ORTHOGONAL EDGE POLYNOMIALS IN THE SOLUTION OF BOUNDARY VALUE PROBLEMS*

BY

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1. Introduction. The method presented below for solving boundary value problems is applicable when the problem can be formulated as minimization of some integral. One constructs orthogonal boundary polynomials f_n appropriate to certain boundary conditions, enlarges them by factors g_n—obtained from Euler-Lagrange variational equations—to product functions

$$\varphi_n = g_n f_n \tag{1a}$$

defined over the entire domain, with boundary values

$$\varphi_{nb} = f_n \tag{1b}$$

and expands the prescribed boundary value—say, Φ_b—of the unknown function Φ into the edge polynomials

$$\Phi_b = \sum c_n f_n \,. \tag{2}$$

Then,

$$\Phi \sim \sum c_n \varphi_n = \sum c_n g_n f_n \tag{3}$$

constitutes an approximate solution of the problem. An interesting aspect of the method is that once the general expression for the polynomials f_n has been found, any eigenvalue and its associated eigenfunction can be determined without prior determination of the lower modes.

Another interesting aspect is that while the functions f_n are—by definition—precisely orthogonal, with respect to some suitable weight function ρ,

$$\langle f_n f_m \rangle \equiv \int_{\text{boundary}} \rho\, f_n f_m \, ds = \delta_{nm} \tag{4}$$

the derivatives $\partial f_n/\partial s$, $\partial f_m/\partial s$ (if s denotes the coordinate along the boundary), while not orthogonal, can be regarded—in the problems to be considered—as being approximately orthogonal, in analogy to the precise orthogonality of the derivatives $\partial \Phi_n/\partial s$, $\partial \Phi_m/\partial s$ of the exact eigenfunctions.

In "first approximation" our approach thus neglects the derivative coupling terms altogether and yields the product eigenfunctions (1a) discussed above. In "second approximation" products of $\partial f_n/\partial s$, $\partial f_m/\partial s$ are retained in the variational integral if

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$m = n$ or $n \pm 1$ (or $m = n$, $n \pm 2$ when due to symmetry conditions $\partial f_n/\partial s$, $\partial f_m/\partial s$ turn out to be precisely orthogonal[3]), and only higher order coupling terms are neglected. In such a case the approximate solution—call it now ψ_n—will appear as a linear combination

$$\psi_n = G_{n,n-1}f_{n-1} + G_{nn}f_n + G_{n,n+1}f_{n+1} \tag{5a}$$

of product eigenfunctions, with boundary value

$$\psi_{nb} = f_n \, .$$

While the polynomials f_n of the second approximation are the same as those of the first approximation, the G_n functions, again determined from a variational equation, will differ from the g_n functions. Higher approximations are constructed similarly.

Functional developments of type (3) where $f_n(x, y)$ is some assumed function, $g_n(x)$ is a function determined from an Euler variational equation, have been employed previously by Kantorovitch[4] and Poritsky[5]. The novel feature of the present approach is the systematic development of orthonormal sets f_n. The determination of these sets greatly simplifies subsequent calculations as one is allowed (in first approximation) to regard the various modes involved as uncoupled.

The method described above was first used to prove St. Venant's principle for a plane rectangular elastic region subject to selfequilibrating edge tractions.[6] It was found that the problem could be formulated as a biharmonic eigenvalue problem. The present paper illustrates use of the "first approximation" technique for a simpler group of problems, those relating to the harmonic equation. Section 6 discusses the wave equation and illustrates also the method of the "second approximation."

2. The potential is an isosceles triangle. In this section we shall solve the equation

$$\nabla^2\Phi = 0 \tag{6}$$

in the triangular region shown in Fig. 1, subject to the following boundary conditions.

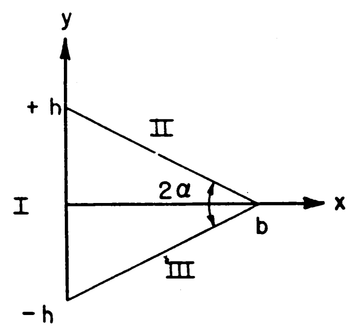

FIG. 1.

[3]As is the case for instance when f_n is an even function, f_{n+1} an odd function.

[4]*Mathematical theory of elasticity*, by I. S. Sokolnikoff, McGraw Hill, 1946, p. 315.

