta b^2)\}^{1/2},$$

while from (4.9), the second term is of higher order, so that

$$\left[(1 + \beta)w + \frac{\theta}{w}\right]_{w=b}^{w=d} = \{2R^{-1}(1 - b^2)(1 - \beta b^2)\}^{1/2}\{1 + o(1)\}, \tag{4.11}$$

and as $(d - b)$ is $O(1 - b)$,

$$bd = 1 + O(R^{-1/2}(1 - b)^{-1/2}). \tag{4.12}$$

In terms of actual speeds, this relation becomes

$$u_1u_2 = a^{*2}\{1 + O(R^{-1/2}(a^* - u_1))^{-1/2}\}, \tag{4.13}$$

where

$$a^* = \left(\frac{2}{\gamma + 1}\right)^{1/2} a_0 , \tag{4.14}$$

and as $R \to \infty$, this becomes the Prandtl relation.

With reference to the beginning of **3.2**, where the solution curves belonging to class II were defined as those for which $b > w_i > 0$, we see from (4.12) that as $d \to \beta^{-1/2}$, $b \to \beta^{1/2}\{1 + O(R^{-1/2}(1 - b)^{-1/2})\}$, and hence we have

$$w_i = \beta^{1/2}\{1 + O(R^{-1/2}(1 - b)^{-1/2})\}. \tag{4.15}$$

4.3. Summary of main solution. From sec. 3 we see that in the w-V plane, a typical solution curve of class II which passes through $P(w_1 , V_1)$ on the curve $V = g_2(w)$, approaches the inviscid solution curve in the ranges $0 \leq w < w_1$ and $1 < w < w_2$. For large R and fixed w_1 we may summarise the behaviour as follows:—

(i) In $0 \leq w \leq w_1 - \epsilon_1$, $V \geq 0$, where $\epsilon_1 = O(R^{-1/2})$, V goes from 0 to $f(w) - \eta_1$, $\eta_1 = O(R^{-1/2})$, and $V - f(w)$ is at most $O(R^{-1/2})$.

(ii) In $1 + \epsilon_3 \leq w \leq w_2 - \epsilon_2$, $V < 0$, $(w_2 = w_1^{-1}[1 + O(R^{-1/2})])$, where $\epsilon_2 = O(R^{-1/2})$, $\epsilon_3 = O(R^{-1/8})$, V goes from $f(1 + \epsilon_3) + \eta_3$ to $f(w_2) - \eta_2$, where η_1, $\eta_2 = O(R^{-1/2})$, and $V - f(w)$ is at most $O(R^{-1/2})$.

(iii) (a) In $w_1 - \epsilon_1 \leq w \leq w + \epsilon_1^{(p)}$, $V > 0$, where $\epsilon_1^{(p)}$ is $O(R^{-p})$, V goes from $f(w_1) - \eta_1$ to $O(R^{1-p})$.

(b) In $w_2 - \epsilon_2^{(p)} \leq w \leq w_2$, $V > 0$, where $\epsilon_2^{(p)}$ $O(R^{-p})$, V goes from $O(R^{1-p})$ to 0.

(c) In $w_1 + \epsilon_1^{(p)} \leq w \leq w_2 - \epsilon_2^{(p)}$, $V > 0$, V is $O(R^{-p})$, being $O(R)$ when $p = 0$, that is, when both $w - w_1$ and $w_2 - w$ are $O(1)$.

(d) In $w_2 \geq w \geq w_2 - \epsilon_2$, $V < 0$, V goes from 0 to $f(w_2) - \eta_2$.

(iv) In $1 + \epsilon_3 \geq w \geq 0$, $V < 0$, the curve crosses $w = 1$, with $| V | > (1 - \beta)^{2/3}R^{1/3}$, and becomes asymptotic to $w = 0$, as given by (2.11).

