VIBRATIONS OF TWISTED BEAMS*

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1. Equations of equilibrium. In recent years the determination of the natural frequencies of twisted, cantilevered beams has received much attention. The analysis of a twisted beam possesses certain added complications over that of an untwisted beam due to the geometry of the structure. W. Prager**[1] has given a very compact vector representation of the equations of static equilibrium of curved, twisted beams. His results will be extended in this paper to include dynamic effects. Since Prager's analysis is not readily available, a short summary of his approach to the problem will be given.

A beam which is twisted but not curved can be described in terms of a straight line, called the *center line*, which is the locus of the centroids of the cross-sectional planes taken normal to the line. A cross-section of the beam is specified by means of the arc length s measured in a positive sense along the center line from a fixed origin 0. With each cross section, a right handed triad of orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ can be associated as indicated in Fig. 1. The vector $\mathbf{i}$ is the unit tangent vector of the center line at the

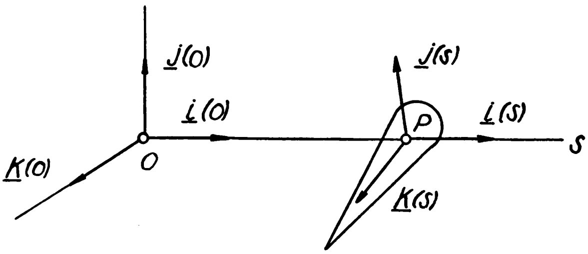

FIG. 1

centroid P of the cross section under consideration and points in the direction of increasing s. The unit vectors $\mathbf{j}$ and $\mathbf{k}$ have the directions of the principal axes of inertia of the cross section. In the case of an untwisted beam the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have fixed directions. For a twisted beam the triad rotates about $\mathbf{i}$ as the point P moves along the center line. It is this change in orientation of the principal axes of the cross section as we move from point to point which creates the difficulty in the problem.

Since the triad $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ moves as a rigid body, its instantaneous motion can be decomposed into the translation of P plus a rotation about P. If the rate of rotation per unit length is characterized by the vector $\boldsymbol{\tau}$, we note that $\boldsymbol{\tau}$ is proportional to $\mathbf{i}$. Hence

$$\boldsymbol{\tau} = \tau \mathbf{i}, \tag{1.1}$$

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**Numbers in square brackets refer to the list of References given at the end of the paper.

where the scalar τ is called the *natural twist* of the beam. It can then be shown that

$$d\mathbf{i}/ds = 0, \qquad d\mathbf{j}/ds = \tau\mathbf{k}, \qquad d\mathbf{k}/ds = -\tau\mathbf{j}. \tag{1.2}$$

Let $\mathbf{f}(s)$ and $\mathbf{c}(s)$ denote the vector intensities at P of the distributed forces and couples applied to the beam. Let the stress resultant $\mathbf{R}(s)$, a force applied at the centroid P of the cross section, and $\mathbf{M}(s)$, a couple, be equipollent to the stresses exerted on this cross section by the portion of the beam lying to the side of increasing values of s. If no concentrated forces or couples are applied, the force and moment equilibrium equations take the form

$$\frac{d\mathbf{R}(s)}{ds} + \mathbf{f}(s) = 0, \tag{1.3}$$

$$\frac{d\mathbf{M}(s)}{ds} + \mathbf{c}(s) + \mathbf{i} \times \mathbf{R}(s) = 0. \tag{1.4}$$

Under the action of the loads $\mathbf{f}$ and $\mathbf{c}$ the elements of the beam will undergo certain displacements. Points lying in the plane cross section at P of the undeformed beam will no longer lie in the same plane, but will form a curved surface called the *warped cross section*. In the theory of thin beams it is customary to exclude shearing stresses of the type which would produce a change in the right angle between two line elements at P in the directions of $\mathbf{j}$ and $\mathbf{k}$. Consequently, it is possible to define a new triad $\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$ in which $\mathbf{j}^*$ and $\mathbf{k}^*$ point along the displaced principal axes of inertia at P^*, the new position of P. Let $\mathbf{u}(s)$ denote the displacement of P. In order to make the triad at P coincide with the new triad at P^*, it is necessary to add to $\mathbf{u}(s)$ a small rotation $\boldsymbol{\theta}(s)$ about the point P^*. If $\mathbf{g}(s)$ and $\mathbf{h}(s)$ denote the linear and angular distortions of the beam, respectively, then the following relations are valid:

$$\mathbf{g}(s) = \frac{d\mathbf{u}(s)}{ds} - \boldsymbol{\theta}(s) \times \mathbf{i}, \tag{1.5}$$

$$\mathbf{h}(s) = \frac{d\boldsymbol{\theta}(s)}{ds}. \tag{1.6}$$

For elastic beams it is assumed that these distortions can be represented as the products of the corresponding stress resultants and certain dyadics $\mathbf{A}$ and $\mathbf{B}$ which depend on the shape of the cross section and on the elastic properties of the material of the beam. This is analogous to the usual stress-strain law in elasticity. More precisely we assume

$$\mathbf{g}(s) = \mathbf{A}\cdot\mathbf{R}(s), \tag{1.7}$$

$$\mathbf{h}(s) = \mathbf{B}\cdot\mathbf{M}(s). \tag{1.8}$$

Moreover, since the $\mathbf{j}$ and $\mathbf{k}$ axes coincide with the principal axes of inertia of the beam, the dyadics $\mathbf{A}$ and $\mathbf{B}$ are in diagonal form. If we write

$$\mathbf{B} = \begin{bmatrix} \beta_0 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \end{bmatrix},$$

the entries β_1 and β_2 are equal to $1/EI_1$ and $1/EI_2$ where E denotes Young's modulus and I_1 and I_2 denote the moments of inertia of the cross section about the vectors **j** and **k**. The coefficient β_0 is the reciprocal of the torsional rigidity of the beam.

Throughout this work it will be assumed that the bending due to shear forces can be neglected. This is a usual assumption in beam theory, and has been discussed by S. Timoshenko for an untwisted beam [2]. Hence the linear distortions $\mathbf{g}(s)$ can be neglected; and consequently it will not be necessary to consider α_0 , α_1 , α_2 . These effects can, in principle, be included in the work to be described; but the resulting system of equations, which would include θ_1 and θ_2 as dependent variables, would be considerably more complicated.

2. Equations of motion. If the beam is assumed to undergo small transverse vibrations only, then the longitudinal motion can be neglected. Using the subscripts 0, 1, 2 to denote components taken in the direction of **i**, **j**, **k** respectively, the previous assumption implies that $u_0 = 0$. Since there are no torsional oscillations, $\theta_0 = 0$. Furthermore, it is customary to neglect the effect of rotational inertia; hence $\mathbf{c} = 0$. Again we point out that this approximation can be removed if desired.