[5]*Reduction of the Solution of Certain Partial Differential Equations to Ordinary Differential Equations*, by H. Poritsky, Proc. Fifth Int. Cong. of Appl. Mech. 1938.

[6]*The end problem of rectangular strips*, by G. Horvay, J.A.M. 1953, p. 87. See also discussion of the paper in the issue of Sept. 1953.

Along the boundaries II, III where

$$y = \pm y_b \equiv \pm h(1 - x/b), \tag{7a}$$

we prescribe

$$\Phi_{II} = \Phi_{III} = 0. \tag{7b}$$

Along the boundary I, where

$$x = 0, \qquad -h \leq y \leq +h, \tag{8a}$$

we prescribe

$$\Phi_I = \text{arbitrary function.} \tag{8b}$$

The integral to be minimized is

$$I = \int_0^b \int_{-y_b}^{+y_b} [(\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2]\, dy\, dx. \tag{9}$$

The lowest degree polynomial $f(y)$ which satisfies

$$f(x, \pm y_b) = 0, \qquad \langle f^2(0, y)\rangle \equiv \int_{-y_b}^{+y_b} f^2(x, y)\, dy \bigg|_{x=0} = \int_{-h}^{+h} f^2(0, y)\, dy = 1 \tag{10}$$

is

$$f_0 = \left(\frac{15}{16h}\right)^{1/2}\left(1 - \frac{y^2}{y_b^2}\right). \tag{11}$$

The higher degree polynomials, obtained by orthonormalization with respect to the lower ones, are

$$\begin{aligned} f_1 &= (105/16h)^{1/2}(y/y_b)[1 - y^2/y_b^2], \\ f_2 &= (3/4)^{1/2}[1 - 7y^2/y_b^2]f_0 , \\ f_3 &= (11/4)^{1/2}[1 - 3y^2/y_b^2]f_1 , \\ f_4 &= (91/128)^{1/2}[1 - 18(y/y_b)^2 + 33(y/y_b)^4]f_0 , \cdots \end{aligned} \tag{12}$$

Introducing the notation

$$F_n = \{F\}_n ,$$

$$\frac{\partial f(x, y)}{\partial x} = f_x , \qquad \frac{\partial f(x, y)}{\partial y} = f_y , \qquad \frac{dg}{dx} = g', \qquad \int_{-y_b}^{+y_b} f f_y\, dy = \langle f f_y \rangle \tag{13}$$

and writing

$$\Phi \sim \sum A_n \varphi_n = \sum A_n g_n(x) f_n(x, y), \tag{14}$$

we write Eq. (9) in the form:

$$I = \sum_{n,m} A_n A_m \int_0^b \int_{-y_b}^{+y_b} [\{g f_y\}_n \cdot \{g f_y\}_m + \{g f_x + g' f\}_n \cdot \{g f_x + g' f\}_m]\, dy\, dx. \tag{15}$$

To first approximation this is

$$I = \sum_n A_n^2 \int_0^b \{\langle f^2\rangle g'^2 + 2\langle ff_x\rangle gg' + \langle f_x^2 + f_y^2\rangle g^2\}_n \, dx. \tag{16}$$

To minimize (16), the Euler-Lagrange equations

$$\frac{d}{dx}\{\langle f^2\rangle g'\}_n - \{\langle ff_{xx} + 2f_x^2 + f_y^2\rangle g\}_n = 0 \tag{17}$$

must be satisfied. It is readily seen that

$$\langle f_n^2\rangle = y_b/h, \qquad \{\langle ff_{xx} + 2f_x^2 + f_y^2\rangle\}_n = c_n/hy_b\,, \tag{18}$$

where c_n is a constant which depends on the order n of the polynomial. The differential equation (17) thus assumes the form

$$[(1 - x/b)g_n']' - c_n g_n/h^2(1 - x/b) = 0 \tag{19}$$

and has the solution

$$g_n = (1 - x/b)^{\nu_n}, \qquad \nu_n = b(c_n)^{1/2}/h. \tag{20}$$

In particular for $n = 0$ we obtain

$$\langle f_0^2\rangle = 1 - \frac{x}{b}\,, \qquad \langle f_0 f_{0xx} + 2f_{0x}^2 + f_{0y}^2\rangle = \left(\frac{5}{2} + \frac{3}{2}\frac{h^2}{b^2}\right)\Big/ h^2\left(1 - \frac{x}{b}\right) \tag{21}$$

and

$$\nu_0^2 = \frac{5}{2}\left(\frac{b}{h}\right)^2 + \frac{3}{2}\,. \tag{22}$$

In first approximation ν_0 is the zeroth eigenvalue, $g_0 f_0$ the zeroth eigenfunction. For the special case $b/h = 10$ there results

$$\nu_0 = 15.9. \tag{23a}$$

This compares with the well-known precise solution

$$\nu_0 = \frac{\pi}{2 \text{ arccot } b/h} = 15.8 \tag{23b}$$

of the sector of vertex angle $2\alpha = 2$ arccot 10.