The part of the solution described in (iii) above approaches an 'ideal' shock solution as $R \to \infty$, for the equation for the conservation of mass, equation (1.39) and equation (4.12) give, in the limit,

$$\rho_1u_1 = \rho_2u_2 , \tag{4.16}$$

$$\frac{\gamma p}{\rho} = a_0^2 - \frac{\gamma - 1}{2} u^2, \tag{4.17}$$

and

$$u_1u_2 = \frac{2}{\gamma + 1} a_0^2 , \tag{4.18}$$

(where the fact that $\xi_P - \xi_{P''} \to 0$ as $R \to \infty$ is used in (4.16)), and these imply the Hugoniot relations; the curve approached is $w = w_1$, $f(w_1) \leq V < \infty$ and $w = w_2$, $f(w_2) \leq V < \infty$.

The actual shock, (as opposed to the limiting 'ideal' shock) has a maximum velocity gradient of order R, which occurs where the solution curve crosses C_1 (within $O(R^{-1})$ of $w = 1$) and its 'thickness' is of order $R^{-1}(1 - w_1)^{-1} \log [R(1 - w_1)^3]$, from (4.9). The 'thickness' has been defined as the distance in the physical plane between P and P'' in Fig. 3. Of course, there is some degree of choice as to the definition of the shock thickness, and this leads to an apparent discrepancy between the above results and the usual result for a plane shock [3], which is that the shock thickness is of order $\mu\rho^{-1}(1 - w_1)^{-1}((1 - w_1)$ is proportional to the shock strength.) The difficulty is resolved if an arbitrary length is introduced in the plane-shock treatment so as to allow the definition of a Reynolds number. Then, for a given shock strength, it is seen that the thickness is proportional to R^{-1} (implied above) only if the velocity gradients at the points measured from are $O(R)$, while if the edges of the shock are taken to be at points where the velocity gradient has fallen to less than $O(R)$, as in the treatment here, the results agree. In practical cases, however, the difference is slight.

From (1.21) and (4.16) to (4.18), the solution approached in the physical plane is given by

$$r = \frac{\kappa}{a_0\rho_0(1 - \beta)^{1/2}} \cdot w^{-1}(1 - \beta w^2)^{(\beta-1)/2\beta}, \qquad 0 \leq w \leq w_1, \tag{4.19}$$

$$r = \frac{\kappa}{a_0\rho_0'(1 - \beta)^{1/2}} \cdot w^{-1}(1 - \beta w^2)^{(\beta-1)/2\beta}, \qquad 1 < w \leq w_1^{-1}, \tag{4.20}$$

where

$$\frac{\rho_0'}{\rho_0} = w_1^2\left\{\frac{1 - \beta w_1^2}{1 - w_1^{-2}}\right\}^{(1-\beta)/(2\beta)}, \tag{4.21}$$

and the suffix $_0$ refers to conditions at the stagnation point at $r = \infty$ (ρ_0 and ρ_0' are different because of the change in entropy through the shock.) From (4.21) we see that a change of w_1 corresponds to a change of the boundary conditions at a given point of the supersonic region, and it is this that determines which of the infinite number of solution curves of class II is selected for given boundary values. This corresponds to a diffuser with constant outlet conditions.

The remainder of the solution curve in the w-V plane, for $0 \leq w \leq 1$, with V negative, approaches in the physical plane the flow starting from a stagnation-point at

$$r = r_s = \frac{\kappa}{a_0\rho_0'}(1 - \beta)^{-1/(2\beta)} \tag{4.22}$$

with infinite pressure and density, and proceeding with infinite velocity gradient up to $w = 1$, where it joins on to the flow given by (4.20) (see Fig. 4). The equations would break down before this second stagnation-point is reached; nevertheless, this theory does not exclude a transition from subsonic flow to supersonic flow without a contraction as, in fact, there is heat addition at the second stagnation point. However, the velocity gradient in this range is so great, that it is difficult to see how this kind of boundary condition could be realised in practice.