Equations (1.3) and (1.4) can be written as

$$\frac{\partial \mathbf{R}}{\partial s} + \mathbf{f} = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{M}}{\partial s} + \mathbf{i} \times \mathbf{R} = 0, \tag{2.2}$$

where we write **R** for $\mathbf{R}(s)$, **M** for $\mathbf{M}(s)$, and similarly for the other vector quantities. Since we are neglecting the distortion due to the shear forces, Eq. (1.5) reduces to

$$\frac{\partial \mathbf{u}}{\partial s} = \boldsymbol{\theta} \times \mathbf{i}; \tag{2.3}$$

and, combining Eqs. (1.6) and (1.8), we have

$$\frac{\partial \boldsymbol{\theta}}{\partial s} = \mathbf{B}\cdot\mathbf{M}. \tag{2.4}$$

Eliminating **R** between Eqs. (2.1) and (2.2) we obtain

$$\frac{\partial^2 \mathbf{M}}{\partial s^2} - \mathbf{i} \times \mathbf{f} = 0. \tag{2.5}$$

Also differentiating Eq. (2.3) with respect to s and substituting $\mathbf{B}\cdot\mathbf{M}$ for $\partial\boldsymbol{\theta}/\partial s$ we have

$$\frac{\partial^2 \mathbf{u}}{\partial s^2} = (\mathbf{B}\cdot\mathbf{M}) \times \mathbf{i},$$

from which **M** can be written as

$$\mathbf{M} = \mathbf{B}^{-1}\cdot\left\{\mathbf{i} \times \frac{\partial^2 \mathbf{u}}{\partial s^2}\right\}. \tag{2.6}$$

It should be noted that $\mathbf{B}^{-1}$ is now a two dimensional dyadic. For the case of pure transverse vibrations the distributed load or reversed effective force is

$$\mathbf{f} = -mA\,\frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{2.7}$$

where m is the mass per unit volume and A is the area of the cross section. Combining all of these results, we obtain the following equation for the displacement vector $\mathbf{u}(s, t)$:

$$\frac{\partial^2}{\partial s^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2}\right\} + mA\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial t^2} = 0. \tag{2.8}$$

We shall consider a beam which is elastically constrained at the end $s = 0$, and is free at the end $s = l$. The boundary conditions at the constrained end ($s = 0$) may be expressed in the following way:

$$\left.\begin{aligned} \mathbf{i}\times\mathbf{u} &= 0, \\ \frac{\partial}{\partial s}\{\mathbf{i}\times\mathbf{u}\} - \boldsymbol{\varepsilon}\cdot\mathbf{M} &= 0, \end{aligned}\right\} \tag{2.9}$$

where $\boldsymbol{\varepsilon}$ is a dyadic with positive entries,

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}.$$

The first of Eqs. (2.9) simply states that $u_1 = u_2 = 0$ at $s = 0$. The second, when written out in component form, yields

$$\frac{\partial u_1}{\partial s} - \epsilon_2 M_2 = 0,$$

or

$$\frac{\partial u_1}{\partial s} - \epsilon_2 EI_2\left\{\frac{\partial^2 u_1}{\partial s^2} - 2\tau\frac{\partial u_2}{\partial s} - \frac{\partial\tau}{\partial s}u_2 - \tau^2 u_1\right\} = 0,$$

and

$$\frac{\partial u_2}{\partial s} + \epsilon_1 M_1 = 0,$$

or

$$\frac{\partial u_2}{\partial s} - \epsilon_1 EI_1\left\{\frac{\partial^2 u_2}{\partial s^2} + 2\tau\frac{\partial u_1}{\partial s} + \frac{\partial\tau}{\partial s}u_1 - \tau^2 u_1\right\} = 0.$$

Physically, this boundary condition implies that the angle of deflection about the $\mathbf{j}$ axis is proportional to the moment about that axis, and similarly for the $\mathbf{k}$ axis. If $\epsilon_1 = 0$, then $\partial u_2/\partial s = 0$ and hence there is perfect clamping about the $\mathbf{j}$ axis. If $\epsilon_1 = \infty$, the beam is simply supported about the $\mathbf{j}$ axis. Similar statements hold concerning ϵ_2. Various combinations of ϵ_1 and ϵ_2 may be taken.

The conditions at $s = l$ are the usual conditions that there be no moment or force transmitted across the free end of the bar. These may be written as

$$\mathbf{M} = 0, \qquad \mathbf{R} = 0,$$

at $s = l$, or

$$\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2} = 0, \qquad \mathbf{i}\times\frac{\partial}{\partial s}\left\{\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2}\right\} = 0.$$

If $\mathbf{u}(s, t)$ is assumed to be of the form

$$\mathbf{u}(s, t) = \mathbf{v}(s)e^{i\lambda t},$$

Eq. (2.9) becomes

$$\frac{d^2}{ds^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right\} - mA\lambda^2\mathbf{i} \times \mathbf{v} = 0, \tag{2.10}$$

with the boundary conditions

$$\mathbf{i} \times \mathbf{v} = 0, \qquad \frac{d}{ds}(\mathbf{i} \times \mathbf{v}) - \boldsymbol{\varepsilon}\cdot\mathbf{B}^{-1}\cdot\frac{d^2}{ds^2}(\mathbf{i} \times \mathbf{v}) = 0 \tag{2.11}$$

at $s = 0$, and

$$\mathbf{B}^{-1}\cdot\frac{d^2}{ds^2}(\mathbf{i} \times \mathbf{v}) = 0, \qquad \frac{d}{ds}\left\{\mathbf{B}^{-1}\cdot\frac{d^2}{ds^2}(\mathbf{i} \times \mathbf{v})\right\} = 0 \tag{2.12}$$

at $s = l$. It will be shown later that Eqs. (2.12) also appear as natural boundary conditions arising from the corresponding variational principle.

In order to obtain a similarity principle, we introduce the dimensionless variables

$$x = s/l, \qquad \mathbf{w} = \mathbf{v}/l, \qquad \delta = \tau l.$$

Then Eq. (2.10) becomes

$$\frac{d^2}{dx^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{w}}{dx^2}\right\} - mAl^4\lambda^2\mathbf{i} \times \mathbf{w} = 0, \tag{2.13}$$

with the boundary conditions

$$\mathbf{i} \times \mathbf{w} = 0, \qquad \frac{d}{dx}(\mathbf{i} \times \mathbf{w}) - \boldsymbol{\varepsilon}'\cdot\mathbf{B}^{-1}\cdot\frac{d^2}{dx^2}(\mathbf{i} \times \mathbf{w}) = 0 \tag{2.14}$$

at $x = 0$, where $\boldsymbol{\varepsilon}' = \epsilon/l$; and

$$\mathbf{B}^{-1}\cdot\frac{d^2}{dx^2}(\mathbf{i} \times \mathbf{w}) = 0, \qquad \frac{d}{dx}\left\{\mathbf{B}^{-1}\cdot\frac{d^2}{dx^2}(\mathbf{i} \times \mathbf{w})\right\} = 0 \tag{2.15}$$

at $x = 1$. If the dimensions of the cross section of a cantilevered beam $[\boldsymbol{\varepsilon}' = 0]$ are changed by a scale factor a, then the area A', and moments of inertia I_1', I_2' of the new beam C' are related to the old ones by $A' = a^2A$, $I_1' = a^4I_1$, $I_2' = a^4I_2$, $l' = al$. If, in addition, $E' = bE$, and $m' = cm$ then we have for the beam C'

$$\frac{d^2}{dx^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{w}}{dx^2}\right\} - \left\{\frac{a^2c}{b}\right\}mAl^4\lambda'^2\mathbf{i} \times \mathbf{w} = 0,$$

and the homogeneous boundary conditions remain the same. From this we see that if λ is the natural frequency of the original beam C then

$$\frac{1}{a}\sqrt{\frac{b}{c}}\,\lambda$$

is the natural frequency of C'. This is exactly the same law of similarity that could be used for an untwisted cantilevered beam.

