Application II: Fluid mixing.

The average interfacial area per unit volume is a significant measure of the "degree of mixedness" in an isotropic field of two molecularly immiscible liquids. It is, however, inaccessible to straightforward experimental determination. Equation (14) permits its calculation from the simpler process of interfacial-intersection counting along a linear traverse through the mixture.

In the homogeneous mixing of two gases or molecularly miscible liquids (e.g. turbulent mixing) it is possible that the notion of interfacial area can be replaced either by the surface area on which the concentration fluctuation is zero, or by the surface on which the concentration gradient magnitude has a local maximum.

EQUILIBRIUM OF MEMBRANES ELASTICALLY SUPPORTED AT THE EDGES*

By V. G. HART (*Dublin Institute for Advanced Studies*)

Abstract. The problem considered is that of finding the statical deflection of a stretched membrane, subjected to a uniform pressure on one side and elastically supported at the edges. The deflection of the membrane is supposed small and the problem reduces to solving Laplace's equation with mixed boundary conditions. Solutions are given for the cases where the bounding curve of the membrane is (i) an equilateral triangle, and (ii) a rectangle.

1. Introduction. We consider the problem of finding the statical deflection of a membrane originally lying in a plane (the neutral plane), when subjected to a uniform pressure on one side, its edge being elastically supported. This means that the edge can move in a direction perpendicular to the neutral plane, but is restrained at any point by a force proportional to the deflection at that point. Small deflections only being considered, the tension is a constant, and the problem reduces to solving the boundary-value problem (3) for an edge of arbitrary shape; solutions are given for (i) a membrane in the form of an equilateral triangle and (ii) a rectangular membrane.

Imagine a membrane stretched to a uniform tension T and bounded by any plane curve B. A uniform pressure P now acts on one side of the membrane which takes up a statical deflected position with deflection w. At any point on the edge the tension gives a component of force per unit length in the direction perpendicular to the neutral plane of amount $-T\,\partial w/\partial n$, ($\partial n$ being the outward normal element to B in the neutral plane), and this is balanced by the elastic force of constraint, which we write w/k, where k is a constant.

The appropriate partial differential equation and boundary condition for w are therefore

$$\Delta w = -P/T \text{ in } S, \qquad (w)_B = -kT(\partial w/\partial n)_B\,, \tag{1}$$

S being the domain of the membrane.

We now make the transformation

$$u = \tfrac{1}{2}r^2 + 2Tw/P, \tag{2}$$

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where r is the distance from any fixed point in the plane of the problem, and from (1) we find that u must satisfy Laplace's equation with mixed boundary conditions, viz.,

$$\Delta u = 0 \text{ in } S, \qquad \left(u + c\frac{\partial u}{\partial n}\right)_B = \frac{1}{2}r^2 + c\frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right), \tag{3}$$

where $c = kT$. Our procedure is to solve (3) for u in the case of the particular boundary B chosen, and then to find the deflection w from (2).

We note that on putting $c = 0$ in (3) the boundary-value problem reduces to the torsion problem, and this fact affords a check on the results obtained.

The Dirichlet integral of w, which is intimately connected with Laplace's equation, represents in this problem the excess potential energy stored in the deflected membrane. The total energy V_1 stored in the membrane is

$$V_1 = \frac{1}{2}T\int_S \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dS + C, \tag{4}$$

where C is the energy stored in the undeflected membrane. On taking up its deflected position, work is also done on the elastic support and the potential energy V_2 contained in this is

$$V_2 = \frac{1}{2k}\int_B w^2\, db, \tag{5}$$

where db is an element of length of the boundary B. Using the boundary-condition on w in (1) and also Green's theorem we find

$$\begin{aligned} V_2 &= -\frac{1}{2}T\int_B w\frac{\partial w}{\partial n}\, db, \\ &= \frac{1}{2}P\int_S w\, dS - \frac{1}{2}T\int_S \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dS, \\ &= \frac{1}{2}P\int_S w\, dS - V_1 + C, \end{aligned} \tag{6}$$

since $\Delta w = -P/T$ in S. Thus the total potential energy V stored in membrane and support is

$$V = V_1 + V_2 = \frac{1}{2}P\int_S w\, dS + C, \tag{7}$$

as is indeed obvious.

