where a and b must be chosen so that $1 - ab\omega_0^2 > 0$. Substituting these values of the Z 's and S in (3), dividing out the common factor

$$W + \frac{Z_3 + Z_5}{Z_3(Z_2 + Z_5)}$$

and simplifying, we obtain

$$Z'(p) = \frac{abp(p^2 + \omega_0^2)W + (a + b - ab^2\omega_0^2)p^2 + a\omega_0^2}{a(p^2 + \omega_0^2)W + p(abp^2 + 1)}. \quad (5)$$

Now it follows from (1) and (2) that $Z(p)$ may be written in terms of the reduced function $R_1(p)$ as

$$Z(p) = Z(k) \cdot \frac{p(p^2 + \omega_0^2)R_1 + (\alpha + k)p^2 + k\omega_0^2}{k(p^2 + \omega_0^2)R_1 + p(p^2 + \alpha k + \omega_0^2)}. \quad (6)$$

A comparison of (5) and (6) shows that if we let $W = R_1$, $b = 1/k$, $a = k/(\alpha k + \omega_0^2)$, we obtain $Z'(p) = Z(p)/Z(k)$. Making this impedance level change in $Z'(p)$, we conclude that $Z(p)$ may be realized as an unbalanced bridge with

$$\begin{aligned} Z_1 &= W + S, & W &= Z(k)R_1(p), & S &= \frac{Z(k)(1 - ab\omega_0^2)(a + b)p}{ab(p^2 + \omega_0^2)}; \\ Z_2 &= a\omega_0^2 Z(k)/p, & Z_3 &= bZ(k)p, & Z_4 &= Z(k)/R_1(p), & Z_5 &= aZ(k)p, \\ & & a &= k/(\alpha k + \omega_0^2), & b &= 1/k. \end{aligned}$$

Thus our algorithm requires five elements and two reduced functions.

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ON TWO METHODS OF GENERATING SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS†

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1. Definite integrals with variable limits of integration. Consider the canonical form of the linear hyperbolic differential equation in two independent variables

$$L(u) \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \quad (1)$$

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It has been well known since the beginnings of the subject (see the detailed discussion in Goursat [4]) that if $U(x, y, \alpha)$ is any¹ solution of (1) containing a parameter α , for $\alpha_1 \leq \alpha \leq \alpha_2$ with α_1 and α_2 fixed constants, then

$$u(x, y) = \int_{\alpha_1}^{\alpha_2} f(\alpha)U(x, y, \alpha) d\alpha, \quad (2)$$

where $f(\alpha)$ is an arbitrary function, is also a solution of (1). In this way solutions of (1) can be generated from functions $f(\alpha)$ of a single variable α .

In 1895, Le Roux [1] showed that one or both of the fixed limits of integration can be replaced by a characteristic variable [say, x or y , in the case of (1)], and that (2) will still be a solution of (1), provided only that $U(x, y, \alpha)$ is chosen so as to satisfy certain additional conditions. Thus, if either limit of integration is replaced by x , then $U(x, y, \alpha)$ must satisfy the characteristic condition

$$\frac{\partial U}{\partial y} + a(\alpha, y)U = 0 \quad \text{on } x = \alpha, \quad (3)$$

which, incidentally, is one of the conditions satisfied by the Riemann function. Similarly, if either limit of integration is replaced by y , then $U(x, y, \alpha)$ must satisfy

$$\frac{\partial U}{\partial x} + b(x, \alpha)U = 0, \quad \text{on } y = \alpha. \quad (4)$$

As a matter of fact, the characteristic variables x and y , or an arbitrary function of just one of them, are the only variable limits which may be used in the definite integral in (2), if $u(x, y)$ is to be again a solution of (1). This is proved by Le Roux [1] for the case in which the functions occurring in (2) are such that the definite integral may be differentiated under the integral sign.

