A NOTE ON THE NUMERICAL SOLUTION OF THE EQUATION $x^m + ax^n + b = 0$*

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In this equation, a and b are real and not equal to zero; m and n are positive integers, $m > n$.**

We give a method of transforming this equation which permits us to obtain the complex non-real roots by solving independently for the argument θ and the modulus r.

Substituting $x = r \exp(i\theta)$ in the above equation yields, for the real and imaginary parts,

$$r^m \cos m\theta + ar^n \cos n\theta + b = 0 \tag{1}$$

and

$$r^m \sin m\theta + ar^n \sin n\theta = 0. \tag{2}$$

Solving (2) for r,

$$r = \left(-\frac{a \sin n\theta}{\sin m\theta}\right)^{\frac{1}{m-n}} \qquad \theta \neq 0. \tag{3}$$

Substituting (3) in (1), we have

$$\left(-\frac{a \sin n\theta}{\sin m\theta}\right)^{\frac{m}{m-n}} \cos m\theta + a\left(-\frac{a \sin n\theta}{\sin m\theta}\right)^{\frac{n}{m-n}} \cos n\theta = -b.$$

Factoring,

$$\left(-\frac{\sin n\theta}{\sin m\theta}\right)^{\frac{n}{m-n}}\left(\cos n\theta - \frac{\sin n\theta}{\sin m\theta} \cos m\theta\right) = -b/a^{\frac{m}{m-n}}.$$

Simplifying, we conclude

$$\left(-\frac{\sin n\theta}{\sin m\theta}\right)^{\frac{n}{m-n}}\left[\frac{\sin (m-n)\theta}{\sin m\theta}\right] = -b/a^{\frac{m}{m-n}}. \tag{4}$$

We have therefore an equation in θ alone. Clearly it can be solved by repeated substitution, and then r can be obtained from (3).

The fact that r is positive can be used conveniently to limit the range which we must test for θ. Equation (3) implies

$$r^{m-n} = -\frac{a \sin n\theta}{\sin m\theta}. \tag{5}$$

Thus $\sin n\theta$ and $\sin m\theta$ have opposite signs if and only if a is positive. On the other hand, if we multiply (4) by $a^{n/m-n}$, it becomes

$$r^n\left[\frac{\sin (m-n)\theta}{\sin m\theta}\right] = -\frac{b}{a}. \tag{6}$$

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**The restriction on m and n is not necessary but the use of more general m and n requires a definition of the corresponding powers of a complex number. This would complicate the discussion.

Hence $\sin (m - n)\theta$ and $\sin m\theta$ have the same sign if and only if a and b have opposite signs. These conditions reduce considerably the amount of test work for θ, as will be shown in the example.

Example. $x^8 + x^6 + 1 = 0.$

According to (3) and (4), we write

$$r = \left(-\frac{\sin 6\theta}{\sin 8\theta}\right)^{1/2} \tag{7}$$

and

$$\left(-\frac{\sin 6\theta}{\sin 8\theta}\right)^3\left(\frac{\sin 2\theta}{\sin 8\theta}\right) = -1,$$

whence

$$\sin^3 6\theta \sin 2\theta = \sin^4 8\theta.$$

The limiting condition for the range of θ is that both $\sin 2\theta$ and $\sin 6\theta$ should be of opposite signs from $\sin 8\theta$. Thus we need only to consider the following ranges for θ.

(a) 22.5° – 30.0°	(e) 202.5° – 210.0°
(b) 67.5° – 90.0°	(f) 247.5° – 270.0°
(c) 90.0° – 112.5°	(g) 270.0° – 292.5°
(d) 150.0° – 157.5°	(h) 330.0° – 337.5°

These ranges can be paired off as (a), (e); (b), (*f*); (c), (g); and (d), (h); or, as (a), (d); (b), (c); (e), (h) and (*f*), (g). For the first group of pairs, the values of the sines of 2θ, 6θ and 8θ are identical for each pair, respectively, whereas for the second group, they are symmetrical and have opposite signs for each pair, respectively. Thus once the test for θ is completed for any two ranges not belonging to the same pair, the other values of θ follow automatically.