3. The potential problem in a semi-infinite strip. The extremely simple problem $\nabla^2\Phi = 0$ in the semi-infinite strip of Fig. 2, with the boundary conditions

$$\begin{aligned} \Phi_{\text{I}} &= \text{prescribed} \\ \Phi_\infty &= 0 \\ \Phi_{\text{II}} &= \Phi_{\text{III}} = 0 \end{aligned} \tag{24a,b,c}$$

provides a simple basis for comparing the polynomial method with the exact solution.

The same polynomials f_n are used here as in Section 2, one must only place

$$y_b = h = 1 \tag{25}$$

in Eqs. (12). The g_n functions are solutions

$$g_n = e^{-\nu_n x} \tag{26a}$$

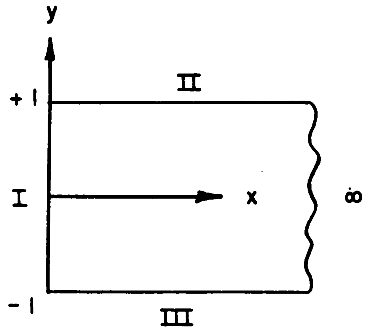

FIG. 2.

of the Euler equations

$$g_n'' - \nu_n^2 g_n = 0. \tag{26b}$$

The resulting eigenvalues are

$$\nu_n = \langle f_n'^2 \rangle^{1/2}; \qquad \nu_0 = 1.58, \nu_1 = 3.24, \nu_2 = 5.05, \cdots \tag{27}$$

as contrasted with the exact values

$$\frac{n+1}{2}\pi = 1.57, 3.14, 4.71, \cdots \tag{28}$$

For the specific example

$$\Phi_{\mathrm{I}} = 1 - y^4 \tag{29}$$

the exact solution becomes

$$\varphi_E = 1.173e^{-\pi x/2} \cos\frac{\pi y}{2} - 0.209e^{-3\pi x/2} \cos\frac{3\pi y}{2} + \cdots \tag{30}$$

while the polynomial method leads to

$$\varphi_A = 1.143e^{-1.58x}(1 - y^2) - 0.143e^{-5.05x}(1 - 8y^2 + 7y^4) \tag{31}$$

In contrast, a one-term Rayleigh approximation

$$\varphi_R = e^{-\nu x}(1 - y^4) \tag{32}$$

gives

$$\varphi_R = e^{-1.79x}(1 - y^4). \tag{33}$$

It is interesting to make a numerical comparison of the solutions E, A, R. At the point $(x, y) = (1, 0)$ they give, respectively

$$\varphi_E = 0.242, \qquad \varphi_A = 0.235, \qquad \varphi_R = 0.167. \tag{34}$$

Function (29) can be considered as the zeroth orthogonal polynomial constructed from the set $1, y^4, y^8, \cdots$. Clearly, the resulting approximation is poorer than when the set $1, y^2, y^4, \cdots$ is used. Since both sets are complete[7] there arises the question of

[7]By Szász' Theorem (Courant-Hilbert, *Methoden der Mathematischen Physik*, I. p. 86) the set $1, y^{\lambda_1}, y^{\lambda_2}, \cdots$ is complete, in a finite interval, 0 to h, whenever $\Sigma\ (\lambda_k)^{-1}$ diverges.

the proper selection of the best set. The answer is: the best set is obtained when the gap between successive powers is the smallest, consistent with boundary, symmetry and finiteness requirements for the functions and their derivatives.

