ON THE AVERAGE SECOND MOMENT OF THE ENERGY SPECTRAL INTENSITY*

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Introduction. In a paper by J. Leray [1] it was proved that the kinetic energy of a two-dimensional incompressible viscous flow in a domain $\mathfrak{D}$ bounded by a regular curve $\mathfrak{S}$ dissipates at least as fast as an exponential law. Following this, J. Kampé de Fériet [2] proved the same proposition by using a different technique and related the dissipation constant with the domain $\mathfrak{D}$. In a later paper [3] he studied the Fourier transform of the two-dimensional vorticity equation and proved that the upper bounds of the Fourier transform of the vorticity and the energy spectral function also decrease in accordance with a similar law.

The present paper is concerned with an incompressible viscous fluid such that the domain of flow extends to infinity. One of the objects of this paper is to study the time variation of the average second moment of the energy spectral intensity which is closely related to the dissipation of energy.

We begin by assuming the velocity and pressure field to belong to a class of functions such that they can be represented by a trigonometric series. For this purpose we introduce

$$K(\mathbf{r}\cdot\mathbf{x}) = \cos(\mathbf{r}\cdot\mathbf{x}) + \sin(\mathbf{r}\cdot\mathbf{x}), \tag{1}^{**}$$

so that

$$\mathbf{v}(\mathbf{x}, t) = \sum \mathbf{A}(\mathbf{r}, t)K(\mathbf{r}\cdot\mathbf{x}), \tag{2}$$

and

$$p(\mathbf{x}, t) = \sum \Phi(\mathbf{r}, t)K(\mathbf{r}\cdot\mathbf{x}), \tag{3}$$

where $\mathbf{v}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ are the velocity and pressure respectively, being functions of the position vector $\mathbf{x}$ in the physical space and the time t, $\mathbf{A}(\mathbf{r}, t)$ is the velocity spectral function and is a vector, $\Phi(\mathbf{r}, t)$ is the pressure spectral function, and $\mathbf{r}$ is a wave number vector. The purpose of using $K(\mathbf{r}\cdot\mathbf{x})$ as defined by (1) instead of the customary exponential function $\exp(\mathbf{r}\cdot\mathbf{x})$ is to avoid complex spectral functions in (2) and (3).

The fundamental equations governing the motion of an incompressible viscous fluid are the Navier-Stokes and the continuity equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}, \tag{4}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{5}$$

where ∇ is the gradient operator and ρ, ν are constants, the density and kinematic viscosity of the fluid. Using (2) and (3), we can transform (4) and (5) into

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) = \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] - \nu r^2\mathbf{A}(\mathbf{r}, t), \tag{6}$$

*Received Sept. 30, 1953.

**This notation, $K(\mathbf{r}\cdot\mathbf{x}) = \cos(\mathbf{r}\cdot\mathbf{x}) + \sin(\mathbf{r}\cdot\mathbf{x})$, was first introduced, to the author's knowledge, by L. S. G. Kovasznay. $\mathbf{r}$ is a vector of magnitude r.

where

$$\mathfrak{F}[\mathbf{A}(\mathbf{r},\ t)] = \tfrac{1}{2}\left[\mathbf{Q}(\mathbf{r},\ t) - \frac{\mathbf{r}}{r^2}\,\mathbf{Q}(\mathbf{r},\ t)\cdot\mathbf{r}\right] \tag{7}$$

and

$$\mathbf{Q}(\mathbf{r},\ t) = \sum_{\mathbf{s}} [\mathbf{A}(\mathbf{r}+\mathbf{s},\ t) + \mathbf{A}(\mathbf{r}-\mathbf{s},\ t) + \mathbf{A}(-\mathbf{r}+\mathbf{s},\ t) - \mathbf{A}(-\mathbf{r}-\mathbf{s},\ t)]\cdot\mathbf{s}\mathbf{A}(-\mathbf{s},\ t). \tag{8}$$

$$\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{r} = 0. \tag{9}$$

A study of the transformed Navier-Stokes and continuity equations (6) and (9) has been made by the author [4]. Here we summarize a few of the conclusions which will be made use of in this paper:

(a) Let the spectral function $\mathbf{A}(\mathbf{r},\ t)$ be defined by the spectral vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ... located at $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, ... in the wave number space at time t, then $\mathfrak{F}[\mathbf{A}(\mathbf{r},\ t)]$ represents the contribution to $\partial\mathbf{A}(\mathbf{r},\ t)/\partial t$ at time t due to the non-linear interaction of the spectral vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ... ; $\mathfrak{F}[\mathbf{A}(\mathbf{r},\ t)]$ is in general not zero at time t for $\mathbf{r} = \pm\mathbf{r}_i \pm \mathbf{r}_j$, $i \neq j$, $i, j = 1, 2, 3, \ldots$ and is zero for $\mathbf{r} \neq \pm\mathbf{r}_i \pm \mathbf{r}_j$. It is to be noticed that $\mathfrak{F}[\mathbf{A}_1(\mathbf{r},\ t) + \mathbf{A}_2(\mathbf{r},\ t)] \neq \mathfrak{F}[\mathbf{A}_1(\mathbf{r},\ t)] + \mathfrak{F}[\mathbf{A}_2(\mathbf{r},\ t)]$ in general, while $\mathfrak{F}[c\mathbf{A}(\mathbf{r},\ t)] = c^2\mathfrak{F}[\mathbf{A}(\mathbf{r},\ t)]$, c being a constant.

(b) For an arbitrary spectral function $\mathbf{A}(\mathbf{r},\ t)$ which satisfies the orthogonality relation (9), $\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{r} = 0$, the following is true:

$$\sum_{\mathbf{r}} \mathbf{A}(\mathbf{r},\ t)\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r},\ t)] = 0. \tag{10}$$

(c) $\sum \mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t) = E(t)$, where $E(t)$ is twice the average kinetic energy of the flow per unit volume. In what follows, we shall call $E(t)$ simply the energy of the flow and $\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t)$ the energy spectral intensity. Therefore, by combining (6) and (10), we obtain

$$\frac{d}{dt}E(t) = -2\nu \sum r^2\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t),$$

i.e.,

$$E(t) = E(0)\exp\left[-2\nu\int_0^t \langle r^2(\tau)\rangle\,d\tau\right], \tag{11}$$

where

$$\langle r^2(t)\rangle = \frac{\sum r^2\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t)}{\sum \mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t)} \tag{12}$$

and is the average second moment of the energy spectral intensity.

From (11) it is seen that the average second moment of the energy spectral intensity, $\langle r^2(t)\rangle$, is intimately related to the energy of the flow, $E(t)$. The paper is mainly devoted to the study of the time variation of this average second moment.

1. From (6) we have

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t) = 2\mathbf{A}(\mathbf{r},\ t)\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r},\ t)] - 2\nu r^2\mathbf{A}(\mathbf{r},\ t)\cdot\mathbf{A}(\mathbf{r},\ t). \tag{13}$$

It is seen that the energy spectral intensity $\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t)$ varies with the time t due to the interaction of spectral vectors and due to viscosity. It follows that the average second moment of the energy spectral intensity may also vary with t due to the interaction and viscosity. We shall, first of all, examine the effect of viscosity alone, ignoring the effect of the interaction. The following conclusion can be easily deduced:

I. *The average second moment of the energy spectral intensity is a decreasing function of time if only the effect of viscosity is taken into consideration and will approach r_0^2 as $t \to \infty$ where $r_0 = \min (r_1, r_2, r_3, \ldots)$.*

When the interaction term is ignored, Eq. (6) becomes simply

$$\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) = -\nu r^2 \mathbf{A}(\mathbf{r}, t), \tag{14}$$

so that

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, 0) \exp(-\nu r^2 t). \tag{15}$$

