CORRECTIONS IN HOT-WIRE CORRELATION MEASUREMENTS*

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Summary. Particularly in small-scale turbulence, hot-wire correlation measurements must be corrected for the effect of finite wire length. End effects, imperfect velocity correlation along the length of the wire and transient longitudinal heat conduction must be considered. An integral equation for these corrections is obtained; with appropriate experimental procedure, this equation may be solved explicitly.

1. Introduction. A hot-wire instrument consists of a wire filament (e.g. tungsten of .005 mm diameter and 1 mm length) heated by an electric current and cooled by the flow of the fluid in which it is immersed. The rate of heat loss of such a wire is found experimentally to be proportional to the temperature difference between wire and stream multiplied by an empirical function of stream velocity. When the stream velocity is fluctuating, as in turbulence, the relationship between instantaneous values of temperature, velocity, and rate of heat loss is assumed to be the same as that between their steady-state values; this assumption, justified by an order-of-magnitude heat transfer calculation and also indirectly by the results of its use, allows a study of turbulence to be made.

It is customary to maintain constant either the temperature or the current in the wire. In the latter case, the response of the instrument to a velocity fluctuation is modified by its own heat capacity; for convenience, a compensating circuit to correct for this effect is often included. Since such circuits result in a decrease in signal-noise ratio, they are occasionally omitted [1]. A constant temperature instrument does not require compensation, and moreover has an inherently faster "response-completion" time; however, the required electronic feed-back circuit decreases the signal-noise ratio. Although each instrument is superior to the other in specific applications, the constant-current type is the more widely used, largely because of its simplicity.

In practice, a number of difficulties arise. Among these are the fact that the two ends of the wire are both at stream temperature because of the large amount of well-cooled metal in the solder joint at each end (Fig. 1); this end effect affects the response of the instrument—particularly if the wire is sufficiently short to respond well to small-scale turbulence. The magnitude of the end-effect has been calculated by Betchov [2] for the special case in which all elements of the wire length experience the same instantaneous velocity fluctuation (in fact, a harmonic fluctuation). Betchov has also discussed the effect of possible non-linearity in the dependence of heat-transfer rate on the temperature difference between wire and fluid; however, not only is the existence of appreciable non-linearity doubtful (for example, certain experiments performed on thin tungsten wires by the present author did not show any discernable non-linearity), but it appears in any case that such non-linearity effect would be completely dominated by end-effects for such short wires as are contemplated herein. It may be remarked that even if non-linearity is included, the perturbation equation corresponding to Eq. (3) is still linear and precisely the same Fourier Transform techniques may be used; the algebra

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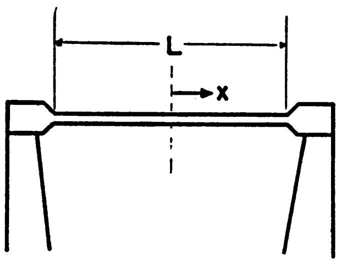
is merely somewhat more complicated (involving elliptic instead of hyperbolic functions).

A second problem arises from the effect of imperfect velocity correlation along the length of the wire. Skramstad [4] has calculated the correction to be applied to measured correlation coefficients in the case where (1) there are no end-effects (2) the effect of transient heat conductivity along the length of the wire is neglected. It has been pointed out by various authors, e.g. Ref. [3], that Skramstad's equation may be obtained by merely manipulating the order of integration of a certain multiple integral; some general remarks on such situations have been made by Uberoi and Kovasznay [5] and by Liepmann [6]. This latter author includes some remarks on combined space-time integral products (Eq. III-17 of Ref. [6])*. Our present purpose is to derive an integral equation for the true correlation function in terms of the apparent function as measured by two identical wires; it will be found necessary to measure the correlation for a sequence of different values of wire separation, and with different values of time displacement. It will turn out that there is a particular way of measuring these correlations which allows the integral equation to be explicitly inverted. Only one component of the correlation tensor is discussed; if the turbulence has isotropy as well as the assumed homogeneity, and if the fluid is essentially incompressible, then it is well known that a knowledge of this one component determines all others. It is assumed in the derivation that the two hot wires used are not sufficiently close to one another to result in mutual interference. Although the weak signals encountered in small-scale turbulence work make the constant-current instrument with its inherently superior signal-noise ratio the natural choice in such work, the constant temperature case will also be discussed.