4. The two-region problem. Consideration of a simple two-region potential problem (Fig. 3) shows an additional advantage in the use of the polynomial method, namely that the often troublesome transcendental boundary equations are replaced by a set of equations which are linear in the unknowns. Consider $\nabla^2\Phi = 0$ in each of two regions, A and B, with boundary conditions:

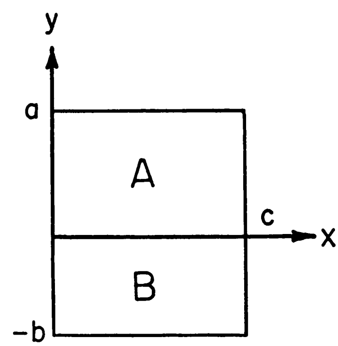

FIG. 3.

$$\Phi = 0 \qquad \text{on the edges} \qquad y = a, y = -b \qquad \text{and} \qquad x = c \tag{35a}$$

$$\Phi = \text{arbitrary on edge} \qquad x = 0 \tag{35b}$$

$$\Phi_A = \Phi_B\ , \ k\Phi_{Ay} = \Phi_{By}\ , \ \Phi_{Ayy} = \Phi_{Byy}\ , \ k\Phi_{Ayyy} = \Phi_{Byyy}\ , \ \cdots \tag{35c}$$

on the interface, $y = 0$. The polynomials $f_n(y)$ to be used have a different form in each of the two regions A and B:

$$f_{A0} = (y - a)(A_{00} + A_{01}y),\ f_{B0} = (y + b)(B_{00} + B_{01}y),$$

$$f_{A1} = (y - a)(A_{10} + A_{11}y + A_{12}y^2),\ f_{B1} = (y + b)(B_{10} + B_{11}y + B_{12}y^2),\ \cdots \tag{36}$$

The four constants in f_0 are determined from the normalization condition and three interface conditions. The six constants in f_1 are determined from the two conditions of orthonormality, and four interface conditions. Successive polynomials are constructed similarly. The appropriate orthonormalization condition is

$$\int_{-b}^{0} f_{Bn}f_{Bm}\,dy + \int_0^a f_{An}f_{Am}\,dy = \delta_{nm}\ . \tag{37}$$

For the particular values of the parameters

$$k = 3(2)^{1/2}, \qquad a = 1, \qquad b = 1/2^{1/2} \tag{38a}$$

one finds

$$\nu_0 = \langle f_0'^2\rangle^{1/2} = 1.75. \tag{38b}$$

This compares with the exact value

$$\nu_0 = 1.67 \tag{39a}$$

obtained by solving

$$k \tan \nu b + \tan \nu a = 0. \tag{39b}$$

5. The potential problem in a sector. The case of the circular sector brings out a novel feature in the orthogonality condition. The geometry is as in Fig. 4. Here the

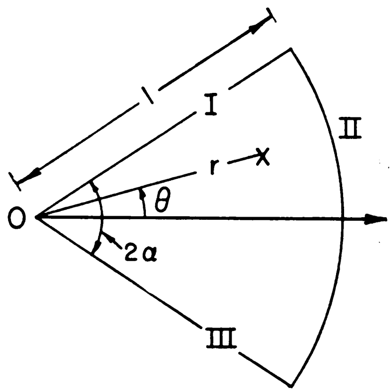

Fig. 4.

integral to be minimized for the k-th approximate eigenfunction

$$\varphi_k = g_k(r) f_k(\theta) \tag{40}$$

is

$$\begin{aligned} I_k &= \int_0^1 \int_{-\alpha}^{+\alpha} [(\partial\varphi_k/\partial r)^2 + (\partial\varphi_k/r\, d\theta)^2]\, d\theta\, r\, dr \\ &= \int_0^1 \int_{-\alpha}^{+\alpha} [(g_k' f_k)^2 + (g_k f_k'/r)^2]\, d\theta\, r\, dr. \end{aligned} \tag{41}$$

For cases where boundary values along the circular arc are specified, we are led to θ polynomials of type f_n considered in Section 3.