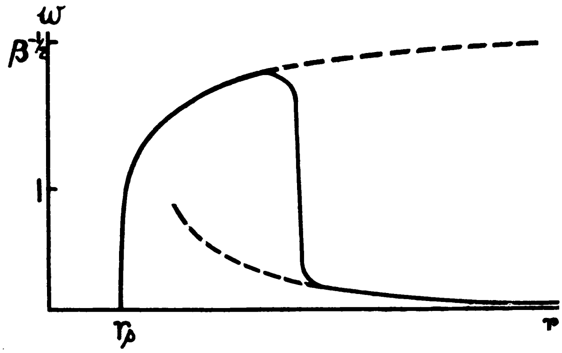

FIG. 4. Typical w-r curve for a viscous gas.

4.4. Some numerical results. This theory could be applied to the flow in the divergent part of a supersonic-subsonic nozzle, neglecting the boundary layer effects. It may be useful for low-density tunnels, as it should be more precise than plane-shock theory. To give some idea of the magnitudes involved, some rough results calculated from **4.1** and **4.3** are given below, in which δ_1 and δ_2 are respectively upper and lower bounds for $(r_1 - r_2)$. The estimates of $(r_1 - r_2)$ could be improved, but not without a considerable amount of labour. The thicknesses involved are of the expected order of magnitude.

w_1	w_2	r_1 cm	T_0 ^0A	$\frac{\rho_1 - \rho_2}{\rho_2}$	ρ_0 gm.cm.$^{-3}$	R	δ_1 micron	δ_2 micron
0.7	1.43	17.98	288	1.04	1.226×10^{-3}	6.02×10^6	6.12	0.93
0.7	1.43	17.98	288	1.04	1.226×10^{-4}	6.02×10^5	48.7	8.0
0.7	1.43	17.98	288	1.04	1.226×10^{-5}	6.02×10^4	402	67
0.9	1.11	15.86	288	0.234	1.226×10^{-3}	6.02×10^6	14.4	2.57
0.9	1.11	15.86	288	0.234	1.226×10^{-4}	6.02×10^5	116	20.5
0.9	1.11	15.86	288	0.234	1.226×10^{-5}	6.02×10^4	899	157

4.5. It may be shown by arguments similar to those employed above that if $P(w_1, V_1)$ lies on $V = g_1(w)$, and $(1 - w_1^2)^{-1}R^{-1/3} = o(1)$, that is, for the curves of class III, the solution curves cross the w-axis before $w = 1$ and then approach $w = 0$ as described before. As $R \to \infty$, the solution curve approaches C_1 in $0 \leq w < w_1$ $(V < 0)$, and the remainder of the curve gives rise to a sort of 'negative shock', similar to that for the curves of class II in $0 < w < 1$, $(V < 0)$.

When $(1 - w_1^2)$ is of order $R^{-1/3}$, (the curves previously excluded) the situation is confused, since there is a transition from curves of the 'shock' type to those of the 'negative shock' type just described as P moves along $V = g_2(w)$ to (w_m, V_m) and then back along $V = g_1(w)$. It is impossible to tell without detailed calculation, possibly involving numerical integration, just where this transition takes place.

It should be noted that none of the solutions obtained has an infinite discontinuity of velocity gradient, even in the limit $R = \infty$, and the flow patterns are quite different from those obtained for the inviscid fluid.

5. The entropy variation.

In this section we will consider the variation of specific entropy over a typical solution curve of the shock type, as an interesting fact arises which, it is believed, was not hitherto known. It is found that there is an entropy maximum within the shock, and in the limit, as the shock becomes infinitely thin, this maximum does not disappear. Of course, the same phenomenon occurs with the plane shock, as is demonstrated.

We define S to be the specific entropy, and start from the well-known equation

$$T\frac{DS}{Dt} = \frac{1}{\rho}\{\Phi + \lambda\nabla^2 T\} \tag{5.1}$$

where Φ is the viscous dissipation function.