2. Transform of energy equation. Consider a long thin wire supported at each end as in Fig. 1. The steady component of fluid velocity is perpendicular to the wire axis,



and insofar as turbulence is concerned, the wire will respond only to that component of velocity fluctuation which is parallel to this steady velocity direction. At any instant, the requirement of energy conservation applied to a wire element of length dx yields:

$$\frac{I^2 r\,dx}{A} + kA\,dx\left(\frac{\partial^2 T}{\partial x^2}\right) = (T - T_a)\,dx\,f(V) + gAc\,dx\left(\frac{\partial T}{\partial t}\right), \tag{1}$$

where x (cm) is measured as in Fig. 1, A (cm^2) is the cross-sectional area of the wire, I (amps) the wire current, r (ohm cm) the resistivity of the wire metal, k (joules/cm sec °C.) the thermal conductivity of the wire metal, T and T_a (°C.) the temperature of the wire element and stream, respectively (the latter is assumed constant), V the fluid velocity, $f(V)$ (in joules/cm sec °C.) the heat transfer coefficient from wire to stream

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per unit length of wire, g (gm/cm^3) the density of the wire metal, c (joules/gm °C.) the specific heat of the wire metal, and t (sec) the time. The resistivity r at temperature T is related to its value r_a at T_a by means of α (/°C.), the temperature coefficient of resistivity:

$$r = r_a[1 + \alpha(T - T_a)].$$

For drawn tungsten wire, some typical values are: $c = .142$, $k = 1.67$, $\alpha = .0037$, $r_a = 8 \times 10^{-6}$ at 20 °C, $g = 19$. For such a wire of $A = 6.15 \times 10^{-7}$, $f(V) = .00254$, .00362, and .00519 at 40, 100, and 250 ft/sec air flow respectively.

The velocity and resistivity may each be replaced by the sum of a steady-state and a fluctuating component:

$$V = V + v(x, t), \qquad r = R(x) + r(x, t),$$

where $R(x)$ is the steady-state solution of Eq. (1), corresponding to $\partial T/\partial t = 0$:

$$R(x) = \frac{r_a f(V)}{kAP_1^2}\left[1 - \frac{I^2 r_a \alpha \cosh(P_1 x)}{Af(V)\cosh(P_1 L/2)}\right], \tag{2}$$

where

$$P_1^2 = \frac{f(V) - I^2 r_a \alpha/A}{kA}.$$

Then a perturbation in Eq. (1) (I is held constant; the case of constant temperature will be discussed subsequently) yields:

$$\frac{\partial^2 r}{\partial x^2} - P_1^2 r - P_2 \frac{\partial r}{\partial t} = Bv\left[1 - \frac{\cosh(P_1 x)}{\cosh(P_1 L/2)}\right], \tag{3}$$

where

$$P_2 = \frac{gc}{k}, \qquad B = \frac{I^2 r_a^2 \alpha(\partial f/\partial V)}{kA^2 f(V) - I^2 r_a \alpha k A}.$$

The boundary conditions are that $r = 0$ at $x = \pm L/2$. Taking the Fourier Transform in time of Eq. (3) gives an ordinary differential equation

$$\frac{d^2 r'}{dx^2} - (P_1^2 + iP_2\lambda)r' = Bv'\left[1 - \frac{\cosh(P_1 x)}{\cosh(P_1 L/2)}\right],$$

where

$$r'(x, \lambda) = \int_{-\infty}^{\infty} r(x, t)\exp(-i\lambda t)\,dt, \qquad v'(x, \lambda) = \int_{-\infty}^{\infty} v(x, t)\exp(-i\lambda t)\,dt,$$

which is easily solved for r' (boundary conditions unchanged). The transform of perturbation voltage e across the wire may then be calculated by

$$e'(\lambda) = \int_{-L/2}^{L/2} Ir'\,dx/A,$$

to give, finally,

$$e'(\lambda) = \frac{BI}{A(P_1^2 + iP_2\lambda)} \int_{-L/2}^{L/2} v'\left(1 - \frac{\cosh(P_1 x)}{\cosh(P_1 L/2)}\right) \cdot \left(-1 + \frac{\cosh[(P_1^2 + iP_2\lambda)^{1/2} x]}{\cosh[(P_1^2 + iP_2\lambda)^{1/2} L/2]}\right) dx. \tag{4}$$