For instance, if $R(\xi, \eta; x, y)$ is the Riemann function² of (1) [so that, for fixed (ξ, η) , the function $R(\xi, \eta; x, y)$ of the "pole variables" (x, y) is a solution of (1)] then, for η fixed, $R(\alpha, \eta; x, y)$ will be a solution of (1) satisfying (3); while, for ξ fixed, $R(\xi, \alpha; x, y)$ will be a solution of (1) satisfying (4); and, finally, $R(\alpha, \alpha; x, y)$ will be a solution of (1) satisfying both (3) and (4). Accordingly, for arbitrary f the definite integrals with variable limits

$$\int_{\alpha_1}^x f(\alpha)R(\alpha, \eta; x, y) d\alpha, \quad \int_{\alpha_1}^y f(\alpha)R(\xi, \alpha; x, y) d\alpha, \quad \int_x^y f(\alpha)R(\alpha, \alpha; x, y) d\alpha,$$

will each be a solution of (1).

2. The integral operator. In several papers Bergman³ has developed a theory of integral operators for obtaining solutions of partial differential equations. As applied to (1), the method is, for the most part (see the concluding paragraph of Sec. 3), based on the following theorem:

¹Throughout this paper precise statements concerning differentiability, continuity, and regions of definition of functions will be omitted, except in those cases where they are of prime importance.

²For the connection between the integral operator of the next section and Riemann's function, see [10].

³For a summary and bibliography see in particular [9] and [11].

Theorem. Let $E(x, y, t)$ be a solution of

$$(1 - t^2)(E_v + aE_t) - \frac{1}{t}(E_v + aE) + 2xtL(E) = 0, \quad (5)$$

such that, for $x \neq 0$,

$$\frac{(1 - t^2)^{1/2}(E_v + aE)}{xt} \quad (6)$$

is continuous for $t = 0$, and tends to zero for each (x, y) as t approaches ± 1 . Then, if f is an arbitrary once continuously differentiable function, the function u defined by

$$u(x, y) = \int_{-1}^{+1} E(x, y, t) f[\tfrac{1}{2}x(1 - t^2)] \frac{dt}{(1 - t^2)^{1/2}} \quad (7)$$

is a solution of (1).

Proof. Writing, for brevity, f instead of $f[\tfrac{1}{2}x(1 - t^2)]$, one has, upon differentiating (7),

$$L(u) = \int_{-1}^{+1} \left\{ L(E) \cdot f + (E_v + aE) \cdot \frac{\partial f}{\partial x} \right\} \frac{dt}{(1 - t^2)^{1/2}}. \quad (8)$$

Now,

$$\frac{1}{2}(1 - t^2) \frac{\partial f}{\partial t} + xt \frac{\partial f}{\partial x} = 0,$$

and when $xt \neq 0$ this equation may be solved for $\partial f / \partial x$. Substituting for $\partial f / \partial x$ in (8) and integrating by parts (cf. (6) at $t = 0$), one obtains

$$L(u) = \int_{-1}^{+1} \left\{ \frac{L(E)}{(1 - t^2)^{1/2}} + \left[\frac{(1 - t^2)^{1/2}}{2xt} (E_v + aE) \right]_t \right\} f dt - \left[\frac{(1 - t^2)^{1/2}}{2xt} (E_v + aE) f \right]_{-1}^{+1}.$$

The desired conclusion now follows from the hypothesis made about (6) as $t \rightarrow \pm 1$, and the fact that

$$\frac{L(E)}{(1 - t^2)^{1/2}} + \left[\frac{(1 - t^2)^{1/2}}{2xt} (E_v + aE) \right]_t = 0,$$

by virtue of (5).

This theorem, as stated and proved above for Eq. (1), does not seem to be stated explicitly in full in the literature. However, for an equation occurring in gas dynamics [when $a = b = 0$, $c = c(x - y)$], see Bergman [6]. Moreover, Ghaffari [8] considers the general equation (1), but does not state the essential conditions on the behavior of (6) at $t = 0$.

3. Definite integrals with variable limits of integration and the integral operator.

The object of the present note is to show that the theorem of Sec. 2 and its obvious modification when x and y are complex (which constitute the kernel of Bergman's operator method), are merely a restatement of part of Le Roux's results mentioned in Sec. 1, and that Le Roux's formulation is simpler in the fluid dynamical application.