The close analogy between the theory presented here and the usual theory of untwisted beams can be brought out more clearly if we set

$$\mathbf{y} = \mathbf{i} \times \mathbf{w}.$$

Then Eqs. (2.13), (2.14) and (2.15) become

$$\frac{d^2}{dx^2} \mathbf{B}^{-1} \cdot \frac{d^2\mathbf{y}}{dx^2} - mAl^4\lambda^2\mathbf{y} = 0,$$

with

$$\mathbf{y} = 0,$$

$$\frac{d\mathbf{y}}{dx} - \boldsymbol{\varepsilon}' \cdot \mathbf{B}^{-1} \cdot \frac{d^2\mathbf{y}}{dx^2} = 0$$

at $x = 0$, and at $x = 1$

$$\mathbf{B}^{-1} \cdot \frac{d^2\mathbf{y}}{dx^2} = 0, \qquad \frac{d}{dx}\left(\mathbf{B}^{-1} \cdot \frac{d^2\mathbf{y}}{dx^2}\right) = 0.$$

It must be pointed out that, although these equations appear to be simply generalizations of the differential equation and boundary conditions of an untwisted beam, the effect of the twist is actually hidden in the differentiation with respect to x. The two scalar equations for w_1 and w_2 corresponding to the vector Eq. (2.13) are

$$\left.\begin{aligned} \frac{d^2}{dx^2}(EI_2p) - 2\delta\frac{d}{dx}(EI_1q) - \frac{d\delta}{dx}(EI_1q) - \delta^2(EI_2)p - mAl^4\lambda^2w_1 &= 0, \\ \frac{d^2}{dx^2}(EI_1q) + 2\delta\frac{d}{dx}(EI_2p) + \frac{d\delta}{dx}(EI_2p) - \delta^2(EI_1q) - mAl^4\lambda^2w_2 &= 0, \end{aligned}\right\} \tag{2.16}$$

where

$$p = \frac{d^2w_1}{dx^2} - 2\delta\frac{dw_2}{dx} - \frac{d\delta}{dx}w_2 - \delta^2w_1,$$

$$q = \frac{d^2w_2}{dx^2} + 2\delta\frac{dw_1}{dx} + \frac{d\delta}{dx}w_1 - \delta^2w_2.$$

It is interesting to note that for a uniform beam of constant twist Eqs. (2.16) reduce to two simultaneous equations with constant coefficients. They are

$$EI_2\frac{d^4w_1}{dx^4} - 2\delta^2(EI_2 + 2EI_1)\frac{d^2w_1}{dx^2} + EI_2\delta^4w_1 - 2\delta(EI_2 + EI_1)\frac{d^3w_2}{dx^3} + 2\delta^3(EI_2 + EI_1)\frac{dw_2}{dx} = mAl^4\lambda^2w_1,$$

$$EI_1\frac{d^4w_2}{dx^4} - 2\delta^2(EI_1 + 2EI_2)\frac{d^2w_2}{dx^2} + EI_1\delta^4w_2 + 2\delta(EI_1 + EI_2)\frac{d^3w_1}{dx^3} - 2\delta^3(EI_2 + EI_1)\frac{dw_1}{dx} = mAl^4\lambda^2w_2.$$

In principle these equations can be solved exactly.

3. Variational principles. It will be shown that the eigenvalues λ^2 and the corresponding eigenvectors $\mathbf{v}$ of Eqs. (2.10), (2.11) and (2.12) may be obtained from a minimum principle. Essentially we are looking for a statement of the Rayleigh type which relates the square of the natural frequency to the ratio of the potential and kinetic energy. The potential energy will consist of two parts; the first term is the usual energy due to bending and to this must be added the contribution of the elastic constraint at the clamped end. A little investigation will show that the following vector functional plays the role of the potential energy

$$D(\mathbf{v}) = \int_0^l \left(\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\cdot\mathbf{B}^{-1}\cdot\left(\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right) ds + \left[\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\cdot\boldsymbol{\varepsilon}\cdot\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\right]_0, \tag{3.1}$$

and the functional

$$H(\mathbf{v}) = \int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \mathbf{v})\, ds, \tag{3.2}$$

the role of the kinetic energy. The subscript 0 denotes the value of the quantity in brackets at $s = 0$.

In addition, we introduce the mixed forms

$$D(\mathbf{v}, \mathbf{u}) = \int_0^l \left(\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\cdot\mathbf{B}^{-1}\cdot\left(\mathbf{i} \times \frac{d^2\mathbf{u}}{ds^2}\right) ds + \left[\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\cdot\boldsymbol{\varepsilon}\cdot\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{u}}{ds^2}\right)\right]_0 \tag{3.1a}$$

and

$$H(\mathbf{v}, \mathbf{u}) = \int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \mathbf{u})\, ds. \tag{3.2a}$$

Since $\mathbf{B}^{-1}$ and $\boldsymbol{\varepsilon}$ are symmetric, positive dyadics it follows immediately that $D(\mathbf{v}, \mathbf{u})$ and $H(\mathbf{v}, \mathbf{u})$ are symmetric functionals. Also $D(\mathbf{v})$ and $H(\mathbf{v})$ are positive for real vectors $\mathbf{v}$. It can be readily seen that

$$D(\mathbf{v} + \mathbf{u}) = D(\mathbf{v}) + 2D(\mathbf{v}, \mathbf{u}) + D(\mathbf{u}), \tag{3.1b}$$

and

$$H(\mathbf{v} + \mathbf{u}) = H(\mathbf{v}) + 2H(\mathbf{v}, \mathbf{u}) + H(\mathbf{u}). \tag{3.2b}$$

The vector $\mathbf{v}(s)$ will be called an admissible vector if it is continuous and has continuous derivatives of at least the fourth order, and also satisfies the constraining boundary conditions Eqs. (2.11). The lowest eigenvalue of Eq. (2.10) subject to the boundary conditions Eqs. (2.12) can then be found from the following minimum principle.

Of all the admissible vectors $\mathbf{v}(s)$, that one which minimizes $D(\mathbf{v})$ under the side condition $H(\mathbf{v}) = 1$ is an eigenvector of the differential equation (2.10) subject to the boundary conditions (2.12). The minimum value of $D(\mathbf{v})$ is the corresponding eigenvalue.

It should be noted that this principle not only characterizes the lowest eigenvalue, but also states that the minimizing vector will satisfy automatically the free end conditions (2.12) as natural boundary conditions. The proof of this minimum principle will follow as a special case of the general minimum principle for the nth eigenvalue, which will be stated and proved later in this section.

An orthogonality relationship between the eigenvectors of Eq. (2.10) from which the reality of the eigenvalues follows, can be established by means of a standard technique.