The value of e may be obtained by inversion as

$$e(t) = \int_{-\infty}^{\infty} dx\, d\epsilon\, d\lambda\, F(x, \lambda)v(x, \epsilon) \exp [i\lambda(t - \epsilon)], \tag{5}$$

where

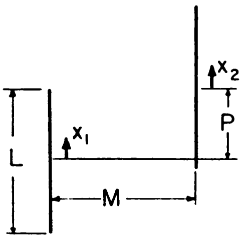
$$F(x, \lambda) = 0 \qquad \text{for} \qquad |x| > L/2,$$

$$F(x, \lambda) = \frac{BI}{2\pi A(P_1^2 + iP_2\lambda)} \left(1 - \frac{\cosh (P_1 x)}{\cosh (P_1 L/2)}\right)\left(-1 + \frac{\cosh [(P_1^2 + iP_2\lambda)^{1/2}x]}{\cosh [(P_1^2 + iP_2\lambda)^{1/2}L/2]}\right)$$
$$\text{for} \qquad |x| < L/2. \tag{6}$$

3. Correlation measurements. By a change of variable of integration, Eq. (5) may be written

$$e(t) = \int_{-\infty}^{\infty} dx\, d\epsilon\, d\lambda\, F(x, \lambda)v(x, t + \epsilon) \exp (-i\lambda\epsilon). \tag{7}$$

Suppose now that two identical wires are arranged as shown in Fig. 2 with the main stream velocity perpendicular to their common plane. The midpoints of the two wires



are displaced horizontally by a distance M and vertically by a distance P; let x_1 and x_2 be distances measured in the same direction from the midpoints along the first and second wires respectively. Define the correlation function C (a, b) to be the time average of the product of the velocity perturbation (i.e., the component parallel to main stream velocity) at one point in space with that at a point distant a in space (in some direction perpendicular to the main stream velocity) and b in time from the first point. Then for points x_1 , x_2 on the two wires,

$$C(a, b) = C\{[M^2 + (P + x_2 - x_1)^2]^{1/2}, b\}.$$

The turbulence field is assumed homogeneous. Now the measured correlation S for a fixed time interval of N seconds is given by the time average of the product of the voltages across the wires, one measured N seconds later than the other:

$$S\ (M, P, N) = \text{average of } [e_1\ (t)\ e_2\ (t + N)].$$

Using Eq. (5), this becomes

$$S(M, P, N) = \int_{-\infty}^{\infty} dx_1\, dx_2\, d\epsilon_1\, d\epsilon_2\, d\lambda_1\, d\lambda_2\, F(x_1 , \lambda_1)F(x_2 , \lambda_2)$$
$$\cdot\ C\{[M^2 + (P + x_2 - x_1)^2]^{1/2}, N + \epsilon_2 - \epsilon_1\} \exp (-i\lambda_1\epsilon_1 - i\lambda_2\epsilon_2).$$

Setting $\epsilon_2 - \epsilon_1 = \epsilon$ and $x_2 - x_1 = x$ and making use of the fact that

$$\int_{-\infty}^{\infty} d\epsilon_1 \exp(-i\lambda_1\epsilon_1 - i\lambda_2\epsilon_2) = 2\pi\Delta(\lambda_1 + \lambda_2),$$

[where Δ is the delta function of the argument $(\lambda_1 + \lambda_2)$] gives

$$S(M, P, N) = 2\pi \int_{-\infty}^{\infty} dx_1\, dx\, d\lambda_2\, d\epsilon\, F(x_1, -\lambda_2)F(x_1 + x, \lambda_2) \cdot C\{[M^2 + (P + x)^2]^{1/2}, N + \epsilon\} \exp(-i\lambda_2\epsilon).$$

Taking the transform of each side gives

$$\begin{aligned}\int_{-\infty}^{\infty} dP\, dN\, S(M, P, N) \exp(i\alpha P + i\beta N) &= 2\pi \int_{-\infty}^{\infty} dP\, dN\, dx_1\, dx\, d\epsilon\, d\lambda_2\, F(x_1, -\lambda_2) \\ &\quad\cdot F(x_1 + x, \lambda_2)C\{[M^2 + (P + x)^2]^{1/2}, N + \epsilon\} \exp(i\alpha P + i\beta N - i\lambda_2\epsilon) \\ &= 2\pi \int_{-\infty}^{\infty} dx_1\, d\epsilon\, d\lambda_2\, d(P + x)\, d(x_1 + x) \\ &\quad\cdot d(N + \epsilon)F(x_1, -\lambda_2)F(x_1 + x, \lambda_2)C\{[M^2 + (P + x)^2]^{1/2}, N + \epsilon\} \\ &\quad\cdot \exp[i\alpha(P + x) - i\alpha(x_1 + x) + i\alpha x_1 + i\beta(N + \epsilon) - i\beta\epsilon - i\lambda_2\epsilon] \\ &= 4\pi^2\left\{\int_{-\infty}^{\infty} d\omega\, d\phi\, C([M^2 + \omega^2]^{1/2}, \phi) \exp(i\alpha\omega + i\beta\phi)\right\} \\ &\quad\cdot \left\{\int_{-\infty}^{\infty} dx\, dy\, F(x, \beta)F(y, -\beta) \exp(i\alpha x - i\alpha y)\right\},\end{aligned}$$