A set of spectral vectors initially located at $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$ will remain at $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$ at any later time, each behaving individually and damped exponentially. Thus

$$\langle r^2(t) \rangle = \frac{\sum r^2 \mathbf{A}(\mathbf{r}, 0) \cdot \mathbf{A}(\mathbf{r}, 0) \exp(-2\nu r^2 t)}{\sum \mathbf{A}(\mathbf{r}, 0) \cdot \mathbf{A}(\mathbf{r}, 0) \exp(-2\nu r^2 t)}, \tag{16}$$

and it can be verified that

$$\frac{d}{dt} \langle r^2(t) \rangle$$

$$= -2\nu \frac{\sum_{i,j} \mathbf{A}(\mathbf{r}_i, 0) \cdot \mathbf{A}(\mathbf{r}_i, 0) \mathbf{A}(\mathbf{r}_j, 0) \cdot \mathbf{A}(\mathbf{r}_j, 0) [r_i^2 - r_j^2]^2 \exp[-2\nu(r_i^2 + r_j^2)t]}{\{\sum \mathbf{A}(\mathbf{r}, 0) \cdot \mathbf{A}(\mathbf{r}, 0) \exp(-2\nu r^2 t)\}^2}. \tag{17}$$

Statement I is thus seen to be true.

2. When viscosity is disregarded the energy of the flow is conserved, as is obvious from (11). Then from (12) we have

$$\frac{d}{dt} \langle r^2(t) \rangle = \frac{1}{\sum \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t)} \frac{d}{dt} \sum r^2 \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t). \tag{18}$$

Thus it is sufficient to examine $d/dt \sum r^2 \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t)$.
It follows from (6) that

$$\frac{d}{dt} \sum r^2 \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t) = 2 \sum r^2 \mathbf{A}(\mathbf{r}, t) \cdot \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)], \tag{19}$$

from which we have:

II. *The time rate of change of the second moment of the energy spectral intensity due to the effect of the interaction of spectral vectors alone is zero if at that instant the spectral vectors are so located at $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$ that $\pm \mathbf{r}_i \pm \mathbf{r}_j \pm \mathbf{r}_k \neq 0$, for all possible $i, j, k = 1, 2, 3, \ldots$, $i \neq j, j \neq k, k \neq i$.*

It is to be noticed, however, that although in this case the first derivative of $\langle r^2(t) \rangle$ is zero at this instant, the second derivative is not zero in general, due to the fact that the interaction of spectral vectors starts to generate new vectors at $\pm \mathbf{r}_i \pm \mathbf{r}_j$.

We proceed to examine the more general case for which the velocity spectral vectors are so located that one or more of $\pm\mathbf{r}_i \pm \mathbf{r}_j \pm \mathbf{r}_k$ may be zero. Consider, for the simple case, that there is only one such set, say, $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0$, then from (7) and (8),

$$\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] = \tfrac{1}{2}\Big\{ \sum_{\mathbf{s}} [\mathbf{A}(\mathbf{r}+\mathbf{s}, t) + \mathbf{A}(\mathbf{r}-\mathbf{s}, t) + \mathbf{A}(-\mathbf{r}+\mathbf{s}, t) - \mathbf{A}(-\mathbf{r}-\mathbf{s}, t)]\cdot\mathbf{s}\mathbf{A}(-\mathbf{s}, t)$$
$$- \frac{\mathbf{r}}{r^2} \sum_{\mathbf{s}} [\mathbf{A}(\mathbf{r}+\mathbf{s}, t) + \mathbf{A}(\mathbf{r}-\mathbf{s}, t) + \mathbf{A}(-\mathbf{r}+\mathbf{s}, t) - \mathbf{A}(-\mathbf{r}-\mathbf{s}, t)]\cdot\mathbf{s}\mathbf{A}(-\mathbf{s}, t)\cdot\mathbf{r}\Big\},$$