For the sake of variety we list the f_n polynomials appropriate to the conditions of vanishing normal edge derivative:

$$[\partial\Phi/r\, d\theta]_{\pm\alpha} = 0. \tag{42}$$

They are

$$\begin{aligned} f_0 &= 1/(2\alpha)^{1/2}, \\ f_1 &= (315/316\alpha)^{1/2}(\theta/\alpha)[1 - \tfrac{1}{3}\theta^2/\alpha^2], \\ f_2 &= (343/384\alpha)^{1/2}[1 - (30/7)(\theta/\alpha)^2 + (15/7)(\theta/\alpha)^4], \cdots \end{aligned} \tag{43}$$

When radial boundary values need representation, then an expansion in polynomials of r becomes necessary. On denoting

$$\langle v(r) \rangle \equiv \int_0^1 v(r)\, dr \tag{44a}$$

and integrating out over the variable r in (41),

$$I_k = \int_{-\alpha}^{+\alpha} [f_k'^2 \langle g_k^2/r \rangle + f_k^2 \langle r g_k'^2 \rangle] \, d\theta \tag{44b}$$

we are lead to the orthonormalization condition

$$\langle g_k g_l / r \rangle \equiv \int_0^1 (g_k g_l / r) \, dr = \delta_{kl} \, . \tag{45}$$

We list the functions $g_k(r)$ for the boundary conditions

$$\partial\Phi_{\mathrm{I}}/\partial r = \partial\Phi_{\mathrm{III}}/\partial r = R(r) = \text{prescribed function of } r, \tag{46a}$$

$$\Phi = 0 \qquad \text{at} \qquad r = 1, \tag{46b}$$

$$\Phi = \Phi/r = 0 \qquad \text{at} \qquad r = 0. \tag{46c,d}$$

Condition (46d), which essentially states that the polynomials $g_k(r)$ must contain a factor r^2, results from the requirement that in the corner, $r \rightarrow 0$, of the sector the transverse variation, $\partial\Phi/r\,\partial\theta$, of the function must be limited (otherwise Φ would not be uniquely determined).

The functions are

$$\begin{aligned} g_1 &= 60^{1/2} r^2 (1 - r), \\ g_2 &= 40^{1/2} r^2 (1 - r)(4 - 7r), \\ g_3 &= 140^{1/2} r^2 (1 - r)(5 - 20r + 18r^2), \\ g_4 &= 840^{1/2} r^2 (1 - r)(4 - 27r + 54r^2 - 33r^3), \cdots \end{aligned} \tag{47}$$

In either case, whether we are concerned with expansions into $g_n(r)$ functions along boundaries I, III, or expansions into $f_n(\theta)$ functions along boundary II, the remaining procedure is the same—insert the polynomials into the integral (41), carry out the integration over the proper variable, and solve the appropriate Euler-Lagrange equation for the second (unknown) factor of the eigenfunction.[8]

In Fig. 5 we plot successive approximations

$$R(r) \sim \sum a_n g_n(r), \qquad a_n \equiv \langle R g_n / r \rangle \tag{48a}$$

to the function

$$R = r \tag{48b}$$

in terms of the functions (47). Since we now have to construct a g_k expansion for a function $R(r)$ which does not satisfy the boundary conditions prescribed for $g_k(r)$, the approximation exhibits a pronounced Gibbs phenomenon near the end, $r = 1$, of the interval.

[8]Solution of the general problem, when non-homogeneous boundary conditions are specified along all edges, is obtained by superposition.

6. The wave equation in a rectangle. The second approximation. In this section we shall treat the vibration problem of a membrane

