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A SUGGESTED MODIFICATION OF NOISE THEORY*

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Abstract. A class of stationary, equilibrium, Markoff processes is demonstrated all of which have the same equilibrium distribution, $W_0(x)$, and correlation function $R(t) = E_0 \exp(-t/\tau_0)$ differing from each other in the number of zero crossings of the system per second. The processes are described by an integral equation characterized by a parameter γ. As γ approaches 1, the integral equation passes over into the Fokker-Planck equation

$$\tau_0 \frac{\partial W}{\partial t} = E_0 \frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial x}(xW).$$

Since the number of zero crossings per second of the system becomes infinite as γ goes to one, the degenerate nature of the Fokker-Planck process is made evident.

1. Introduction. Any stationary Markoffian motion of a system in one dimension may be described by an equation of the form

$$\frac{\partial W(x, t)}{\partial t} = -W(x, t) \int A(x, x')\, dx' + \int W(x', t) A(x', x)\, dx', \tag{1}$$

where $W(x, t)dx$ is the probability of finding the system in the interval $(x, x + dx)$ at time t. $A(x, x')dx'$ is the probability per unit time that the system if at x, will jump to the interval $(x', x' + dx')$. Equation (1) describes the manner in which changes in local probability density can occur. It is seen that

$$\frac{d}{dt} \int W(x, t)\, dx = 0 \tag{2}$$

so that probability is conserved.

The stationary Markoff character of the motion is assured by the formulation in terms of a time independent transition function $A(x, x')$. By stationary Markoff is meant that if at $t = 0$, the system is at x_0, its subsequent distribution is completely described by the second order conditional probability:

$$W(x, t) = P_2(x_0 \mid x; t). \tag{3}$$

For some $A(x, x')$ the process will be an equilibrium process in the sense that

$$\lim_{t \to \infty} P_2(x_0 \mid x; t) = W_0(x) \tag{4}$$

is independent of x_0.

If $A(x, x')$ is not localized about $x = x'$ the stochastic motion described is discontinuous. The system jumps about from point to point. As $A(x, x')$ becomes more localized about $x = x'$, the jumps become on the average smaller, i.e., the motion becomes "more continuous".

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In this paper a class of transition function $A_\gamma(x, x')$ will be studied. It will be shown that, as $\gamma \to 1$ and $A_\gamma(x, x')$ becomes increasingly localized about $x = x'$, the integral equation (1) passes over into the Fokker-Planck equation

$$\tau_0 \frac{\partial W}{\partial t} = E_0 \frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial x}(xW). \tag{5}$$

The nature of the motion described by the Fokker-Planck equation as indicated by this limiting process will be discussed, and the physical implications of this study examined.

2. Properties of the integral equation. Consider the process described by the transition function

$$A(x, x') = \frac{1}{T_{mf}} (\beta/\pi)^{1/2} \exp [-\beta(x' - \gamma x)^2]. \tag{6}$$

Here, T_{mf}, the mean free time of the system, is independent of x, i.e.

$$\int_{-\infty}^{+\infty} A(x, x')\, dx' = 1/T_{mf} \tag{7}$$

is the mean number of transitions of the system per unit time; β describes the dispersion occurring at each transition. The larger β, the more "continuous" is the motion; γ is a parameter describing the relaxation or damping of the system, i.e. if $\langle x \rangle$ is the mean value of x,

$$\langle x \rangle = \int xW(x, t)\, dx \tag{8}$$

then

$$\frac{d\langle x \rangle}{dt} = \frac{-\langle x \rangle}{T_{mf}/(1 - \gamma)} \tag{9}$$

as is readily deduced from Eq. (1). The decay time of system is, then

$$\tau = T_{mf}/(1 - \gamma) \tag{10}$$

The closer γ is to one, the smaller is the damping and the longer the decay time. τ is also the correlation time of the system, for

$$R(t) = \int W_0(x_0)x_0P_2(x_0 \mid x; t)x\, dx_0\, dx.$$

Since from (9) $\langle x \rangle = x_0 \exp(-t/\tau)$, we have

$$R(t) = \int W_0(x_0)x_0^2 \exp(-t/\tau)\, dx_0 ,$$

i.e.

$$R(t) = E \exp(-t/\tau), \tag{11}$$

where

$$E = \int W_0(x)x^2\, dx. \tag{12}$$

Equation (1) with $A(x, x')$ given by (6) may be solved in the following way. Use is made of the expansion

$$\exp[-(x - \gamma y)^2/(1 - \gamma^2)] = \{[\pi(1 - \gamma^2)]^{1/2} \exp[+(y^2 - x^2)/2]\} \cdot \sum_0^\infty \gamma^n \psi_n(x)\psi_n(y), \tag{13}$$

where $\psi_n(x)$ is the set of orthonormal Hermite functions

$$\psi_n(x) = \left(\frac{1}{2^n n! \pi^{1/2}}\right)^{1/2} \exp(x^2/2)(-d/dx)^n \exp(-x^2). \tag{14}$$