so that

$$r^2\mathbf{A}(\mathbf{r}, t)\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]$$
$$= \frac{r^2}{2} \sum_{\mathbf{s}} \{[\mathbf{A}(\mathbf{r}+\mathbf{s}, t) + \mathbf{A}(\mathbf{r}-\mathbf{s}, t) + \mathbf{A}(-\mathbf{r}+\mathbf{s}, t) - \mathbf{A}(-\mathbf{r}-\mathbf{s}, t)]\cdot\mathbf{s}\mathbf{A}(-\mathbf{s}, t)\cdot\mathbf{A}(\mathbf{r}, t)\}$$
$$= -\frac{r^2}{2} \sum_{i=1,2,3} \{[\mathbf{A}(\mathbf{r}-\mathbf{r}_i, t) + \mathbf{A}(\mathbf{r}+\mathbf{r}_i, t) + \mathbf{A}(-\mathbf{r}-\mathbf{r}_i, t) - \mathbf{A}(-\mathbf{r}+\mathbf{r}_i, t)]\cdot\mathbf{r}_i\mathbf{A}(\mathbf{r}_i, t)\cdot\mathbf{A}(\mathbf{r}, t)\}.$$

Writing $\mathbf{A}(\mathbf{r}_i, t) = \mathbf{A}_i$, we see that

$$\sum_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} r^2\mathbf{A}(\mathbf{r}, t)\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] = -\tfrac{1}{2}\left\{\begin{array}{l} r_2^2\mathbf{A}_3\cdot\mathbf{r}_1\mathbf{A}_1\cdot\mathbf{A}_2 + r_3^2\mathbf{A}_2\cdot\mathbf{r}_1\mathbf{A}_3\cdot\mathbf{A}_1 \\ \quad + r_3^2\mathbf{A}_1\cdot\mathbf{r}_2\mathbf{A}_2\cdot\mathbf{A}_3 + r_1^2\mathbf{A}_3\cdot\mathbf{r}_2\mathbf{A}_1\cdot\mathbf{A}_2 \\ \quad + r_1^2\mathbf{A}_2\cdot\mathbf{r}_3\mathbf{A}_3\cdot\mathbf{A}_1 + r_2^2\mathbf{A}_1\cdot\mathbf{r}_3\mathbf{A}_2\cdot\mathbf{A}_3 \end{array}\right\}. \tag{20}$$

We evaluate the right hand side of (20) by choosing the coordinate axes so that $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ all lie in the plane of R_1–R_2, as shown in Fig. 1. The angle between $\mathbf{r}_i$, $\mathbf{r}_j$ is

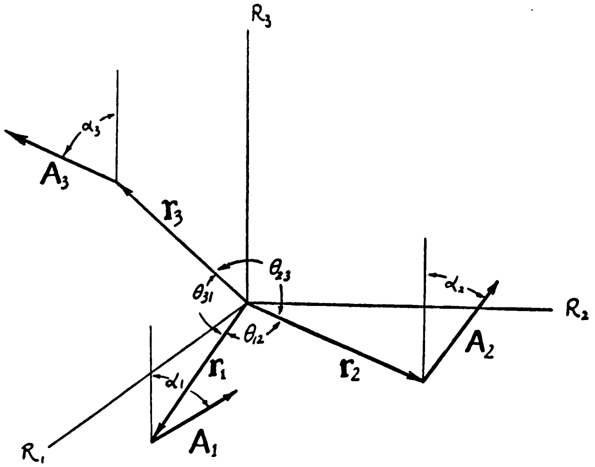

FIG. 1.