Let $\mathbf{v}(s)$ and $\mathbf{u}(s)$ be two eigenfunctions of Eq. (2.10), and λ^2 and μ^2, where $\lambda^2 \neq \mu^2$, the respective eigenvalues. Then

$$\frac{d^2}{ds^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right\} - mA\lambda^2\mathbf{i} \times \mathbf{v} = 0,$$

$$\frac{d^2}{ds^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{u}}{ds^2}\right\} - mA\mu^2\mathbf{i} \times \mathbf{u} = 0.$$

Multiplying the first equation by $\mathbf{i} \times \mathbf{u}$, the second by $\mathbf{i} \times \mathbf{v}$, subtracting and integrating from 0 to l yields

$$\int_0^l \left\{\frac{d^2}{ds^2}\left[\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right]\cdot(\mathbf{i} \times \mathbf{u}) - \frac{d^2}{ds^2}\left[\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{u}}{ds^2}\right]\cdot(\mathbf{i} \times \mathbf{v})\right\} ds$$
$$- (\lambda^2 - \mu^2) \int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \mathbf{u})\, ds = 0.$$

The first integral is seen to be zero, if it is integrated twice by parts; thus

$$(\lambda^2 - \mu^2)H(\mathbf{v}, \mathbf{u}) = 0.$$

Since $\lambda^2 \neq \mu^2$ the following orthogonality relation holds:

$$H(\mathbf{v}, \mathbf{u}) = 0. \tag{3.3}$$

This may also be written as

$$D(\mathbf{v}, \mathbf{u}) = 0, \tag{3.4}$$

which follows from the fact that

$$\int_0^l \left[\frac{d^2}{ds^2}\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\right]\cdot(\mathbf{i} \times \mathbf{u})\, ds = \lambda^2 \int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \mathbf{u})\, ds.$$

The right hand side is zero, and integrating the left hand side twice by parts and using the boundary conditions, Eqs. (2.11) and (2.12) gives

$$\int_0^l \left(\mathbf{i} \times \frac{d^2\mathbf{u}}{ds^2}\right)\cdot\mathbf{B}^{-1}\cdot\left(\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right) ds + \left[\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{u}}{ds^2}\right)\cdot\boldsymbol{\varepsilon}\cdot\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\right]_0 = 0$$

which is the desired result.

A computation completely parallel to that just given also shows that if $\mathbf{v}$ is an eigenvector of the differential equation and λ^2 is the corresponding eigenvalue, then $\lambda^2 = D(\mathbf{v})/H(\mathbf{v})$. Combining this result with the minimum principle, we see that the minimum principle actually characterizes the lowest eigenvalue.

Now assume that λ^2 is complex; that is, $\lambda^2 = \sigma + i\rho$, $[i = \sqrt{-1}]$, where σ and ρ are real, and $\rho \neq 0$. Corresponding to λ^2 there will be a complex eigenvector $\mathbf{v} = \mathbf{r} + i\mathbf{t}$ where $\mathbf{r}$ and $\mathbf{t}$ are real vector functions and $\mathbf{t} \not\equiv 0$. Since Eqs. (2.10), (2.11) and (2.12) are linear with real coefficients, it follows that $\overline{\lambda^2} = \sigma - i\rho$ is also an eigenvalue with corresponding eigenvector $\bar{\mathbf{v}} = \mathbf{r} - i\mathbf{t}$. If $\overline{\lambda^2} \neq \lambda^2$, then the orthogonality condition implies

$$\int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \bar{\mathbf{v}})\, ds = 0.$$

Now

$$\int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \mathbf{v})\, ds = \int_0^l mA(-\bar{v}_2\mathbf{j} + \bar{v}_1\mathbf{k})\cdot(-v_2\mathbf{j} + v_1\mathbf{k})\, ds$$

$$= \int_0^l mA\{|v_1|^2 + |v_2|^2\}\, ds.$$

This integral cannot vanish, consequently

$$\bar{\lambda}^2 - \lambda^2 = 0 \tag{3.5}$$

which implies

$$2i\rho = 0$$

or that $\rho = 0$. Thus the eigenvalues are real and positive.

The general variational principle may be stated as follows. Of all the admissible vectors $\mathbf{v}$, that one which minimizes $D(\mathbf{v})$ under the side condition

$$H(\mathbf{v}) = 1$$

and

$$H(\mathbf{v}, \mathbf{u}^i) = 0, \qquad i = 1, 2, \cdots, n - 1,$$

where $\mathbf{u}^i$ is the ith eigenvector of Eq. (2.10), is an eigenvector of the differential equation (2.10) subject to the boundary conditions (2.12). The minimum value of $D(\mathbf{v})$ under these side conditions on $\mathbf{v}$ is then the corresponding eigenvalue λ_n^2 and $\lambda_n^2 \geqq \lambda_{n-1}^2 \geqq \cdots \geqq \lambda_1^2$.

In order to prove this statement, let $\mathbf{v}$ be the minimizing vector and λ_n^2 the corresponding minimum value. Then

$$D(\mathbf{v}) = \lambda_n^2 H(\mathbf{v}).$$

Let r be an arbitrary admissible vector that satisfies the condition $H(\mathbf{u}^i, \mathbf{r}) = 0$ for $i = 1, 2, \cdots, n - 1$. Let a be an arbitrary parameter. Then since $\mathbf{v}$ minimizes $D(\mathbf{v})/H(\mathbf{v})$ it follows that

$$D(\mathbf{v} + a\mathbf{r}) \geqq \lambda_n^2 H(\mathbf{v} + a\mathbf{r}).$$

This can be written as

$$\{D(\mathbf{v}) - \lambda_n^2 H(\mathbf{v})\} + 2a\{D(\mathbf{v}, \mathbf{r}) - \lambda_n^2 H(\mathbf{v}, \mathbf{r})\} + a^2\{D(\mathbf{r}) - \lambda_n^2 H(\mathbf{r})\} \geqq 0.$$

The fact that this expression assumes a minimum value for $a = 0$ implies that

$$D(\mathbf{v}, \mathbf{r}) - \lambda_n^2 H(\mathbf{v}, \mathbf{r}) = 0. \tag{3.6}$$

Now $\mathbf{r}$ may be chosen as

$$\mathbf{r} = \mathbf{t} + \sum_{i=1}^{n-1} a_i \mathbf{u}^i,$$

where $\mathbf{t}$ is an arbitrary admissible vector and the a_i are determined by the condition that

$$H(\mathbf{r}, \mathbf{u}^j) = H(\mathbf{t}, \mathbf{u}^j) + a_j H(\mathbf{u}^j) = 0,$$

which yields

$$a_i = -H(\mathbf{t}, \mathbf{u}^i).$$

Substituting for $\mathbf{r}$ in Eq. (3.6) we have

$$D(\mathbf{v}, \mathbf{t}) - \lambda_n^2 H(\mathbf{v}, \mathbf{t}) + \sum_{i=1}^{n-1} a_i[D(\mathbf{v}, \mathbf{u}^i) - \lambda_n^2 H(\mathbf{v}, \mathbf{u}^i)] = 0.$$