Suppose we take (a) and (b). Owing to the periodicity of the sine function, it can be determined by inspection that θ lies within 25.5° – 30.0° and 72.0° – 77.5°. With the aid of a sine table, the respective ranges can be narrowed down further by inspection to 25.5° – 26.5° and 74.0° – 75.0°. From this point the location of θ can be carried on by computation.

The approximate values of θ are:

26.05077°	206.05077°
74.70037°	254.70037°
105.29963°	285.29963°
153.94923°	333.94923°

Substituting in (7), we obtain the respective approximate values for r:

.91911	.91911
1.08800	1.08800
1.08800	1.08800
.91911	.91911

Therefore the respective approximate values for x are:

$.82573 + .40364i$	$-.82573 - .40364i$
$.28709 + 1.04944i$	$-.28709 - 1.04944i$
$-.28709 + 1.04944i$	$.28709 - 1.04944i$
$-.82573 + .40364i$	$.82573 - .40364i$

NOTE ON THE CIRCLE THEOREM OF HYDRODYNAMICS*

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The circle theorem [1, 2] is concerned with the irrotational, two dimensional flow of an incompressible, inviscid fluid in the z plane. Let $f(z)$ be the complex potential of flow. Then if there are no singularities within a radius, a, of the origin and *no rigid boundaries* in the plane, the appropriate flow function after introducing a circle $|z| = a$ about the origin is given by $g(z) = f(z) + f^*(a^2/z)$ (where * denotes conjugation).

The purpose of this note is to show that the restriction on rigid boundaries may be somewhat relaxed.

That some restriction is necessary may be seen from the following example. Consider the uniform flow $g(w) = w$. From the circle theorem, $h(w) = w + 1/w$ represents the flow with a unit circle about the origin. By the transformation $z = w + 3i$, we see that $H(z) = z - 3i + (z - 3i)^{-1}$ represents the flow about the circle centered at $3i$. Then if a second unit circle is introduced about the origin, $F(z) = z + (z - 3i)^{-1} + z^{-1} + (z^{-1} + 3i)^{-1}$ should give the flow for the two circles. It can easily be verified that while $z = e^{i\theta}$ is a streamline, $z = e^{i\theta} + 3i$ is not.

A rigid boundary will be called generally admissible if for every flow $f(z)$ having this boundary as a streamline and satisfying the remaining conditions of the circle theorem, the function $g(z) = f(z) + f^*(1/z)$ has both the unit circle and the boundary as streamlines. The boundary will be called conditionally admissible if this is true only for certain complex potentials $f(z)$.

To determine a necessary and sufficient condition for rigid boundaries to be generally admissible, consider any complex potential $f(z) = \alpha(x, y) + i\beta(x, y)$. Then

$$[f(z)]^* = \alpha(x, y) - i\beta(x, y) = f^*(z^*) = f^*(x - iy),$$

$$f^*(1/z) = f^*\left(\frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}\right) = \alpha\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) - i\beta\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

$$\mathcal{I}[g(z)] = \mathcal{I}[f(z) + f^*(1/z)] = \beta(x, y) - \beta\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right).$$

Clearly if $x^2 + y^2 = 1$, $\mathcal{I}[g(z)] = 0$, and hence the boundary of the unit circle is a streamline. Suppose C: $x = x(\tau)$, $y = y(\tau)$, $\tau_0 \leq \tau \leq \tau_1$, is a rigid boundary. Then $\mathcal{I}[f(z)] = \beta[x(\tau), y(\tau)] =$ constant for $\tau_0 \leq \tau \leq \tau_1$. In order that this rigid boundary be admissible, it is necessary and sufficient that $\mathcal{I}[g(z)]$ be constant along C. Hence $\beta[x/(x^2 + y^2),$

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