For this problem, the equation becomes

$$T\rho u\frac{dS}{dr} = \frac{4}{3}\mu\left\{\frac{u^2}{r^2} - \frac{u}{r}\frac{du}{dr} + \left(\frac{du}{dr}\right)^2\right\} + \lambda\left\{\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right\}, \tag{5.2}$$

and in terms of the non-dimensional quantities from (1.12), (1.13), (1.14), and (1.22),

$$\frac{R\theta}{\mathrm{R}}\frac{dS}{d\xi} = 4\left\{w^2 + \frac{1}{2}wV + \frac{1}{4}V^2\right\} + \frac{3V}{8\beta\sigma}\left\{V\frac{d^2\theta}{dw^2} + \frac{dV}{dw}\frac{d\theta}{dw}\right\}. \tag{5.3}$$

When $\sigma = \sigma_0$, (5.3) becomes

$$\frac{\theta}{\mathrm{R}}\frac{dS}{d\xi} = -\frac{1}{w}\left\{\frac{1}{2}V\left[1 - \left(1 + \frac{4}{R(1+\beta)}\right)w^2\right] - w\left[1 - \beta\left(1 + \frac{4}{R(1+\beta)}\right)w^2\right]\right\} \tag{5.4}$$

$$\doteqdot -\frac{wV}{R}\frac{dV}{dw}. \tag{5.5}$$

The use of (5.5) instead of (5.4) is justified in the same way as the basic approximation used in **1.4**, as differences in w of order R^{-1} only are neglected.

In Fig. 3 we see, then, that $dS/d\xi$ is very small over the regions where the solution curve lies close to the inviscid curve, becomes of order 1 in the interior of the shock and vanishes very close to the point of maximum velocity gradient. In fact, if w_V and w_S denote respectively the values of w at which the velocity gradient and entropy have their maxima, then it may be shown that

$$w_V = 1 - O(R^{-1}(1 - w_1)^{-2}), \tag{5.6}$$

$$w_S = w_V - O(R^{-1}(1 - w_1)^{-2}) \tag{5.7}$$

$$= 1 - O(R^{-1}(1 - w_1)^{-2}). \tag{5.8}$$

By examining the signs of V and dV/dw, it is seen that $dS/d\xi < 0$ in $0 \leq w < w_S$, $V > 0$, and $dS/d\xi > 0$ over the remainder of the curve. This seemingly paradoxical result of decreasing entropy is explained by the fact that as the heat conductivity is not zero, fluid elements are no longer isolated systems. In the range $0 \leq w < w_S$, $V > 0$ that is, on the subsonic 'side' of the shock, heat is continually being conducted backward in the sense of decreasing r, but a given fluid element in this range gives out more heat at the rear than it takes in from the front, and so has a net loss of heat energy.

For a perfect gas, with the neglect of a constant,

$$S = C_v \log(p/\rho^\gamma), \tag{5.9}$$

which becomes, in our notation,

$$S - S_0 = C_v \log \{\theta(wr)^{2\beta/(1-\beta)}\}, \tag{5.10}$$

where

$$S_0 = \frac{C_v}{1-\beta} \log \{\mathrm{R}T_0\kappa^{-2\beta}(1+\beta)^{\beta}\}. \tag{5.11}$$

So, at the second stagnation point, $S = -\infty$, and with this additional information, the sketch curve of entropy variation with ξ (Fig. 5) may be drawn. If we denote by

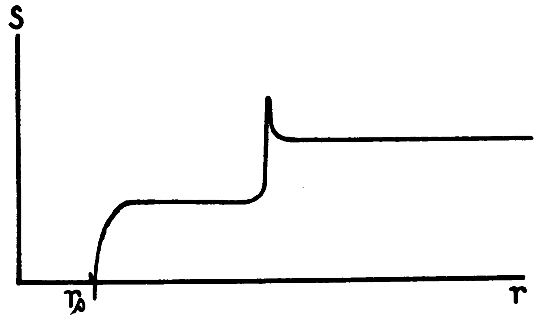

FIG. 5. Entropy variation for a typical shock solution.