denoted by θ_{ij}, measured from $\mathbf{r}_i$ to $\mathbf{r}_j$ in the clockwise direction when facing the R_3-axis. Since the velocity spectral vector $\mathbf{A}_i$ is perpendicular to the wave number vector $\mathbf{r}_i$

because of continuity, its direction is completely fixed by the angle α_i which is measured from a parallel vector to the R_3-axis to the spectral vector $\mathbf{A}_i$ in the counter-clockwise direction when facing in the direction of the corresponding wave number vector. Using this choice of axes it can be verified that:

$$\sum_{r_1, r_2, r_3} r^2 \mathbf{A}(\mathbf{r}, t) \cdot \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] = \tfrac{1}{2} A_1 A_2 A_3 r_1 r_2 r_3 \left\{ \begin{aligned} &\sin (\theta_{23} - \theta_{31}) \cos \alpha_1 \cos \alpha_2 \sin \alpha_3 \\ &+ \sin (\theta_{12} - \theta_{23}) \cos \alpha_1 \sin \alpha_2 \cos \alpha_3 \\ &+ \sin (\theta_{31} - \theta_{12}) \sin \alpha_1 \cos \alpha_2 \cos \alpha_3 \end{aligned} \right\}. \tag{21}$$

Equation (21) is also true for $\pm \mathbf{r}_i \pm \mathbf{r}_j \pm \mathbf{r}_k = 0$. Obviously, we can generalize the result so as to include the case in which there are more sets such that $\pm \mathbf{r}_i \pm \mathbf{r}_j \pm \mathbf{r}_k = 0$, $i, j, k = 1, 2, 3, \cdots, i \neq j, j \neq k, k \neq i$. Hence we have,

III. *When only the effect of the interaction of spectral vectors is considered, the time rate of change of the average second moment of the energy spectral intensity is given by the following formula:*

$$\frac{d}{dt} \langle r^2(t) \rangle = \frac{1}{\sum_i \mathbf{A}_i \cdot \mathbf{A}_i} \cdot \sum_{\pm r_i \pm r_j \pm r_k = 0} A_i A_j A_k r_i r_j r_k \left\{ \begin{aligned} &\sin (\theta_{jk} - \theta_{ki}) \cos \alpha_i \cos \alpha_j \sin \alpha_k \\ &+ \sin (\theta_{ij} - \theta_{jk}) \cos \alpha_i \sin \alpha_j \cos \alpha_k \\ &+ \sin (\theta_{ki} - \theta_{ij}) \sin \alpha_i \cos \alpha_j \cos \alpha_k \end{aligned} \right\}, \tag{22}$$

where the summation is extended to all sets of vectors $\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k$ *such that* $\pm \mathbf{r}_i \pm \mathbf{r}_j \pm \mathbf{r}_k = 0$, $i, j, k = 1, 2, 3, \cdots, i \neq j, j \neq k, k \neq i$.

Nothing can be said about the sign of $d\langle r^2(t) \rangle / dt$ for the general three-dimensional flow. However, if the motion of the fluid is two-dimensional, then $d\langle r^2(t) \rangle / dt$ is zero and $\langle r^2(t) \rangle$ remains constant with time. Hence:

IV. *The average second moment of the energy spectral intensity of an unbounded two-dimensional incompressible non-viscous flow remains constant with time.*

3. We now examine the time rate of change of $\langle r^2(t) \rangle$ due to the combined effect of the interaction of spectral vectors and the viscosity. As a matter of fact it follows from previous results that

$$\frac{d}{dt} \langle r^2(t) \rangle = \frac{1}{\sum_i \mathbf{A}_i \cdot \mathbf{A}_i} \sum_{\pm r_i \pm r_j \pm r_k = 0} A_i A_j A_k r_i r_j r_k \left\{ \begin{aligned} &\sin (\theta_{jk} - \theta_{ki}) \cos \alpha_i \cos \alpha_j \sin \alpha_k \\ &+ \sin (\theta_{ij} - \theta_{jk}) \cos \alpha_i \sin \alpha_j \cos \alpha_k \\ &+ \sin (\theta_{ki} - \theta_{ij}) \sin \alpha_i \cos \alpha_j \cos \alpha_k \end{aligned} \right\}$$
$$- \frac{2\nu}{[\sum_i \mathbf{A}_i \cdot \mathbf{A}_i]^2} \sum_{i,j} \mathbf{A}_i \cdot \mathbf{A}_i \mathbf{A}_j \cdot \mathbf{A}_j (r_i^2 - r_j^2)^2. \tag{23}$$