By hypothesis $H(\mathbf{v}, \mathbf{u}^i) = 0$; and following the argument used when $\mathbf{v}$ was an eigenvector we can show that $D(\mathbf{v}, \mathbf{u}^i) = 0$. Consequently

$$D(\mathbf{v}, \mathbf{t}) - \lambda_n^2 H(\mathbf{v}, \mathbf{t}) = 0$$

where $\mathbf{t}$ is an arbitrary admissible vector. This may be written as

$$\int_0^l \left(\mathbf{i} \times \frac{d^2\mathbf{t}}{ds^2}\right)\cdot\mathbf{B}^{-1}\cdot\left(\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right) ds + \left[\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{t}}{ds^2}\right)\cdot\boldsymbol{\varepsilon}\cdot\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\right]_0 - \lambda_n^2 \int_0^l mA(\mathbf{i} \times \mathbf{v})\cdot(\mathbf{i} \times \mathbf{t})\, ds = 0.$$

Integration twice by parts yields

$$\int_0^l \left\{\frac{d^2}{ds^2}\left[\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right] - mA\lambda_n^2(\mathbf{i} \times \mathbf{v})\right\}\cdot(\mathbf{i} \times \mathbf{t})\, ds + \left[\left(\mathbf{i} \times \frac{d\mathbf{t}}{ds}\right)\cdot\mathbf{B}^{-1}\cdot\left(\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\right]_{s=l} - \left[(\mathbf{i} \times \mathbf{t})\cdot\frac{d}{ds}\left(\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right)\right]_{s=l} = 0.$$

Since the vector $\mathbf{t}$ is chosen arbitrarily in $0 < s < l$ and also at $s = l$, it follows immediately that

$$\frac{d^2}{ds^2}\left[\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right] - mA\lambda_n^2\mathbf{i} \times \mathbf{v} = 0,$$

and at $s = l$,

$$\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2} = 0, \qquad \frac{d}{ds}\left\{\mathbf{B}^{-1}\cdot\mathbf{i} \times \frac{d^2\mathbf{v}}{ds^2}\right\} = 0.$$

These are identical with the differential equation (2.10) and the boundary conditions (2.12). Furthermore $\lambda_n^2 \geqq \lambda_{n-1}^2 \geqq \cdots \geqq \lambda_1^2$, for the class of functions used in minimizing $D(\mathbf{v})$ for λ_n^2 is more restrictive than any previous class. Also it can be readily shown that λ_n^2 is actually the nth eigenvalue in order of increasing magnitude. For if μ^2 is an eigenvalue different from $\lambda_1^2, \cdots, \lambda_{n-1}^2$ and $\mathbf{t}$ is the corresponding eigenvector then

$$\mu^2 = \frac{D(\mathbf{t})}{H(\mathbf{t})}$$

and $H(\mathbf{t}, \mathbf{u}^i) = 0$ for $i = 1, \cdots n - 1$. Thus $\mathbf{t}$ is an admissible vector in the minimizing problem considered. Therefore

$$\frac{D(\mathbf{t})}{H(\mathbf{t})} \geqq \frac{D(\mathbf{v})}{H(\mathbf{v})}$$

or

$$\mu^2 \geqq \lambda_n^2 .$$

Consequently λ_n^2 is the nth eigenvalue.

Finally $D(\mathbf{v})$ and $H(\mathbf{v})$ may be expressed in scalar form as

$$D(\mathbf{v}) = \int_0^l (EI_1Q^2 + EI_2P^2)\, ds + [\epsilon_1(EI_1Q)^2 + \epsilon_2(EI_2P)^2]_0 , \tag{3.7}$$

$$H(\mathbf{v}) = \int_0^l mA(v_1^2 + v_2^2)\, ds, \tag{3.8}$$

where

$$\left.\begin{aligned} P &= \frac{d^2v_1}{ds^2} - 2\tau\frac{dv_2}{ds} - \frac{d\tau}{ds}v_2 - \tau^2v_1 \\ Q &= \frac{d^2v_2}{ds^2} + 2\tau\frac{dv_1}{ds} + \frac{d\tau}{ds}v_1 - \tau^2v_2 . \end{aligned}\right\} \tag{3.9}$$

The differential equation (2.8) and the energy principle derived in this section can be extended to include torsional vibrations about the **i** direction coupled with transverse vibrations. If the axis of twist and the axis of centroids do not coincide we obtain coupled transverse-torsional oscillations. If they do coincide, the differential equations of motion uncouple and we have torsional oscillations superimposed on the transverse vibrations.

4. Rotating twisted beam. The vector differential equation derived in Section 2 for transverse vibrations will now be extended to a twisted beam which is rotating. The bending moment $\mathbf{M}_E(s)$ due to the internal forces, which might be called the elastic moment, must be equal to the moment due to the external forces. This moment is the sum of three moments; $\mathbf{M}_I(s)$, the moment due to the inertial forces relative to a rotating frame; $\mathbf{M}_T(s)$, the moment due to rotation; and $\mathbf{M}_C(s)$, the moment due to the Coriolis forces.

In order to compute these moments the right handed coordinate system X_0, X_1, X_2, with unit vectors **I**, **J**, **K** is introduced (Fig. 2). The **I** axis coincides with the **i** axis. The

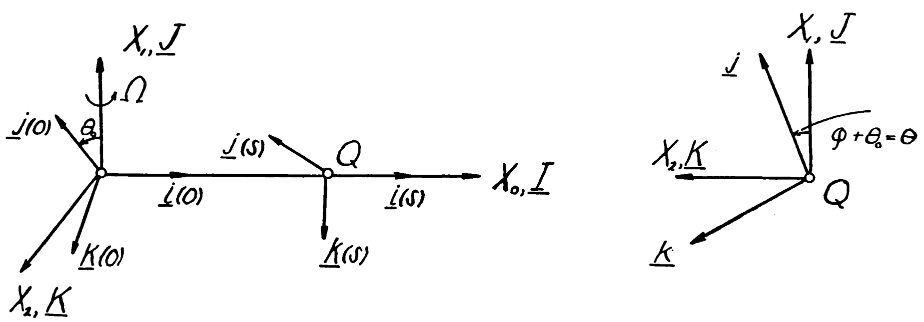

FIG. 2

beam is assumed to rotate with constant angular velocity Ω about the **J** axis. If θ is measured as indicated we have

$$\tau\mathbf{i} = \frac{d\theta}{ds}\,\mathbf{i},$$

or

$$\theta(s) = \int_0^s \tau \, ds + \theta_0 .$$

When the beam vibrates the centroid of the cross section Q moves to a new point Q'. The position vector $\mathbf{R}(s)$ of Q' may be written as

$$\begin{aligned} \mathbf{R} &= s\mathbf{I} + U_1(s)\mathbf{J} + U_2(s)\mathbf{K}, \\ &= s\mathbf{i} + u_1(s)\mathbf{j} + u_2(s)\mathbf{k}, \end{aligned}$$

where the transformations from one coordinate system to the other are

$$\left.\begin{aligned} U_1 &= u_1 \cos\theta - u_2 \sin\theta & u_1 &= U_1 \cos\theta + U_2 \sin\theta \\ U_2 &= u_1 \sin\theta + u_2 \cos\theta & u_2 &= -U_1 \sin\theta + U_2 \cos\theta \\ \mathbf{J} &= \cos\theta\mathbf{j} - \sin\theta\mathbf{k} & \mathbf{j} &= \cos\theta\mathbf{J} + \sin\theta\mathbf{K} \\ \mathbf{K} &= \sin\theta\mathbf{j} + \cos\theta\mathbf{k} & \mathbf{k} &= -\sin\theta\mathbf{J} + \cos\theta\mathbf{K} \end{aligned}\right\} \quad (4.1)$$