S_1, S_2 and $S_{\max}$ respectively, the entropy when $w = w_1$, w_2 and w_S, we have from (5.10)

$$S_1 - S_0 = C_v \log \{\theta_1(w_1r_1)^{2\beta/(1-\beta)}\},$$

$$S_2 - S_0 = C_v \log \{\theta_2(w_2r_2)^{2\beta/(1-\beta)}\},$$

and

$$S_{\max} - S_0 = C_v \log \{\theta_S(w_Sr_S)^{2\beta/(1-\beta)}\}.$$

In the limit, as $R \to \infty$, $w_2 \to w_1^{-1}$, $w_S \to 1$ from (5.9), and r_1, r_2 and r_S tend to a common value, so that

$$S_{\max} - S_1 = C_v \log \left\{\frac{1-\beta}{1-\beta w_1^2} w_1^{2\beta/(\beta-1)}\right\}, \tag{5.12}$$

$$S_1 - S_2 = C_v \log \left\{\frac{1-\beta w_1^2}{w_1^2 - \beta} w_1^{2(1+\beta)/(1-\beta)}\right\}, \tag{5.13}$$

the last being the usual result for an ideal shock.

For plane flow, with constant viscosity, we find the solution

$$\frac{(w - w_1)^{w_1}}{(w_2 - w)^{w_2}} = D \exp\left\{-\frac{3}{4}\frac{m}{\mu}\frac{(w_2 - w_1)}{1+\beta} x\right\}, \tag{5.14}$$

where $m = \rho u$ and $\sigma = 3/4$, [4], which is seen to give a shock-type flow, the 'speeds' w_1, w_2 being attained at $x = -\infty$ and $x = \infty$ respectively. In this case

$$\frac{\theta}{\mathrm{R}}\frac{dS}{dx} = -\frac{3m}{4\mu(1+\beta)} \cdot \frac{1-w^2}{w^2}(w_2 - w)(w - w_1). \tag{5.15}$$

This clearly exhibits the entropy maximum which now falls exactly at $w = 1$, (the point of inflexion of the w-x curve) while the relations (5.13) and (5.14) for the entropy differences $S_{max} - S_1$, and $S_1 - S_2$ now hold exactly.

6. Viscosity varying with temperature.

In this section we consider briefly the case of μ varying directly with temperature. The results are qualitatively the same for the shock type curves, but the whole flow is now automatically confined to the region of positive temperature.

The equations to be investigated are (1.15) and (1.16) (with $C = 1$ as before). Assume that

$$\mu = \mu_0 \theta, \tag{6.1}$$

and hence

$$R = R_0/\theta, \tag{6.2}$$

where

$$R_0 = \frac{3\kappa}{\mu_0(1 + \beta)}, \qquad \text{a constant.} \tag{6.3}$$

Equation (1.16) now has the form

$$\theta - 1 + \beta\left\{1 + \frac{4\theta}{R_0(1 + \beta)}\right\}w^2 - \frac{\theta}{R_0(1 + \beta)}\left\{\frac{3}{\sigma}\frac{d}{d\xi}(\theta - 1) + 8\beta w\frac{dw}{d\xi}\right\} = 0, \tag{6.4}$$

and to find an integrating factor as in **1.4**, we must take

$$\sigma = \frac{3}{4} \cdot \frac{R_0(1 + \beta) + 4\theta}{R_0(1 + \beta) + 4\beta w^2}, \tag{6.5}$$

which is not constant. However, for the range of w in question the departure of σ from the constant value 3/4 is so small that it may be neglected for practical purposes. With this value of σ, if we again define

$$\begin{aligned} E &= \theta - 1 + \beta\left\{1 + \frac{4}{R(1 + \beta)}\right\}w^2 \\ &= \theta - 1 + \beta\left\{1 + \frac{4\theta}{R_0(1 + \beta)}\right\}w^2, \end{aligned} \tag{6.6}$$

then equation (6.4) has the formal solution

$$E = A \exp\left[\int^{\xi} \frac{R_0(1 + \beta)}{4\theta}\left\{1 + \frac{4\theta}{R_0(1 + \beta)}\right\} d\xi\right]. \tag{6.7}$$

Physically, θ must be bounded near the stagnation point at ∞, and if we assume that it is integrable in this neighbourhood, then A must be zero for E to remain finite as $\xi \to \infty$, and so, as in **1.4**, $E \equiv 0$, and hence

$$\theta = R_0(1 + \beta)\frac{1 - \beta w^2}{R_0(1 + \beta) + 4\beta w^2}. \tag{6.8}$$