Then

$$A(x', x) = \frac{\alpha}{T_{mf}} \exp\left[-\frac{\alpha^2}{2}(x^2 - x'^2)\right] \sum_0^\infty \gamma^n \psi_n(\alpha x)\psi_n(\alpha x'), \tag{15}$$

where

$$\alpha = [\beta(1 - \gamma^2)]^{1/2}.$$

In terms of

$$\phi(x, t) = \exp\left[\frac{\alpha^2}{2} x^2\right] W(x, t)$$

Eq. (1) becomes

$$\frac{\partial \phi}{\partial t} = -\frac{\phi}{T_{mf}} + \frac{\alpha}{T_{mf}} \int \sum_0^\infty \psi_n(\alpha x)\psi_n(\alpha x')\gamma^n \phi(x', t)\, dx'.$$

If we expand ϕ in terms of the orthonormal set $\alpha^{1/2}\psi_n(\alpha x)$

$$\phi(x, t) = \sum_0^\infty a_n \alpha^{1/2} \psi_n(\alpha x),$$

our equation separates into

$$\frac{da_n}{dt} = -\frac{a_n}{T_{mf}} + \frac{\gamma^n a_n}{T_{mf}}. \tag{16}$$

If, at $t = 0$,

$$W(x, 0) = \delta(x - x_0) = \sum_0^\infty \alpha \psi_n(\alpha x)\psi_n(\alpha x_0),$$

then

$$a_n(0) = \alpha^{1/2} \exp\left(\frac{\alpha^2}{2} x_0^2\right) \psi_n(\alpha x_0),$$

and

$$P_2(x_0 \mid x; t) = \exp\left[\frac{\alpha^2}{2}(x_0^2 - x^2)\right] \sum_0^\infty \{\alpha \psi_n(\alpha x)\psi_n(\alpha x_0) \exp[-t(1 - \gamma^n)/T_{mf}]\}. \tag{17}$$

We see that

$$\lim_{t\to\infty} P_2(x_0 \mid x;\, t) = W_0(x) = \pi^{-1/2}\alpha \exp(-\alpha^2 x^2)$$
$$= \pi^{-1/2}[\beta(1-\gamma^2)]^{1/2} \exp[-\beta(1-\gamma^2)x^2] \tag{18}$$

and

$$E = \langle x^2 \rangle = [2\beta(1-\gamma^2)]^{-1}. \tag{19}$$

3. Passage to the Fokker-Planck process. The Fokker-Planck equation, (5), contains two parameters, the correlation time τ_0 and the equilibrium mean square, $E_0 = \langle x^2 \rangle$.

Our transition function $A(x, x')$ contains three variables τ, E, γ. Let us maintain the values $\tau = \tau_0$, and $E = E_0$ and permit γ to pass through a set of values approaching one. To do so we adjust T_{mf} and β to the value of γ through the equations

$$T_{mf} = \tau_0(1-\gamma) \tag{20}$$

and

$$\beta = \frac{1}{2E_0(1-\gamma^2)} \simeq \frac{1}{4E_0(1-\gamma)}. \tag{21}$$

These parameter values define a set of processes A_γ all of which will have the same equilibrium distribution and the same correlation function as the corresponding Fokker-Planck process.

Indeed, the corresponding Fokker-Planck process is just the limit of the process A_γ as $\gamma \to 1$.

This may be seen in two ways. First one may pass to the limit $\gamma = 1$ in the conditional probability function.

Since for the process A_γ

$$\alpha = \left(\frac{E_0}{2}\right)^{1/2}, \qquad \lim \frac{1-\gamma^n}{T_{mf}} = \frac{n}{\tau_0},$$

we have

$$\lim_{\gamma\to 1} P_{2\gamma}(x_0 \mid x;\, t) = \exp\left[\frac{\alpha^2}{2}(x_0^2 - x^2)\right] \sum_0^\infty \{\alpha\psi_n(\alpha x)\psi_n(\alpha x_0) \exp(-nt/\tau_0)$$
$$= \{2E_0\pi[1 - \exp(-2t/\tau_0)]\}^{-1/2} \exp\left\{\frac{-[x - x_0\exp(-t/\tau_0)]^2}{2E_0[1-\exp(-2t/\tau_0)]}\right\} \tag{22}$$

and this is indeed the second order conditional probability of our equation (5).

It can also be seen directly that the integral equation passes over into the Fokker-Planck equation.