If B is a circle, the solution of (3) is very simple, viz.,

$$u = \tfrac{1}{2}a^2 + ca, \tag{8}$$

where a is the radius of the circle, and the origin is taken at its centre. The corresponding deflection is

$$w = \frac{Pa^2}{4T}\left[1 - \frac{r^2}{a^2} + 2\frac{c}{a}\right]. \tag{9}$$

2. Membrane in the form of an equilateral triangle. We now consider the case where the membrane is an equilateral triangle of height $3a$. We take the origin and axes as in Fig. 1. To solve the boundary-value problem (3) for this domain, we seek a har-

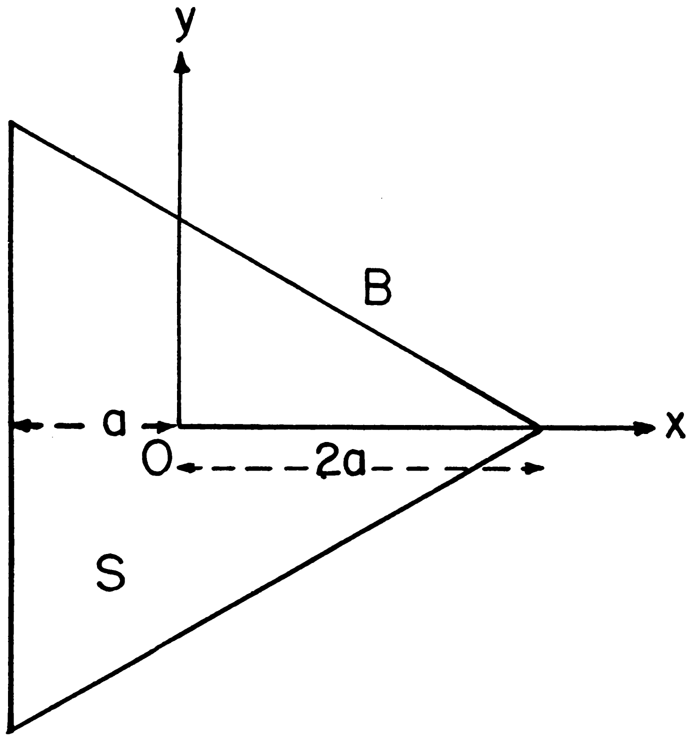

Fig. 1

monic function possessing the symmetry properties of the triangle and also satisfying the boundary conditions. It is obvious that a function with one of the symmetries of the triangle—invariance under a rotation of axes through $2\pi/3$—will satisfy the boundary conditions on all three sides of the triangle if it satisfies them on one—say on $x = -a$. Bearing in mind the other symmetry of the triangle (reflection in $0x$), we accordingly choose

$$u = A + B(z^3 + z^{*3}), \qquad z = x + iy, \qquad z^* = x - iy, \tag{10}$$

which is harmonic and has both the symmetries. We adjust the constants A and B to satisfy the boundary condition on $x = -a$, which reads:

$$\left(u - c\frac{\partial u}{\partial x}\right)_{x=-a} = \frac{1}{2}(a^2 + y^2) + ca. \tag{11}$$

On substituting from (10) we find A and B by comparing coefficients of y

$$A = \frac{a(2a^2 + 6ac + 3c^2)}{3(a + c)}, \qquad B = \frac{1}{12(a + c)}. \tag{12}$$

The problem (3) is now solved for u, and by (2) the deflection

$$w = \frac{Pa^2}{2T}\left[\frac{3c^2/a^2 + 6c/a + 2}{3(1 + c/a)} + \frac{x^3 - 3xy^2}{6a^3(1 + c/a)} - \frac{1}{2a^2}(x^2 + y^2)\right]. \tag{13}$$

The level lines are approximately circles for large values of the dimensionless parameter c/a, i.e. for a very weak elastic support on the edge.

The total potential energy stored in the membrane and support is by (7)

$$V = C + \frac{3^{3/2}P^2a^4}{40T}\left[\frac{10\,c^2/a^2 + 15\,c/a + 3}{1 + c/a}\right]. \tag{14}$$

3. Rectangular membrane. We consider now the case of a rectangular membrane (length $2a$, breadth $2b$). The axes are taken so that the sides have the equations $x = \pm a$, $y = \pm b$.

To solve the boundary-value problem (3), we introduce an auxiliary function v by the transformation

$$u = \tfrac{1}{2}(x^2 - y^2) + b^2 + 2cb + v, \tag{15}$$

and (3) becomes the following boundary-value problem for v

$$\Delta v = 0 \quad \text{in} \quad S, \qquad \begin{cases} \text{(A)} \quad v \pm c\dfrac{\partial v}{\partial y} = 0 & \text{on} \quad y = \pm b, \\ \text{(B)} \quad v \pm c\dfrac{\partial v}{\partial x} = y^2 - b^2 - 2cb & \text{on} \quad x = \pm a; \end{cases} \tag{16}$$

note the homogeneous boundary conditions on $y = \pm b$.