$$\nabla^2\Phi = K^2\ddot{\Phi} \tag{49}$$

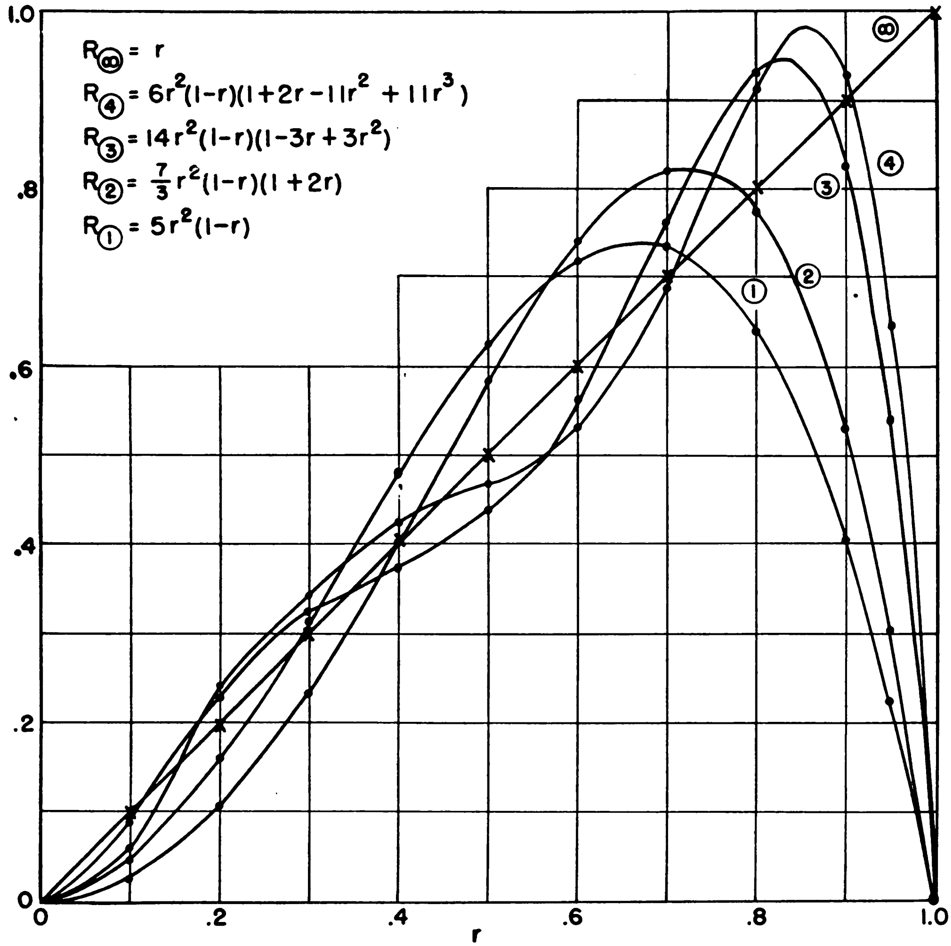

FIG. 5.

in a simple rectangle region. The example will illustrate the use of orthogonal polynomials for a problem where more than two independent variables are involved; it will also demonstrate the use of the "second approximation."

Let the problem be as follows: Solve (49) for $\Phi(x, y, t)$ in the rectangular region, Fig. 6, subject to the initial condition

$$\Phi(x, y, 0) = \text{prescribed} \tag{50}$$

and to the following boundary conditions (at all t):

$$\Phi_{II} = \Phi_{IV} = 0, \tag{51a}$$

$$\Phi_{I} = \partial\Phi_{III}/\partial x = 0. \tag{51b}$$

Problem (49) is equivalent to the requirement that one minimize the integral:

$$I = \int_0^\infty \int_0^b \int_{-a}^a [(\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2 - K^2(\partial\Phi/\partial t)^2]\, dy\, dx\, dt \tag{52}$$

subject to (50), (51). We write in "first approximation"

$$\Phi(x, y, t) \sim \sum C_{kl}\varphi_{kl}(x, y, t) = \sum C_{kl} g_k(x) f_l(y) \tau_{kl}(t), \tag{53}$$

where $g_k(x)$ and $f_l(y)$ are members of appropriate sets of orthogonal polynomials, the functions τ_{kl} will be determined by solving Euler's equations for the time variable after the integration over x and y has been carried out, and the C_{kl} are to be determined from the initial conditions.

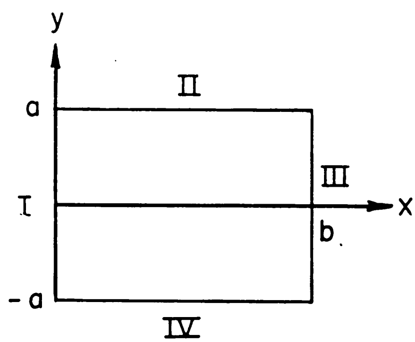

FIG. 6.

The $f_l(y)$ functions should go to zero at $y = \pm a$; thus they are the same polynomials as those of Section 2, except that y_b and h are now replaced by a. The $g_k(x)$ must be zero for $x = 0$ and have zero derivative for $x = b$.
This gives rise to the set

$$\begin{aligned} g_0 &= (15/2b)^{1/2}(x/b)[1 - x/b], \\ g_1 &= (273/2b)^{1/2}(x/b)[1 - (61/26)(x/b) + (16/13)(x/b)^2], \cdots \end{aligned} \tag{54}$$