The integral equation can always be rewritten formally* as a partial differential equation of infinite order

$$\frac{\partial W(x, t)}{\partial t} = \sum_1^\infty \frac{\partial^n}{\partial x^n}(A_n(x)W(x, t)), \tag{23}$$

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*See p. 246 Keilson and Storer, Q. Appl. Math. 10, 243-253 (1952).

where

$$A_n(x) = \frac{1}{n!} \int (x - x')^n A(x, x')\, dx'. \tag{24}$$

In the limit $\gamma \to 1$, $A_1(x)$ and $A_2(x)$ do not vanish, but all higher moments do vanish. It is readily found that

$$A_1(x) = x/\tau_0$$

and

$$A_2(x) = \frac{(1-\gamma)^2 x^2}{2T_{mf}} + \frac{(1-\gamma^2)E_0}{2T_{mf}} \to \frac{E_0}{\tau_0}.$$

The higher moments

$$A_n(x) = \frac{1}{n!} \int \{(1-\gamma)x + (\gamma x - x')\}^n \frac{1}{T_{mf}} \left(\frac{\beta}{\pi}\right)^{1/2} \exp\,[-\beta(x' - \gamma x)^2]\, dx'$$

contain terms

$$(1-\gamma)^m x^m \langle(\gamma x - x')^p\rangle,$$

where either $m > 2$ or $p > 2$. A simple examination reveals that all such terms go to zero as γ approaches one.

4. Zero crossings of the system A_γ. It is easily seen that the number of times per second that the system with its motion characterized by $A(x, x')$ will cross zero is given by

$$\begin{aligned} j_+(0) &= \int_{-\infty}^{0} W_0(x)\, dx \int_0^\infty A(x, x')\, dx' \\ &= \frac{(1-\gamma^2)^{1/2}}{\pi T_{mf}} \int_{-\infty}^{0} \int_0^\infty \exp\,[-x^2 - y^2 - 2\gamma xy]\, dx\, dy \\ &= \frac{(1-\gamma^2)^{1/2}}{\pi T_{mf}} \int_A \exp\,[-S^2(1+\gamma) - T^2(1-\gamma)]\, ds\, dt, \end{aligned}$$

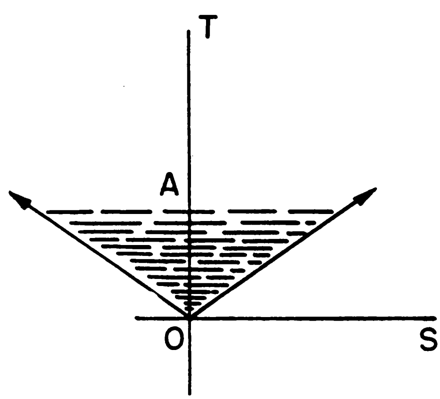

FIG. 1

where A is the shaded area shown in Fig. 1, i.e.

$$j_+(0) = \frac{1}{\pi T_{mf}} \tan^{-1} \left\{ \left(\frac{1 - \gamma}{1 + \gamma} \right)^{1/2} \right\}. \tag{25}$$

For the process A_γ,

$$j_+(0) = \frac{1}{\pi \tau_0 (1 - \gamma)} \tan^{-1} \left\{ \left(\frac{1 - \gamma}{1 + \gamma} \right)^{1/2} \right\}. \tag{26}$$

As γ approaches one, $j_+(0)$ becomes infinite. This will be true not only for zero but for all x. The implication is plain. The Fokker-Planck process is a degenerate process in which the one sided current density of the system is infinite. A Fokker-Planck model for the velocity motion of a colloid particle would describe an infinite number of changes of direction of the particle per unit time. Such a model used to describe voltage fluctuations would imply an infinite number of polarity reversals per second. Since a process $A\gamma$ will afford the same correlation function and equilibrium distribution, *and* finite polarity reversal frequency, it is suggested that such a model may better describe noise, and that the number of zero crossings be regarded as an independent macroscopic physical quantity on an equal footing with τ_0, E_0.

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EVALUATION OF CONSTANTS IN CONFORMAL REPRESENTATION*

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In using the Schwarz-Christoffel transformation [1],

$$dz = K \prod_{i=1}^{n} (\zeta - \zeta_i)^{(\alpha_i/\pi) - 1} d\zeta = K f(\zeta)\, d\zeta$$

whereby the upper half ζ-plane is mapped into a simple connected polygon, the evaluation of the unknown constant K (if complex $K = ce^{i\lambda}$, c, λ real), is oftentimes tedious. We shall show a simple method of evaluating the unknown constant K by examples, proving first a

THEOREM: *By the Schwarz-Christoffel transformation if ζ_i in the ζ-plane corresponds to two points P_i, Q_i in the z-plane and $\zeta = \zeta_i$ is a simple pole of $f(\zeta)$, then*

$$K = \frac{\text{dist}\,(P_i, Q_i)}{\pi i R(\zeta = \zeta_i)}$$

R, denoting residue and dist (P_i, Q_i), denoting the distance between the two points P_i and Q_i.

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