If $\mathbf{a}$ is the total acceleration of the centroid Q, it is easy to show [3] that

$$\begin{aligned} \mathbf{a}_I &= \frac{\partial^2 U_1}{\partial t^2}\mathbf{J} + \frac{\partial^2 U_2}{\partial t^2}\mathbf{K}, \\ \mathbf{a}_T &= -\Omega^2\{s\mathbf{I} + U_2(s)\mathbf{K}\}, \\ \mathbf{a}_C &= 2\Omega\frac{\partial U_2}{\partial t}\mathbf{I} \end{aligned}$$

Multiplying $\mathbf{a}$ by $-mA$ gives the effective force intensity at the cross section, and hence the moment at the cross section s due to the action of an infinitesimal element of the beam at ξ, $\xi > s$, is

$$-mA[\mathbf{R}(\xi) - \mathbf{R}(s)]\, d\xi \times \mathbf{a}.$$

If second order terms are neglected, that is, products of the U's and their derivatives, the $\mathbf{I}$ component of $\mathbf{M}_I(s)$ and the Coriolis moment, $\mathbf{M}_C(s)$, must be neglected [4]. Consequently we set

$$\mathbf{M}_E(s) = \mathbf{M}_I(s) + \mathbf{M}_T(s) \quad (4.2)$$

where

$$\left.\begin{aligned} \mathbf{M}_I(s) &= \int_s^l mA(\xi)(\xi - s)\frac{\partial^2 U_2(\xi)}{\partial t^2}\, d\xi\, \mathbf{J} - \int_s^l mA(\xi)(\xi - s)\frac{\partial^2 U_1(\xi)}{\partial t^2}\, d\xi\, \mathbf{K} \\ \mathbf{M}_T(s) &= \Omega^2 \int_s^l mA(\xi)\{\xi[U_2(\xi) - U_2(s)] - (\xi - s)U_2(\xi)\}\, d\xi\, \mathbf{J} \\ &\qquad - \Omega^2 \int_s^l mA(\xi)\xi[U_1(\xi) - U_1(s)]\, d\xi\, \mathbf{K}. \end{aligned}\right\} \quad (4.3)$$

From Section 2 we have

$$\mathbf{M}_E(s) = \mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2}.$$

Noticing that $d\mathbf{J}/ds = d\mathbf{K}/ds = 0$, and differentiating Eq. (4.2) twice with respect to s we obtain

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2}\right\} &= \Omega^2\left\{-\frac{\partial}{\partial s}\left[\frac{\partial U_2}{\partial s}L(s)\right] - mA(s)U_2(s)\right\}\mathbf{J} \\ &+ \Omega^2\frac{\partial}{\partial s}\left[\frac{\partial U_1}{\partial s}L(s)\right]\mathbf{K} + mA(s)\frac{\partial^2 U_2(s)}{\partial t^2}\mathbf{J} - mA(s)\frac{\partial^2 U_1(s)}{\partial t^2}\mathbf{K},\end{aligned} \tag{4.4}$$

where

$$L(s) = \int_s^l mA(\xi)\xi\, d\xi.$$

Either the capital U's and $\mathbf{J}$, $\mathbf{K}$ can be transformed into the small u's and $\mathbf{j}$ and $\mathbf{k}$ or vice versa. Doing the former we obtain as our equation of motion

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\left\{\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2}\right\} &- \Omega^2\frac{\partial}{\partial s}\left\{L(s)\mathbf{i}\times\frac{\partial\mathbf{u}}{\partial s}\right\} \\ &- mA(s)\Omega^2\boldsymbol{\Theta}\cdot(\mathbf{i}\times\mathbf{u}) + mA(s)\mathbf{i}\times\mathbf{u} = 0,\end{aligned} \tag{4.5}$$

where the dyadic $\boldsymbol{\Theta}(s)$ is defined as

$$\boldsymbol{\Theta}(s) = \begin{pmatrix}\cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}.$$

If the beam is cantilevered about both axes at $s = 0$ the constraining boundary conditions are simply

$$\mathbf{i}\times\mathbf{u} = 0, \qquad \mathbf{i}\times\frac{\partial\mathbf{u}}{\partial s} = 0; \tag{4.6}$$

and, since the moment and its derivative due to the rotation vanish at $s = l$, the requirement that there be no shear and no moment transmitted across the free end requires

$$\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2} = 0, \qquad \frac{\partial}{\partial s}\left\{\mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{\partial^2\mathbf{u}}{\partial s^2}\right\} = 0. \tag{4.7}$$

Assuming a solution of the form

$$\begin{aligned}\mathbf{u}(s, t) &= \mathbf{v}(s)e^{i\lambda t} \\ &= [v_1(s)\mathbf{j} + v_2(s)\mathbf{k}]e^{i\lambda t} \\ &= [V_1(s)\mathbf{J} + V_2(s)\mathbf{K}]e^{i\lambda t}\end{aligned}$$

and substituting

$$\mathbf{w}(s) = \mathbf{i}\times\mathbf{v},$$

we obtain the following differential equation

$$\frac{d^2}{ds^2}\left\{\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{w}}{ds^2}\right\} - \Omega^2\frac{d}{ds}\left\{L(s)\frac{d\mathbf{w}}{ds}\right\} - mA(s)\Omega^2\mathbf{\Theta}\cdot\mathbf{w} - mA(s)\lambda^2\mathbf{w} = 0. \tag{4.8}$$

The corresponding scalar differential equations for the displacements v_1 and v_2 are

$$\frac{d^2}{ds^2}(EI_2P) - 2\tau\frac{d}{ds}(EI_1Q) - \frac{d\tau}{ds}(EI_1Q) - \tau^2(EI_2P)$$

$$- \Omega^2\left\{L(s)P(s) - mA(s)\left[\left(\frac{dv_1}{ds} - \tau v_2\right)s - v_1\sin^2\theta - v_2\sin\theta\cos\theta\right]\right\} - mA(s)\lambda^2 v_1 = 0,$$

$$\frac{d^2}{ds^2}(EI_1Q) + 2\tau\frac{d}{ds}(EI_2P) + \frac{d\tau}{ds}(EI_2P) - \tau^2(EI_1Q)$$

$$- \Omega^2\left\{L(s)Q(s) - mA(s)\left[\left(\frac{dv_2}{ds} + \tau v_1\right)s - v_1\sin\theta\cos\theta - v_2\cos^2\theta\right]\right\} - mA(s)\lambda^2 v_2 = 0,$$

where P and Q are defined by Eq. (3.9).