First of all, notice that $E(x, y, t)$ can be considered—without any loss of generality—to be an even function of t , for otherwise $E(x, y, t)$ may be replaced by its even part $\frac{1}{2}[E(x, y, t) + E(x, y, -t)]$ in (7), without altering the value of the definite integral. Hence, instead of (7), consider

$$u(x, y) = 2 \int_0^1 E(x, y, t) f\left[\frac{1}{2}x(1 - t^2)\right] \frac{dt}{(1 - t^2)^{1/2}}. \quad (9)$$

Then, for $x \neq 0$, the change of integration variable

$$\alpha = x(1 - t^2), \quad \text{i.e.} \quad t = \left(\frac{x - \alpha}{x}\right)^{1/2}, \quad (10)$$

transforms Eq. (9) into

$$u(x, y) = \int_0^x E\left[x, y, \left(\frac{x - \alpha}{x}\right)^{1/2}\right] f\left(\frac{1}{2}\alpha\right) \frac{d\alpha}{[\alpha(x - \alpha)]^{1/2}}.$$

Now, if $U(x, y, \alpha)$ is defined as

$$U(x, y, \alpha) = \frac{E\left[x, y, \left(\frac{x - \alpha}{x}\right)^{1/2}\right]}{[\alpha(x - \alpha)]^{1/2}},$$

so that

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{ E_x + \frac{\alpha}{2x^{3/2}(x - \alpha)^{1/2}} E_t - \frac{1}{2(x - \alpha)} E \right\}, \\ \frac{\partial U}{\partial y} &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} E_y, \\ \frac{\partial^2 U}{\partial x \partial y} &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{ E_{xy} + \frac{\alpha}{2x^{3/2}(x - \alpha)^{1/2}} E_{yt} - \frac{1}{2(x - \alpha)} E_y \right\}, \end{aligned}$$

then

$$\begin{aligned} L(U) &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{ L(E) + \frac{\alpha}{2x^{3/2}(x - \alpha)^{1/2}} (E_{yt} + aE_t) - \frac{1}{2(x - \alpha)} (E_y + aE) \right\}, \\ &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{ L(E) + \frac{\alpha/x}{2x[(x - \alpha)/x]^{1/2}} (E_{yt} + aE_t) \right. \\ &\quad \left. - \frac{1}{2x[(x - \alpha)/x]} (E_y + aE) \right\}, \\ &= 0, \end{aligned}$$

by (5), since the t argument of E is just $[(x - \alpha)/x]^{1/2}$. Hence $U(x, y, \alpha)$ is a solution of (1) containing a parameter α . Further

$$\left(\frac{\partial U}{\partial y} + aU\right) = \frac{1}{[\alpha(x - \alpha)]^{1/2}} (E_y + aE),$$

so that

$$\frac{(1 - t^2)^{1/2}}{xt} (E_y + aE) = \frac{\alpha}{x} \left(\frac{\partial U}{\partial y} + aU\right).$$

Now, the theorem uses the continuity, at $t = 0$, of the function on the left hand side of the last equation. At first glance this appears to be a weaker requirement than the vanishing of the right hand side at $\alpha = x$, which is needed by Le Roux. However, as has already been remarked, E can be assumed to be an even function of t , so that $(1 - t^2)^{1/2}(E_x + aE)/xt$ is odd in t ; hence the continuity of the latter function at $t = 0$ implies that it must of necessity vanish there.

Thus the function $E(x, y, t)$ used by Bergman may be written as the sum of an even part $[E(x, y, t) + E(x, y, -t)]/2$, which satisfies Le Roux's condition (3) after the transformation (10) is performed; and an odd part $[E(x, y, t) - E(x, y, -t)]/2$, which plays no essential role, since it contributes nothing to the definite integral. This remark about the odd part of E has apparently been overlooked in working out some particular examples.