Now,

$$\begin{aligned} \langle g_k g_l \rangle &\equiv \int_0^b g_k g_l\, dx = \delta_{kl}, \\ \langle f_k f_l \rangle &\equiv \int_{-a}^{+a} f_k f_l\, dy = \delta_{kl}, \end{aligned} \tag{55a,b}$$

and, in first approximation,

$$\begin{aligned} \langle g_k' g_l' \rangle &= \langle g_k'^2 \rangle \delta_{kl}, \\ \langle f_k' f_l' \rangle &= \langle f_k'^2 \rangle \delta_{kl}, \end{aligned} \tag{56a,b}$$

so that the integral (52) to be minimized becomes

$$I = \sum_{k,l} I_{kl}, \tag{57a}$$

$$I_{kl} = \int_0^\infty [\{\langle g_k'^2 \rangle + \langle f_l'^2 \rangle\} \tau_{kl}^2 - K^2 \dot{\tau}_{kl}^2]\, dt. \tag{57b}$$

The integrals (57b) lead to the Euler-Lagrange equations

$$\ddot{\tau}_{kl} + \nu_{kl}^2 \tau_{kl} = 0, \tag{58a}$$

where

$$K^2 \nu_{kl}^2 = \langle g_k'^2 \rangle + \langle f_l'^2 \rangle. \tag{58b}$$

Assuming that initial conditions require that

$$\tau_{kl}(0) = 1, \qquad \dot{\tau}_{kl}(0) = 0 \tag{59a}$$

it follows that

$$\tau_{kl} = \cos \nu_{kl} t \tag{59b}$$

The first two values of $\langle g_k'^2 \rangle^{1/2}$ and first six values of $\langle f_l'^2 \rangle^{1/2}$ are listed below and compared with the appropriate true eigenvalues $(k + \frac{1}{2})\pi$ and $(l + 1)\pi/2$ respectively.

k	$b\langle g_k'^2 \rangle^{1/2}$	$(k + 1/2)\pi$	% error
0	1.58	1.57	0.6
1	4.85	4.71	3.0
l	$a\langle f_l'^2 \rangle^{1/2}$	$(l + 1)\pi/2$	% error
0	1.58	1.57	0.6
1	3.24	3.14	3.2
2	5.05	4.71	7.2
3	7.04	6.28	12
4	9.19	7.85	16
5	11.51	9.42	22

One arrives at the second approximation ψ_{kl} by writing, in accordance with (5a, b),

$$\psi_{kl}(x, y, t) = g_{k-1}T_{k-1,l}f_l + g_kT_{k,l-2}f_{l-2} + g_kT_{kl}f_l + g_kT_{k,l+2}f_{l+2} + g_{k+1}T_{k+1,l}f_l\,, \tag{60a}$$

$$\psi_{kl}(x, y, 0) = g_k(x)f_l(y). \tag{60b}$$

These functions, orthonormal at $t = 0$, are suitable for expanding the initial value (50) of Φ. To simplify the notation we shall consider the special case of

$$\psi_{03} = g_0(T_{01}f_1 + T_{03}f_3 + T_{05}f_5) + g_1T_{13}f_3 \tag{61}$$

and to further simplify the calculations we shall neglect the last term, $g_1T_{13}f_3$ of this expression. Insertion of (61) into the variational integral (52) and minimization leads to the equations of motion

$$\begin{aligned}
[\langle g_0'^2 \rangle + \langle f_1'^2 \rangle]T_{01} + \langle f_1'f_3' \rangle T_{03} + \langle f_1'f_5' \rangle T_{05} + K^2\ddot{T}_{01} &= 0,\\
\langle f_1'f_3' \rangle T_{01} + [\langle g_0'^2 \rangle + \langle f_3'^2 \rangle]T_{03} + \langle f_3'f_5' \rangle T_{05} + K^2\ddot{T}_{03} &= 0,\\
\langle f_1'f_5' \rangle T_{01} + \langle f_3'f_5' \rangle T_{03} + [\langle g_0'^2 \rangle + \langle f_5'^2 \rangle]T_{05} + K^2\ddot{T}_{05} &= 0.
\end{aligned} \tag{62}$$