The boundary conditions are

$$\mathbf{w} = 0, \qquad \frac{d\mathbf{w}}{ds} = 0 \tag{4.9}$$

at $s = 0$, and at $s = l$

$$\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{w}}{ds^2} = 0, \qquad \frac{d}{ds}\left(\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{w}}{ds^2}\right) = 0. \tag{4.10}$$

The variational principle stated in Section 3 may be generalized to give a minimum principle which yields the eigenvalues and corresponding eigenfunctions of Eqs. (4.8), (4.9), and (4.10). The functional $D''(\mathbf{w}) = D''(\mathbf{i}\times\mathbf{v})$ which plays the role of the potential energy is

$$D''(\mathbf{i}\times\mathbf{v}) = \int_0^l \left(\mathbf{i}\times\frac{d^2\mathbf{v}}{ds^2}\right)\cdot\{\mathbf{M}''_E(s) - \mathbf{M}''_T(s)\}\,ds, \tag{4.11}$$

where

$$\mathbf{M}''_E(s) = \mathbf{B}^{-1}\cdot\mathbf{i}\times\frac{d^2\mathbf{v}}{ds^2},$$

and

$$M''_T(s) = \Omega^2\int_s^l mA(\xi)\{\xi[V_2(\xi) - V_2(s)] - (\xi - s)V_2(\xi)\}\,d\xi\,\mathbf{J}$$

$$- \Omega^2\int_s^l mA(\xi)\xi[V_1(\xi) - V_1(s)]\,d\xi\,\mathbf{K}$$

$$= -\Omega^2 L(s)\mathbf{i}\times\mathbf{v}(s) + \Omega^2\int_s^l mA(\xi)(\mathbf{i}\times\mathbf{v})\,d\xi$$

$$- \Omega^2\int_s^l mA(\xi)(\xi - s)V_2(s)\,d\xi\,\mathbf{J}.$$

Substituting these expressions in Eq. (4.11) and integrating by parts in the proper manner we obtain

$$D''(\mathbf{w}) = \int_0^l \left\{\frac{d^2\mathbf{w}}{ds^2}\cdot\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{w}}{ds^2} + \Omega^2L(s)\,\frac{d\mathbf{w}}{ds}\cdot\frac{d\mathbf{w}}{ds} - mA(s)\,\Omega^2\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{w}\right\} ds, \tag{4.12}$$

or in scalar form

$$D''(\mathbf{i}\times\mathbf{v}) = \int_0^l \left\{EI_1Q^2 + EI_2P^2 + \Omega^2L(s)\left[\left(\frac{dv_2}{ds} + \tau v_1\right)^2 + \left(\frac{dv_1}{ds} - \tau v_2\right)^2\right] - mA(s)(v_1\sin\theta + v_2\cos\theta)^2\right\} ds.$$

The functional $H''(\mathbf{w})$ which plays the role of the kinetic energy is

$$H''(\mathbf{w}) = \int_0^l mA(s)\mathbf{w}\cdot\mathbf{w}\,ds \tag{4.13}$$

or in scalar form

$$H''(\mathbf{i}\times\mathbf{v}) = \int_0^l mA(s)(v_1^2 + v_2^2)\,ds.$$

We introduce the mixed forms

$$D''(\mathbf{w},\mathbf{t}) = \int_0^l \left\{\frac{d^2\mathbf{w}}{ds^2}\cdot\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{t}}{ds^2} + \Omega^2L(s)\,\frac{d\mathbf{w}}{ds}\cdot\frac{d\mathbf{t}}{ds} - mA(s)\mathbf{w}\cdot\boldsymbol{\Theta}\cdot\mathbf{t}\right\} ds, \tag{4.12a}$$

and

$$H''(\mathbf{w},\mathbf{t}) = \int_0^l mA(s)\mathbf{w}\cdot\mathbf{t}\,ds \tag{4.13a}$$

Since $\boldsymbol{\Theta}$ is a symmetric dyadic both $D''(\mathbf{w}, \mathbf{t})$ and $H''(\mathbf{W}, \mathbf{t})$ are symmetric functionals. It also follows immediately that

$$D''(\mathbf{w}+\mathbf{t}) = D''(\mathbf{w}) + 2D''(\mathbf{w},\mathbf{t}) + D''(\mathbf{t}), \tag{4.12b}$$

$$H''(\mathbf{w}+\mathbf{t}) = H''(\mathbf{w}) + 2H''(\mathbf{w},\mathbf{t}) + H''(\mathbf{t}). \tag{4.13b}$$

The vector $\mathbf{w}(s)$ will be called an admissible vector if it is continuous and has continuous derivatives of at least the fourth order, and also satisfies the constraining boundary conditions Eq. (4.9). It must be pointed out that due to the presence of the minus sign in the last term of $D''(\mathbf{w})$ it is not obvious that $D''(\mathbf{w})$ is positive for all admissible $\mathbf{w}$, and it appears to be difficult to prove that such is the case. A careful investigation of $D''(\mathbf{w})$ reveals that the troublesome term arises from $V_2(s)$. It is felt, but not justified, that this term is always at least balanced by the other terms in the functional.

The general minimum principle is the following: Of all the admissible vectors $\mathbf{w}(s)$, that one which minimizes $D(\mathbf{w})$ under the side conditions $H(\mathbf{w}) = 1$, and $H(\mathbf{w}), \mathbf{t}^i) = 0$ for $i = 1, 2, \cdots, n - 1$ where $\mathbf{t}^i$ is the ith eigenvector of the differential equation is an eigenvector of Eq. (4.8) and satisfies Eqs. (4.10) as natural boundary conditions. The minimum value of $D(\mathbf{w})$ under these side conditions on $\mathbf{w}$ is then the corresponding eigenvalue λ_n^2 and $\lambda_n^2 \geqq \lambda_{n-1}^2 \geqq \cdots \geqq \lambda_1^2$. The proof of this principle is analogous to the proof of the variational principle given in Section 3, and will not be repeated.

An orthogonality relationship between the eigenvectors of Eq. (4.8) can be developed in the usual way. If $\mathbf{w}(s)$ and $\mathbf{t}(s)$ are two eigenvectors, and λ^2 and μ^2 their respective eigenvalues where $\lambda^2 \neq \mu^2$, then $\mathbf{w}$ and $\mathbf{t}$ satisfy the orthogonality condition

$$H''(\mathbf{w}, \mathbf{t}) = \int_0^l mA(s)\mathbf{w}\cdot\mathbf{t}\, ds = 0, \tag{4.14}$$

which implies, as before, that

$$D''(\mathbf{w}, \mathbf{t}) = 0. \tag{4.15}$$

The reality of the eigenvectors and eigenvalues of Eq. (4.8) can now be proved in a manner analogous to the proof given in Section 3.

If we assume $\tau = 0$, and set $v_2 \equiv 0$, H'' and D'' reduce to

$$D''(v_1) = \int_0^l \left\{\frac{d^2v_1}{ds^2}\left(EI_2 \frac{d^2v_1}{ds^2}\right) + \Omega^2 L(s) \frac{dv_1}{ds}\cdot\frac{dv_1}{ds} - mA\Omega^2 v_1^2 \sin^2 \theta\right\} ds,$$

$$H''(v_1) = \int_0^l mAv_1^2\, ds.$$

Applying our variational principle we obtain the differential equation

$$\frac{d^2}{ds^2}\left\{EI_2 \frac{d^2v_1}{ds^2}\right\} - \Omega^2 \frac{d}{ds}\left\{L(s) \frac{dv_1}{ds}\right\} - mA\Omega^2 v_1 \sin^2 \theta_0 - mA\lambda^2 v_1 = 0,$$

and

$$\frac{d^2v_1}{ds^2} = \frac{d}{ds}\left(EI_2 \frac{d^2v_1}{ds^2}\right) = 0$$

at $s = l$. These are the differential equation and boundary conditions developed by Lo and Renbarger for a rotating blade vibrating flexurally in a plane making an angle of $(\pi/2 - \theta_0)$ with the plane of rotation [5].