If the independent variables x and y in (1) are both *complex*, and the coefficients a, b, c are analytic functions of these two complex variables, then there is no essential distinction between equations of hyperbolic and elliptic type. In this case, which is the one usually considered by Bergman, the function E is taken to be analytic in the two complex variables x and y , and the function f (since it is differentiated with respect to x) is taken to be an analytic function of its (complex) argument. The formal calculation leading to the proof of the theorem is exactly the same as in the real case considered above. By this passage to complex x and y , however, the class of non-analytic solutions of a hyperbolic differential equation is excluded from direct consideration. It should also be remarked, see [5], that for certain purposes it is convenient, in the case when x and y are both complex, to consider t as complex also, and to integrate from $t = -1$ to $t = +1$ along an arbitrary curve in the t -plane. If this curve does not pass through the origin, the Le Roux condition (3) need not be fulfilled, and hence, in these circumstances, Bergman's method is not equivalent to Le Roux's. However, in concrete applications the path of integration from $t = -1$ to $t = +1$ seems to be invariably taken along the real axis, as in the theorem of Sec. 2.

4. The application of the integral operator method to fluid dynamics. One of the major applications of Bergman's integral operator method is to the equations (in a "modified hodograph" plane) governing the plane steady irrotational flow of a perfect compressible fluid. Under a suitable change of independent variables the equation for, say, the stream function ψ , is of the type (1), with coefficients a, b, c depending on the difference $x - y$ alone.

The method of Le Roux reduces in this case to finding a solution $U^*(x, y)$ of (1) which satisfies either the condition (3) on the *single* characteristic $x = 0$, or the condition (4) on the *single* characteristic $y = 0$, or both. Once this solution $U^*(x, y)$ has been found, one sets

$$U(x, y, \alpha) = U^*(x - \alpha, y - \alpha), \quad (11)$$

and then a family of solutions depending on one arbitrary function is given at once by (2), with α_1 and/or α_2 suitably replaced by x and/or y , as the case may be. For, since the partial differential equation is invariant when the variables x and y are replaced by $x - \alpha$ and $y - \alpha$ respectively, the function $U(x, y, \alpha)$ defined by (11) is again a solution, and will satisfy the corresponding Le Roux conditions on $x = \alpha$ and/or $y = \alpha$.

A special case of Eq. (1) with coefficients depending on $x - y$ alone is the Euler-Poisson equation (see Darboux [2])

$$u_{xy} - \frac{\beta'}{x-y} u_x + \frac{\beta}{x-y} u_y = 0, \quad (12)$$

which for $\beta = \beta' = 1/6$ serves as a first approximation (Tricomi) to the equation for the stream function of compressible flow near the sonic line. Darboux uses the procedure just outlined in the previous paragraph, and an argument equivalent to that of Le Roux, to construct the general solution of Eq. (12); and from this, Tricomi [3], has constructed a simple definite integral representation for the solution of the Cauchy problem for (12) with $\beta = \beta' = 1/6$, the function u and its normal derivative being given on the sonic line $x = y$.

Besides the general compressibility equation, Bergman considers (in his terminology) a "simplified" compressibility equation [7]. In effect this "simplified" equation is that of Tricomi, under change of variables. In dealing with this "simplified" equation, he first finds solutions of the corresponding equation (5) for E in terms of hypergeometric functions, and is led to a complicated representation of a class of solutions of this equation.

Thus, even for this "simplified" equation, it appears that Bergman's method is more complicated than that of Le Roux. Indeed, Bergman's method consists in finding particular solutions of an associated equation for E , Eq. (5), in three independent variables, whereas the method of Le Roux consists in determining particular solutions of the originally given differential equation in two independent variables.

The complications involved are apparent from the following concluding remark: Bergman looks for particular functions $E(x, y, t)$ which involve series whose general term is a power of $t^2 x$ times a function of $x - y$. This is easily seen to be equivalent, in view of the change of variable (10), to just looking for particular solutions of the original differential equation using series whose general term is a power of $(x - \alpha)$ times a function of $x - y$, or (because of the invariance of the differential equation when x and y are replaced by $x - \alpha$ and $y - \alpha$ respectively), just a power of x times a function of $x - y$.

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