We propose to solve (16) for v and then find the deflection w through the transformations (15) and (2).

Since v is harmonic we take as the solution of (16)

$$v = \sum_{n=0}^{\infty} A_n \cosh(\lambda_n x)\cos(\lambda_n y), \tag{17}$$

where λ_n and A_n are to be determined. Application of the boundary condition (16A) gives the following equation for λ_n

$$\cot(b\lambda_n) = c\lambda_n\,. \tag{18}$$

This equation determines an infinite sequence of values. We shall use only the positive values, numbering them in order of increasing magnitude. For large values of n we have

$$\lambda_n \sim \frac{n\pi}{b} + \frac{1}{cn\pi}\,. \tag{19}$$

The functions $\cos\lambda_n y$ form an orthogonal set; direct calculation gives by use of (18):

$$\int_{-b}^{b} \cos(\lambda_m y)\cos(\lambda_n y)\,dy = \begin{cases} 0 & \text{for} \quad m \neq n, \\ b + c\sin^2(\lambda_n b) & \text{for} \quad m = n. \end{cases} \tag{20}$$

We expand the right-hand side of the boundary condition (16B) in terms of these orthogonal functions, writing

$$y^2 - b^2 - 2bc = \sum_{n=0}^{\infty} B_n \cos \lambda_n y, \qquad (-b \leq y \leq b); \tag{21}$$

the coefficients are found by using (20) and are

$$B_n = -4 \sin (\lambda_n b)/[\lambda_n^3(b + c \sin^2 \lambda_n b)]. \tag{22}$$

The boundary condition (16B) is now applied to v as defined by (17) and this requires that A_n should satisfy

$$y^2 - b^2 - 2cb = \sum_{n=0}^{\infty} A_n(\cosh \lambda_n a + c\lambda_n \sinh \lambda_n a) \cos \lambda_n y, \qquad (-b \leq y \leq b). \tag{23}$$

Comparing (23) with (21) we see that

$$A_n = B_n/(\cosh \lambda_n a + c\lambda_n \sinh \lambda_n a), \tag{24}$$

where B_n is defined by (22). The coefficients A_n in (17) are thus determined and so the solution v of (16) is found.

Using (15) and (2) we find the deflection to be

$$w = \frac{Pb^2}{2T}\left[1 + 2\frac{c}{b} - \left(\frac{y}{b}\right)^2 - 4\sum_{n=0}^{\infty} \frac{\sin^2 \lambda_n b}{(\lambda_n b)^3[1 + (\sin 2\lambda_n b/2\lambda_n b)](\tan \lambda_n b + \tanh \lambda_n a)} \cdot \frac{\cos (\lambda_n y) \cosh (\lambda_n x)}{\cos (\lambda_n b) \cosh (\lambda_n a)}\right], \tag{25}$$

where we have simplified the coefficients A_n in (24) by the use of (18).

As already remarked we get the torsion problem on putting $c = 0$; Eq. (25) is then easily seen to reduce to the known solution for the torsion of a beam of rectangular section, λ_n now being $(2n + 1)\pi/2b$.

Due to the presence of the term $\cosh (\lambda_n x)/\cosh (\lambda_n a)$ which has the asymptotic value $\exp [\lambda_n(|x| - a)]$ when n is large [λ_n being given by (19)], the series in (25) is uniformly and absolutely convergent in both x and y for the intervals $|x| < a$, $|y| < b$. Also if we differentiate twice with respect to x or y, the resulting series is still uniformly and absolutely convergent for the same reason. Thus the operation of term-by-term differentiation is legitimate, and so w, as in (25), is the solution of the boundary-value problem (1) for a rectangular boundary.

The potential energy stored in the membrane and support is

$$V = \frac{2P^2b^4}{3T}\left[\frac{a}{b}\left(1 + 3\frac{c}{b}\right) - 6\sum_{n=0}^{\infty} \frac{\sin^2 \lambda_n b}{(\lambda_n b)^5[1 + (\sin 2\lambda_n b/2\lambda_n b)]} \cdot \frac{\tan (\lambda_n b) \tanh (\lambda_n a)}{(\tan \lambda_n b + \tanh \lambda_n a)}\right] + C. \tag{26}$$

This statical problem for the rectangular membrane as discussed above is mathematically similar to a dynamical problem treated by Rayleigh (*Theory of sound*, Vol. I, pp. 200-204) viz., that of a vibrating string with ends elastically supported; this involves the equation (18) for the computation of proper frequencies in a certain limiting case.

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