Therefore we have

V. *The average second moment of the energy spectral intensity of an unbounded two-di-*

mensional incompressible viscous flow is a decreasing function of time; therefore, there exists a lower bound of energy:

$$E(t) \geqq E(0) \exp \{-2\nu\langle r^2(0)\rangle t\}, \tag{24}$$

where $\langle r^2(0)\rangle$ denotes the initial average second moment.

For the three-dimensional flow it has not been possible to establish a similar energy bound valid for all time t. However, the following shows that such a bound exists at least for the initial stage of dissipation. Starting with (6) we have

$$\begin{aligned}\frac{d}{dt}\sum r^2\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t) + 2\nu\sum r^4\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t)\\ &= 2\sum r^2\mathbf{A}(\mathbf{r}, t)\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\\ &\leqq 2\{\sum r^4\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t)\sum\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\}^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\sum r^2\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t) + 2\nu\left\{[\sum r^4\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t)]^{\frac{1}{2}} - \left[\frac{\sum\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]}{4\nu^2}\right]^{\frac{1}{2}}\right\}^2\\ \leqq \frac{\sum\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]}{2\nu},\end{aligned}$$

and therefore

$$\frac{d}{dt}\sum r^2\mathbf{A}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t) \leqq \frac{\sum\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]}{2\nu}. \tag{25}$$

Here we need a lemma

Lemma. *Let $\mathbf{A}(\mathbf{r}_i, t) = \mathbf{A}_i$, $i = 1, 2, 3, \ldots$, then*

$$\sum\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]\cdot\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] \leqq 2\sum A_i^2\sum r_i^2A_i^2. \tag{26}$$

Proof. It has been shown by the author [4] that given two spectral vectors such that $\mathbf{A}(\mathbf{r}_1, t) = \mathbf{A}_1$, $\mathbf{A}(\mathbf{r}_2, t) = \mathbf{A}_2$ then

$$\begin{aligned}|\mathfrak{F}[\mathbf{A}(\mathbf{r}, t]| = \tfrac{1}{2}A_1A_2|\sin\theta_{12}|\left\{\frac{(r_1^2 - r_2^2)^2\sin^2\alpha_1\sin^2\alpha_2}{r_1^2 + r_2^2 + 2r_1r_2\cos\theta_{12}}\right.\\ \left. + (r_1\cos\alpha_1\sin\alpha_2 - r_2\cos\alpha_2\sin\alpha_1)^2\right\}^{\frac{1}{2}}\end{aligned} \tag{27}$$

for $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$ and $\mathbf{r} = -\mathbf{r}_1 - \mathbf{r}_2$,

$$\begin{aligned}|\mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]| = \tfrac{1}{2}A_1A_2|\sin\theta_{12}|\left\{\frac{(r_1^2 - r_2^2)^2\sin^2\alpha_1\sin^2\alpha_2}{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_{12}}\right.\\ \left. + (r_1\cos\alpha_1\sin\alpha_2 + r_2\cos\alpha_2\sin\alpha_1)^2\right\}^{\frac{1}{2}}\end{aligned} \tag{28}$$

for $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r} = -\mathbf{r}_1 + \mathbf{r}_2$, where the wave number vectors $\mathbf{r}_1$, $\mathbf{r}_2$ lie in the plane of R_1-R_2, and the various angles are measured using the same convention as before, Fig. 2.