Finally we wish to develop a similarity principle which may be used for experimental work. Introducing the dimensionless variables

$$x = s/l, \qquad \mathbf{y} = \mathbf{w}/l, \qquad \delta = \tau l. \tag{4.16}$$

Eqs. (4.8), (4.9) and (4.10) become

$$\left(\frac{1}{\Omega^2 l^4}\right) \frac{d^2}{dx^2}\left\{\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{y}}{dx^2}\right\} - \frac{d}{dx}\left\{K(x) \frac{d\mathbf{y}}{dx}\right\} - mA\,\boldsymbol{\psi}\cdot\mathbf{y} - mA \frac{\lambda^2}{\Omega^2}\mathbf{y} = 0,$$

with

$$\mathbf{y} = 0, \qquad \frac{d\mathbf{y}}{dx} = 0$$

at $x = 0$, and

$$\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{y}}{dx^2} = 0, \qquad \frac{d}{dx}\left(\mathbf{B}^{-1}\cdot\frac{d^2\mathbf{y}}{dx^2}\right) = 0$$

at $x = 1$. Also

$$K(x) = \int_x^1 mA(\eta)\eta\, d\eta; \qquad \boldsymbol{\psi} = \begin{pmatrix} \cos^2\psi & -\sin\psi\cos\psi \\ -\sin\psi\cos\psi & \sin^2\psi \end{pmatrix}$$

with

$$\psi(x) = \int_0^x \delta \, d\eta + \theta_0 .$$

Consider another beam which we designate by C' which has the same θ_0 and δ as the first beam, but which is rotating with an angular velocity Ω'. If the dimensions of C' are related to those of C by a similiarity factor a, then the area A', and moments of inertia I_1' and I_2' of C' are related to those of C by $A' = a^3 A$, $I_1' = a^4 I_1$, $I_2' = a^4 I_2$, and if $l' = el$, $E' = bE$ and $m' = cm$, then we have for the beam C'

$$\left(\frac{ba^2}{c\Omega'^2 e^4 l^4}\right) \frac{d^2}{dx^2} \left\{\mathbf{B}^{-1} \cdot \frac{d^2\mathbf{y}}{dx^2}\right\} - \frac{d}{dx}\left\{K(x) \frac{d\mathbf{y}}{dx}\right\} - mA \boldsymbol{\psi} \cdot \mathbf{y} - mA \frac{\lambda'^2}{\Omega'^2} \mathbf{y} = 0.$$

The homogeneous boundary conditions remain the same; consequently if

$$\frac{ba^2}{ce^4 \Omega'^2} = \frac{1}{\Omega^2},$$

then

$$\frac{\lambda'^2}{\Omega'^2} = \frac{\lambda^2}{\Omega^2}.$$

5. Numerical example. As an example, the natural frequency of a twisted, uniform, non-rotating, cantilevered beam will be computed using the energy principles developed in Section 3. For simplicity we assume the beam has a constant natural twist, that is τ equals constant. The energy principle may be written for $\mathbf{v}$ an admissible vector,

$$\lambda^2 \leqq D(\mathbf{v})/H(\mathbf{v}). \tag{5.1}$$

Writing $D(\mathbf{v})$ and $H(\mathbf{v})$ in scalar form, introducing dimensionless variables, and using the fact that the beam is uniform we obtain

$$\beta^2 = \frac{mAl^4}{EI_2} \lambda^2$$
$$\leqq \frac{\int_0^1 \left\{\frac{d^2w_1}{dx^2} - 2\delta \frac{dw_2}{dx} - \delta^2 w_1\right\}^2 dx + \gamma \int_0^1 \left\{\frac{d^2w_2}{dx^2} + 2\delta \frac{dw_1}{dx} - \delta^2 w_1\right\}^2 dx}{\int_0^1 (w_1^2 + w_2^2) \, dx}, \tag{5.2}$$

where $\gamma = I_1/I_2$. It is known that $\beta^2 = 12.36$ for a beam with no twist [6].

Of particular interest is the calculation of the natural frequency of a propeller blade, in which case the ratio of width to thickness of the beam is very large, that is, I_1/I_2 is large. Use of the variational principle in this case requires some delicacy in the choice of the approximating functions. Since γ is large, $h_1 = (\mathbf{i} \times \partial^2\mathbf{u}/\partial s2)_1$, which is called the first bend, will be small. Consequently we shall choose trial functions which make the contribution to the potential energy from the h_1 term small. Now

$$\mathbf{M} = \mathbf{B}^{-1} \cdot \mathbf{i} \times \frac{\partial^2 \mathbf{u}}{\partial s^2},$$

and expanding this we obtain

$$M_1 = EI_1h_1 ,$$
$$= EI_1\left(\frac{\partial^2 u_2}{\partial s^2} + 2\tau \frac{\partial u_1}{\partial s} + \frac{\partial \tau}{\partial s} u_1 - \tau^2 u_2\right).$$

From this we see that we wish to choose trial functions that will make the coefficient of γ in Eq. (6.2) as small as possible. Let us choose w_1 and w_2 as fourth degree polynomials,

$$w_1(x) = x^4 + a_3x^3 + a_2x^2,$$
$$w_2(x) = fx^4 + dx^3 + bx^2.$$

For the particular computations in question, the constants a_2 and a_3 were determined by satisfying the conditions

$$\frac{d^2w_1}{dx^2} - \delta^2 w_1 = 0, \qquad \frac{d}{dx}\left(\frac{d^2w_1}{dx^2} - \delta^2 w_1\right) = 0$$

at $x = 1$ for $\delta = \pi/16$. These equations result from requiring that there be no moment and shear in the **j** direction upon assuming $w_2 = 0$. In this case we obtain $a_3 = -3.974054$, $a_2 = 5.980107$. The constants b and d are determined from the conditions that the linear and constant term in $(d^2w_2/dx^2 + 2\delta dw_1/dx - \delta^2 w_1)^2$ should vanish. This gives

$$b = 0, \qquad d = -\frac{2}{3}\,\delta a_2 .$$

The constant f is left arbitrary to be used as a minimizing parameter. If ρ is defined as the ratio of the natural frequency of the twisted beam to the natural frequency of the untwisted beam, we obtain the following results*

δ	I_1/I_2	β^2	ρ
$\pi/16$	48	12.65	1.01
	64	12.72	1.01
	144	13.07	1.03
$\pi/8$	48	13.39	1.04
	64	13.69	1.05
	144	15.01	1.10
1/2	48	14.21	1.07
	64	14.64	1.09
	144	16.85	1.17

These results appear to agree closely with experimental results [7,8]. To obtain better approximations for larger values of δ and I_1/I_2 , w_1 and w_2 should be chosen to be polynomials of higher degree, thus allowing us to require that the coefficient of γ in Eq. (6.2) vanish to a higher degree in x.

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Some of the computational aspects of this problem are being investigated by Henry E. Fettis of the Flight Research Laboratory, Wright Field. He has noted the delicacy of the calculation problem and has proposed transformations of the energy principle which appear to improve the situation. He has also investigated several different computational schemes. These results will be published at a later date.

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