AN INTEGRAL EQUATION GOVERNING ELECTROMAGNETIC WAVES*

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1. Preliminaries. We shall consider in this paper the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

in two independent variables ξ and η. In electromagnetic theory the problem arises of determining, in the exterior D of a simple closed curve C, a solution u of (1) with prescribed boundary values or prescribed normal derivatives on C. The function u is supposed in addition to fulfil at infinity the radiation condition

$$\lim_{r \to \infty} r^{1/2}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2}$$

where $r = (\xi^2 + \eta^2)^{1/2}$ is the distance from the origin. Thus u will be complex-valued and $u(\xi, \eta)e^{-ikt}$ will be the actual wave-function for the problem considered. We can think of u as representing a two-dimensional wave scattered by the finite object C, and the type of boundary condition to be imposed along C then depends on whether we treat the electric or the magnetic field.

We shall investigate in detail the case when the boundary values of u are given along C. However, all our statements carry over almost verbatim when it is the normal derivatives which are prescribed instead. The uniqueness of the solution of (1) and (2) follows easily from the work of Rellich [2], but the existence of the solution as presented by Weyl [3] and Müller [1] depends on a careful discussion of the eigenfunctions of (1) for the interior of C. The object of the present note is to reduce the boundary value problem for (1) in the exterior domain D to a new Fredholm integral equation whose study is independent of the interior domain and its eigenvalues. This will be accomplished by introducing a suitable parametrix for the problem, constructed by means of conformal mapping. Our integral equation has, furthermore, the advantage that it can be solved numerically by iteration in certain important cases.

The middle section of the paper is devoted to the development of this integral equation. In the concluding section, we discuss the uniqueness question from a new point of view and indicate a variational formula for the estimation of the scattering cross section of C.

We shall assume that the frequency $k = 1$ in the following without loss of generality, since this reduction corresponds merely to a change of scale.

2. The parametrix method. Our analysis of the equation (1) is based on the notion of a parametrix $S(\zeta, \tau)$. Such a parametrix S is not a solution of (1), but it satisfies the following requirements. It is regular as a function of ζ in D, except at $\zeta = \tau$, where

$$S(\zeta, \tau) + \log|\zeta - \tau|$$

remains continuous together with its first partial derivatives. At infinity S satisfies the radiation condition

$$\lim_{r \to \infty} r^{1/2}\left(\frac{\partial S}{\partial r} - iS\right) = 0$$

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as a function of ζ, and along C we have $S = 0$ for each fixed τ in D.

Suppose that u is any solution of (1) and (2) in D with $k = 1$, as was agreed above. Then from the radiation condition we obtain

$$\lim_{r\to\infty} \int_{|\zeta|=r} \left(u \frac{\partial S}{\partial n} - S \frac{\partial u}{\partial n}\right) ds = 0,$$

where s and n represent arc length and inner normal along the path of integration. Hence by Green's theorem and by (1),

$$\int_C u(\zeta) \frac{\partial S(\zeta, \tau)}{\partial n} ds = 2\pi u(\tau) - \iint_D u(\Delta S + S)\, d\xi\, d\eta. \tag{3}$$

There are many elementary ways in which S can be constructed, and in each case (3) provides an integral equation for the determination of u.

We choose S in a special and particularly useful manner. Let

$$\zeta = z + f(z) \tag{4}$$

be the conformal transformation of the exterior $|z| > R$ of a circle of radius R onto D, normalized so that

$$f(z) = \sum_{m=0}^{\infty} \frac{a_m}{z^m} \tag{5}$$

is regular at infinity. Furthermore, denote by $G(z, w)$ the Green's function for (1) in the region $|z| > R$. To be precise, G satisfies the equation

$$\Delta G + G = 0 \tag{6}$$

as a function of z, except at $z = w$, where

$$G(z, w) + \log |z - w|$$

remains continuous; G satisfies the radiation condition at ∞; and $G = 0$ for $|z| = R$. This particular Green's function has a variety of explicit representations in terms of Bessel functions, all of which are obtained by the standard procedure of separation of variables in polar coordinates.

With ζ given by (4) and with $\tau = w + f(w)$, we set

$$S(\zeta, \tau) = G(z, w). \tag{7}$$

The function S so constructed by application of a conformal transformation on the arguments of the known Green's function G is evidently a parametrix of the type defined above. Indeed, S is regular in D except for a logarithmic infinity at $\zeta = \tau$, S vanishes on C, and S even satisfies the radiation condition, since the conformal transformation (4) has a derivative 1 at infinity and therefore leaves this condition invariant.