whence

$$\int_{-\infty}^{\infty} d\omega\, d\phi\, C([M^2 + \omega^2]^{1/2}, \phi) \exp(i\alpha\omega + i\beta\phi) = \frac{\displaystyle\int_{-\infty}^{\infty} dP\, dN\, S(M, P, N) \exp(i\alpha P + i\beta N)}{4\pi^2 \left| \displaystyle\int_{-\infty}^{\infty} dx\, F(x, \beta) \exp(i\alpha x) \right|^2}.$$

Inversion yields

$$C([M^2 + d^2]^{1/2}, s) = \int_{-\infty}^{\infty} dP\, dN\, d\alpha\, d\beta \frac{S(M, P, N) \exp[i\alpha(P - d) + i\beta(N - s)]}{8\pi^3 \left| \displaystyle\int_{-L/2}^{L/2} dx\, F(x, \beta) \exp(i\alpha x) \right|^2}. \quad (8)$$

Since $F(x, \beta)$ is known from Eq. (6), Eq. (8) provides an explicit expression for the true correlation function C in terms of the measured correlation S. No compensation circuits are contemplated in these measurements. It is worth emphasizing that for fixed M a sequence of measurements for different values of P are necessary; this is the device which allows an explicit solution to be written. The integration with respect to x, α, and β may be performed explicitly if desired; Eq. (8) then has the form

$$C([M^2 + d^2]^{1/2}, s) = \int_0^{\infty} dP\, dN\, S(M, P, N)W(P - d, N - s),$$

where the function W is probably best handled by tabulation of values as calculated from Eq. (8).

Note also that S could be written as a function of the distance between the midpoints of the two parallel wires and of the angle between their axes and the line joining their midpoints.

4. Constant temperature case. If the total resistance of the wire is maintained constant by means of a feedback circuit, then (letting r again denote the perturbation in resistivity)

$$\int_{-L/2}^{L/2} r\,dx/A = \int_{-L/2}^{L/2} \left(\frac{\partial r}{\partial t}\right)\frac{dx}{A} = 0$$

and, if p is the perturbation in power input to the wire,

$$p(t) = \int_{-L/2}^{L/2} \frac{R(x) - R_a}{R_a\alpha}\left(\frac{\partial f}{\partial v}\right)v(x, t)\,dx + \frac{kA}{R_a\alpha}\left[\left(\frac{\partial r}{\partial x}\right)_{-L/2} - \left(\frac{\partial r}{\partial x}\right)_{L/2}\right]. \tag{9}$$

The last term may be calculated by writing the non-integrated perturbation equation

$$\frac{pRR_a\alpha}{R_0A^2k} + \frac{\partial^2 r}{\partial x^2} - P_1^2 r - P_2\frac{\partial r}{\partial t} = \left(\frac{(R - R_a)(\partial f/\partial v)}{kA}\right)v, \tag{10}$$

where R_0 is the total (constant) wire resistance. A time transform yields

$$\frac{RR_a\alpha}{R_0A^2k}p'(x, \lambda) + \frac{\partial^2 r'(x, \lambda)}{\partial x^2} - (P_1^2 + iP_2\lambda)r'(x, \lambda) = \frac{(R - R_a)(\partial f/\partial v)}{kA}v'(x, \lambda).$$

Insertion of p' from the transform of Eq. (10) gives a certain ordinary differential equation whose explicit solution may be written down subject to the conditions that $r'(-L/2, \lambda) = r'(L/2, \lambda) = 0$. The equation may be differentiated and $(\partial r'/\partial x)_{L/2} - (\partial r'/\partial x)_{-L/2}$ calculated from the result (it will appear on both sides). The expression for this quantity may be inverted and substituted into Eq. (10) which now has the form of Eq. (5); the subsequent analysis is analogous.

There is little point in reproducing this latter analysis in detail because of the fact that the situation of present interest—viz., small-scale turbulence—is precisely that in which the high noise level of the constant temperature instrument makes it rather unsuitable in comparison with (uncompensated) constant current equipment.

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