With change of independent variable to w, and neglect of terms of order R_0^{-1} compared with 1, as in **1.4**, eq. (1.16) takes the form

$$\frac{w^2 V}{R_0}(1 - \beta w^2)\frac{dV}{dw} = \frac{1}{2}V(1 - w^2) - w(1 - \beta w^2) + 2\beta R_0^{-1} w^3 V^2 \tag{6.9}$$

and, with

$$Z = V\theta, \tag{6.10}$$

$$\frac{w^2 Z}{R_0}\frac{dZ}{dw} = \frac{1}{2} Z(1 - w^2) - w(1 - \beta w^2)^2. \tag{6.11}$$

The solution curves for (6.11) are sketched in Fig. 6, and it is seen that in the range

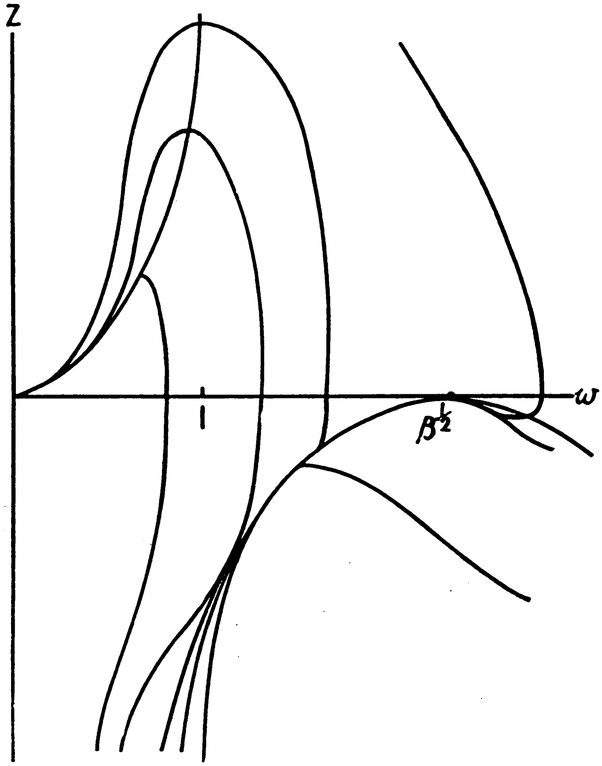

FIG. 6. Solution curves for $\mu \propto T$ with $Z = VO$.

$0 \leq w < \beta^{-1/2}$, their behaviour is very similar to those discussed before in Fig. 3. In fact, the resemblance between (6.11) and (1.41) is striking, and exactly the same technique may be used to discuss this equation. It is obvious that for the 'shock' type curves, similar results will be obtained for Z as were obtained previously for V, and as the variation of θ is only of order 1 for the range $0 \leq w \leq \beta^{-1/2}$, we can carry over the same qualitative results for V for this new equation. (Only coefficients will be affected, orders of magnitude will be unaltered.) It is only near $w = \beta^{-1/2}$ that the results are different, as θ vanishes here. To investigate the behaviour more closely, we may now turn to the $(w\text{-}V)$ plane.

From (6.9), dV/dw is zero when

$$\frac{1}{2} V(1 - w^2) - w(1 - \beta w^2) + \frac{2\beta w^3 V^2}{R_0} = 0, \tag{6.12}$$

provided that neither $V = 0$, $w = 0$ nor $w = \beta^{-1/2}$. Equation (6.12) has the two solutions

$$V_+ = \frac{R_0}{4\beta w^3}\left\{-\frac{1}{2}(1 - w^2) + \left[\frac{1}{4}(1 - w^2)^2 + \frac{8\beta w^4}{R_0}(1 - \beta w^2)\right]^{1/2}\right\}, \tag{6.13}$$

and

$$V_- = \frac{R_0}{4\beta w^3}\left\{-\frac{1}{2}(1 - w^2) - \left[\frac{1}{4}(1 - w^2)^2 + \frac{8\beta w^4}{R_0}(1 - \beta w^2)\right]^{1/2}\right\}, \tag{6.14}$$