We substitute (7) into (3) and evaluate the integrals in the z-plane. The area element $d\xi\, d\eta$ and the Laplacian ΔS are altered in the transformation by multiplication and division, respectively, with the Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = |1 + f'(z)|^2.$$

In particular, the expression $\Delta S\, d\xi\, d\eta$ is a conformal invariant, as is also $(\partial S/\partial n)\, ds$. Therefore we obtain

$$\int_{|z|=R} U(z) \frac{\partial G(z, w)}{\partial n}\, ds = 2\pi U(w) - \iint_{|z|>R} U(z)[\Delta G + G \mid 1 + f' \mid^2]\, dx\, dy,$$

where $U(w)$ stands for $u(\tau)$ and likewise

$$U(z) = u(\zeta). \tag{8}$$

Application of (6) yields

$$\int_{|z|=R} U(z) \frac{\partial G(z, w)}{\partial n}\, ds = 2\pi U(w) - \iint_{|z|>R} U(z)G(z, w)\{2\, \mathrm{Re}\, f'(z) + \mid f'(z) \mid^2\}\, dx\, dy. \tag{9}$$

In order to derive from (9) a Fredholm integral equation for U with a symmetric kernel, we introduce the notations

$$p(z) = \{2\, \mathrm{Re}\, f'(z) + \mid f'(z) \mid^2\}^{1/2}, \tag{10}$$

$$V(z) = p(z)U(z), \tag{11}$$

$$K(z, w) = \frac{1}{2\pi} p(z)p(w)G(z, w). \tag{12}$$

We notice that when $u(\zeta)$ is prescribed on C, the expression

$$g(w) = \frac{p(w)}{2\pi} \int_{|z|=R} U(z) \frac{\partial G(z, w)}{\partial n}\, ds \tag{13}$$

will be known. Multiplying (9) on both sides by $p(w)/2\pi$, we arrive at the final Fredholm integral equation

$$g(w) = V(w) - \iint_{|z|>R} V(z)K(z, w)\, dx\, dy \tag{14}$$

for V, or in other words for U, and this is the basis of our entire discussion.

With $w = a + ib$, the symmetric kernel K of (14) satisfies the square-integrability condition

$$\iint_{|w|>R} \iint_{|z|>R} \mid K(z, w) \mid^2 dx\, dy\, da\, db$$
$$= \frac{1}{4\pi^2} \iint_{|w|>R} \iint_{|z|>R} \mid G(z, w) \mid^2 \mid p(z)p(w) \mid^2 dx\, dy\, da\, db < \infty, \tag{15}$$

since $\mid z \mid p(z)$ is bounded at infinity, by (5) and (10), and since $\mid z \mid^{1/2} \mid w \mid^{1/2} G(z, w)$ is bounded outside a fixed neighborhood of $z = w$, by standard estimates of the Bessel functions. Hence the Fredholm theory is immediately applicable to (14), and we deduce that for any given g we can find a unique square-integrable solution V, provided that the homogeneous equation

$$0 = V - \iint_{|z|>R} VK\, dx\, dy \tag{16}$$

has no non-trivial eigenfunction V. But an eigenfunction V transforms by (11) and (8) into a solution u of (1) which vanishes on C and satisfies the radiation condition (2), as can easily be verified by direct calculation from (16). Rellich's uniqueness theorem [2] then shows that $u \equiv 0$, whence $V \equiv 0$ and there are, indeed, no eigenfunctions.

Thus, given u on C, we can calculate g, substitute into (14), solve for V, and find u as a solution of (1) and (2) by using (8) and (11). This completes our proof of the existence of the solution of the boundary value problem. The hypotheses on the smoothness of C which are required in the argument are only those needed for the demonstration of the uniqueness theorem. In particular, our integral equation yields the solution of the problem for curves C with a finite number of corners, and the behavior of u at the corners can be derived from (14). Furthermore, the same proof is valid when it is the normal derivatives of u which are assigned on C. For this case, we have only to replace the Green's function G throughout the argument by the corresponding Neumann's function $N(z, w)$ for (1) and (2) in the region $|z| > R$. The function N is characterized by the same requirements as we imposed upon G, except that we replace the boundary condition $G = 0$ by the boundary condition $\partial N/\partial n = 0$ along $|z| = R$. For the circle, the Neumann's function has an explicit expansion in terms of Bessel functions, obtained by separation of variables.

From the practical standpoint, the integral equation (14) has some interest. If the domain D differs sufficiently little from the exterior of a circle, the function p depending on the conformal transformation (4) will be so small that

$$\iint_{|z|>R} |K(z, w)| \, dx \, dy \leq \epsilon < 1. \tag{17}$$

Hence the smallest eigenvalue of (14) will have a modulus exceeding 1 and the resolvent kernel can be used to solve the equation in a Neumann-Liouville series. In fact, the successive approximations defined by

$$V_m = g + \iint_{|z|>R} V_{m-1} K \, dx \, dy \tag{18}$$

converge geometrically when (17) holds. The *a priori* bound

$$|V| \leq \frac{\max |g|}{1 - \epsilon}, \tag{19}$$

which follows immediately in this case, is also significant, since the maximum principle fails for the partial differential equation (1) and more elementary estimates of u are therefore not available.