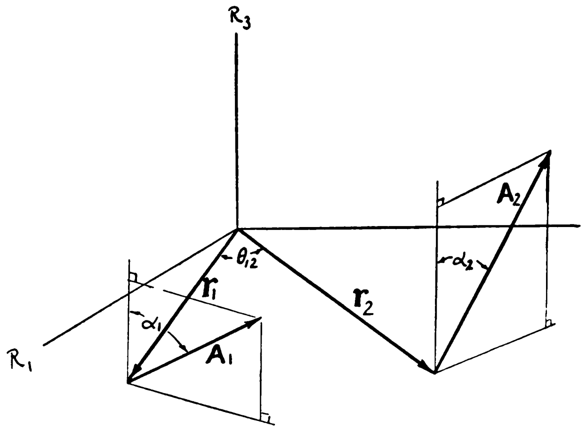

FIG. 2.

Using these results, we have

$$\sum \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] \cdot \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)]$$
$$\leqq \tfrac{1}{2} A_1^2 A_2^2 \sin^2 \theta_{12}[2(r_1 + r_2)^2 \sin^2 \alpha_1 \sin^2 \alpha_2 + 2r_2^2 \sin^2 \alpha_1 \cos^2 \alpha_2 + 2r_1^2 \cos^2 \alpha_1 \sin^2 \alpha_2]$$
$$\leqq A_1^2 A_2^2 (r_1 + r_2)^2$$
$$\leqq 2A_1^2 A_2^2 (r_1^2 + r_2^2).$$

Now let $\mathbf{A}(\mathbf{r}_i, t) = \mathbf{A}_i$, $i = 1, 2, 3, \ldots$, then

$$\sum \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] \cdot \mathfrak{F}[\mathbf{A}(\mathbf{r}, t)] \leqq 2 \sum_{i,j} A_i^2 A_j^2 (r_i^2 + r_j^2)$$
$$\leqq 2 \sum_i A_i^2 \sum_i A_i^2 r_i^2,$$

and the lemma is proved.

From (25) and (26) we have

$$\sum r^2 \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t) \leqq \sum r^2 \mathbf{A}(\mathbf{r}, 0) \cdot \mathbf{A}(\mathbf{r}, 0) \exp \left\{ \frac{1}{\nu} \int_0^t \sum \mathbf{A}(\mathbf{r}, \tau) \cdot \mathbf{A}(\mathbf{r}. \tau) \, d\tau \right\}, \tag{29}$$

i.e.,

$$r^2(t) \leqq r^2(0) \exp \left\{ \frac{1}{\nu} \int_0^t E(\tau) \, d\tau \right\}, \tag{30}$$

where $r^2(t)$ is the second moment of the energy spectral intensity. We have, then

$$\frac{d}{dt} E(t) \geqq -2\nu r^2(0) \exp \left\{ \frac{1}{\nu} \int_0^t E(\tau) \, d\tau \right\}, \tag{31}$$

from which we have

VI. *For an unbounded incompressible viscous flow the energy is bounded below given by*

$$E(t) \geqq E(0) \left\{ 1 - 2\nu^2 \frac{\langle r^2(0) \rangle}{E(0)} \left[\exp \left\{ \frac{E(0)}{\nu} t \right\} - 1 \right] \right\}, \tag{32}$$

and the average second moment of energy spectral intensity is bounded above given by

$$\langle r^2(t)\rangle \leqq \langle r^2(0)\rangle\left\{1 - 2\nu^2 \frac{\langle r^2(0)\rangle}{E(0)}\left[\exp\left\{\frac{E(0)}{\nu}t\right\} - 1\right]\right\}^{-1} \exp\left\{\frac{E(0)}{\nu}t\right\} \tag{33}$$

for

$$t \leqq \frac{\nu}{E(0)} \ln\left[1 + \frac{E(0)}{2\nu^2\langle r^2(0)\rangle}\right]. \tag{34}$$

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