For a natural mode of vibration

$$T_{0i} = T_i \cos \nu t, \tag{63a}$$

$$K^2\nu^2 = \langle g_0'^2 \rangle + \alpha^2 = (5/2b)^2 + \alpha^2, \tag{63b}$$

the secular equation system becomes

$$\begin{aligned} \left(\frac{21}{2} - \alpha^2\right)T_1 - \frac{3}{2}\,11^{1/2}T_3 + \frac{5}{4}\,6^{1/2}T_5 &= 0 \\ -\frac{3}{2}\,11^{1/2}T_1 + \left(\frac{99}{2} - \alpha^2\right)T_3 - \frac{15}{4}\,66^{1/2}T_5 &= 0 \\ \frac{5}{4}\,6^{1/2}T_1 - \frac{15}{4}\,66^{1/2}T_3 + \left(\frac{265}{2} - \alpha^2\right)T_5 &= 0 \end{aligned} \tag{64}$$

This yields the frequency equation

$$-\alpha^6 + 192.5\alpha^4 + 7507.5\alpha^2 + 56306.25 = 0, \tag{65}$$

with solutions

$$\alpha_1^2 = 3.141595^2, \qquad \alpha_3^2 = 6.32440^2, \qquad \alpha_5^2 = 11.94288^2. \tag{66}$$

Assigning now to α^2 in (64) successively each of the values (66), and solving the corresponding systems, we obtain

$$\begin{aligned} {}^1T_1 : {}^1T_3 : {}^1T_5 &= 1 : 0.131442 : 0.007686, \\ {}^3T_1 : {}^3T_3 : {}^3T_5 &= -0.134007 : 1 : 0.333781, \\ {}^5T_1 : {}^5T_3 : {}^5T_5 &= 0.035561 : -0.329016 : 1, \end{aligned} \tag{67}$$

where the superscript of T refers to the particular frequency α_i^2 with which the amplitude triplet ${}^iT_1$, ${}^iT_3$, ${}^iT_5$ is associated. Consequently T_{0i} has the form

$$T_{0i} = A_1\,{}^1T_i \cos(\nu_1 t + \gamma_1) + A_3\,{}^3T_i \cos(\nu_3 t + \gamma_3) + A_5\,{}^5T_i \cos(\nu_5 t + \gamma_5) \tag{68}$$

The six constants $A_1, \cdots, \gamma_5$ are to be determined from the six initial conditions

$$\begin{aligned} T_{01}(0) = 0, \qquad T_{03}(0) = 1, \qquad T_{05}(0) = 0, \\ \dot{T}_{01}(0) = \dot{T}_{03}(0) + \dot{T}_{05}(0) = 0. \end{aligned} \tag{69}$$

One obtains

$$A_1 = 0.12920, \quad A_3 = 0.88545, \quad A_5 = -0.29654, \quad \gamma_1 = \gamma_3 = \gamma_5 = 0. \tag{70}$$

This then completely determines the function ψ_{03} of (61) (if we neglect, as proposed, the T_{13} term). Separating out from the expression of ψ_{03} the coefficient of $\cos \nu_{03} t$ one is led to the second approximation, φ_{03}^*, of the true second eigenfunction Φ_{03} (in the approximation that the $g_1 f_3 T_{13}$ contribution is neglible):

$$\varphi^*_{30} = Ng_0(^3T_1f_1 + {}^3T_3f_3 + {}^3T_5f_5) \cos \nu_{30}t$$

$$= N(x/b)(1 - x/b)(y/a)(1 - y^2/a^2)[1.941 + 1.654(y/a)^2 - 18.719(y/a)^4]$$

$$\cdot \cos \{[(5/2b)^2 + (6.32/a)^2]^{1/2}t/K\} \qquad (71)$$

(N is the normalization factor.)

Since the expansion of the initial value $\Phi(x, y, 0)$ of Φ in the functions (60b) is perfectly straightforward (and hence the present procedure does provide one avenue of improved solution), one may ignore, until further study has been made, the question as to what is preferable: Use simple second approximation functions of type (61) which —at $t = 0$—violate slightly the orthogonality condition

$$\iint \Phi_{kl}(x, y)\Phi_{k'l'}(x, y)\, dy\, dx = \delta_{kk'}\delta_{ll'} \qquad (72)$$

or use the conventional approach involving an infinite secular system,[9] and extract from this system a set of unwieldy but precisely orthogonal approximate eigenfunctions. These precisely orthogonal functions will be successive approximations to the true eigenfunctions according to whether we retain in the infinite secular system only terms along the principal diagonal, also terms along the adjacent sub- and superdiagonals, also terms along the next sub- and superdiagonals, etc.

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[9]See e.g. *Eigenwertaufgaben,* by L. Collatz, Akad. Verlagsges., 1949, p. 398.