and provided that $(1 - w^2)^2R_0 \gg 1$, that is $| 1 - w |^{-1}R_0^{-1/2} = o(1)$, we have

$$\begin{aligned} V_+ &= f(w)(1 + o(1)), \qquad 0 \le w < 1, \\ &= \frac{(w^2 - 1)R_0}{4\beta w^3}(1 + o(1)), \qquad 1 < w \le \beta^{-1/2}, \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} V_- &= \frac{-(1 - w^2)R_0}{4\beta w^3}(1 + o(1)), \qquad 0 \le w < 1, \\ &= f(w)(1 + o(1)), \qquad 1 < w \le \beta^{-1/2}. \end{aligned} \tag{6.16}$$

Thus V_+ in $0 \le w < 1$, and V_- in $1 < w \le \beta^{-1/2}$, lie very close to the curve C_1 which occurred before, with a small neighbourhood of $w = 1$ of order $R^{-1/2}$ excluded (this small neighbourhood lies inside the neighbourhood of order $R^{-1/3}$ previously excluded for the important curves). There is no longer an infinity at $w = 1$, and V_+ and V_- cross $w = 1$ at a height of order $R^{1/2}$.

For large V of order R,

$$\frac{w^2V}{R_0}\frac{d^2V}{dw^2} = \frac{2\beta wV^2}{(1 - \beta w^2)^2}\left\{1 - w^2 + \frac{4\beta w^3}{R_0}V + o(1)\right\}, \tag{6.17}$$

and we see that in $1 < w < \beta^{-1/2}$, $d^2V/dw^2 > 0$ above V_+ and $d^2V/dw^2 < 0$ below V_+, so that we may now proceed to sketch the solution curves, as in Fig. 7. The solution curves which do not bend round to cross the w axis, now go off to $V = +\infty$ as $w \to \beta^{-1/2} - 0$, and in fact, all the possible flows starting with $0 \le w < \beta^{-1/2}$ are confined to this range; the line $w = \beta^{-1/2}$ has become a barrier. The speed at which the acceleration is a maximum is now only within $O(1)$ of $w = 1$, and lies between $w = 1$ and $w = \beta^{-1/2}$, but the entropy maximum still lies very close to the point of maximum acceleration.

When $w_1 < \beta^{-1/2}$ (in the previous notation), an incomplete shock is formed, starting from $w = \beta^{-1/2}(\theta = 0)$ with infinite acceleration. This again leads to an impossible boundary condition as regards the production of this flow in practice. Apart from this, the flow patterns are fundamentally the same as those for constant viscosity.

The case of the vortex source may be discussed in an approximate manner with similar results to those already obtained.

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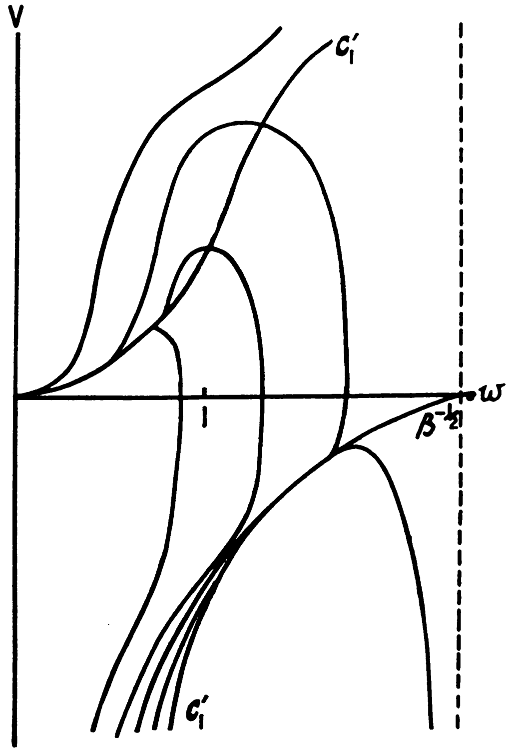

FIG. 7. Solution curves for $\mu \propto T$ in the w-V plane. C_1' is the curve of zero slope.

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