The case (17) of convergence of the successive approximations (18) to the solution V of (14) has an implication for the existence of solutions of the boundary value problem for (1), since no hypothesis of smoothness on the boundary curve C is required when (17) holds. Finally, we point out that in the development of our method, the conformal transformation (4) and the explicit Green's function G could be replaced by the corresponding quantities for any region in which one knows how to solve the equation (1).

3. Remarks about uniqueness and the scattering cross section. For solutions u of the Helmholtz equation in D not necessarily satisfying the radiation condition, one can define a norm by the formula

$$|| u ||^2 = \lim_{r\to\infty} \int_{|z|=r} \left\{ | u |^2 + \left| \frac{\partial u}{\partial n} \right|^2 \right\} ds. \tag{20}$$

If u_+ is the solution of (1) and (2) with boundary values h on C and if u_- is the solution of (1) and (2) with boundary values $\bar{h}$ on C, it then follows that the function

$$u_0 = \frac{u_+ + \bar{u}_-}{2} \tag{21}$$

is characterized by the property that among all solutions u of (1) with the boundary values h on C it has the smallest norm,

$$|| u_0 || = \min || u ||. \tag{22}$$

Indeed, writing $v = u - u_0$, we find

$$|| u ||^2 = || u_0 ||^2 + || v ||^2,$$

since by the radiation condition

$$\lim_{r\to\infty} \int_{|z|=r} \left\{ u_0 \bar{v} + \frac{\partial u_0}{\partial n} \frac{\partial \bar{v}}{\partial n} \right\} ds = \frac{1}{2i} \int_C \left\{ \bar{v} \left(\frac{\partial u_+}{\partial n} - \frac{\partial \bar{u}_-}{\partial n} \right) - (u_+ - \bar{u}_-) \frac{\partial \bar{v}}{\partial n} \right\} ds$$
$$= 0.$$

The radiation condition also yields for the minimum value of the norm the relation

$$|| u_0 ||^2 = \lim_{r\to\infty} \int_{|z|=r} | u_+ |^2 \, ds$$
$$= \lim_{r\to\infty} \int_{|z|=r} | u_- |^2 \, ds.$$

The extremal characterization (22) of u_0 is a refinement of the uniqueness theorem for the solution u of (1) and (2) with given boundary values on C. For if $h = 0$ then $|| u_0 || = 0$ and therefore $u_0 \equiv 0$, whence if u satisfies (2) and vanishes on C it must vanish identically, since we can put both $u_+ = u_- = u$ and $u_+ = u_- = iu$ in (21).

A case of special interest for (22) occurs when $h = -e^{ix}$. The function u_+ then represents the scattered field due to a wave of the form e^{ix} incident on C, while u_- represents the scattered field resulting from an incident wave e^{-ix}. The estimate (22) shows that the total scattering cross section σ of C is given by

$$\sigma = \min || u ||^2 \tag{23}$$

among all solutions u of (1) with the boundary values $-e^{ix}$. This characterization of σ indicates clearly the dependence of the scattering cross section on both u_+ and u_-, in view of the form (21) of the extremal function.

Another formula exhibiting the dependence of σ on both u_+ and u_- is obtained when we shift C by an infinitesimal amount δn along its inner normal and attempt to express the scattering cross section σ^* of the shifted curve C^* in terms of quantities associated with C. We continue to take $h = -e^{ix}$ and we introduce the total fields

$$\varphi_+ = e^{ix} + u_+ ,$$
$$\varphi_- = e^{-ix} + u_- ,$$

which vanish on C. We denote by φ_+^* and φ_-^* , and so forth, the corresponding quantities associated with C^*, and we obtain by an easy application of Green's theorem and the radiation condition

$$\begin{aligned}
\sigma^* - \sigma &= \lim_{r\to\infty} \int_{|z|=r} [| u_+^* |^2 - | u_+ |^2]\, ds \\
&= \mathcal{I}m \int_{C^*} \bar{u}_+^* \frac{\partial u_+^*}{\partial n}\, ds - \mathcal{I}m \int_C \bar{u}_+ \frac{\partial u_+}{\partial n}\, ds \\
&= \mathcal{I}m \int_{C^*} u_-^* \frac{\partial \varphi_+^*}{\partial n}\, ds - \mathcal{I}m \int_C u_+ \frac{\partial \varphi_-}{\partial n}\, ds \\
&= \mathcal{I}m \int \left(u_-^* \frac{\partial \varphi_+^*}{\partial n} - \varphi_+^* \frac{\partial u_-^*}{\partial n} - u_+ \frac{\partial \varphi_-}{\partial n} + \varphi_- \frac{\partial u_+}{\partial n} \right) ds \\
&= \mathcal{I}m \int \left(u_-^* \frac{\partial e^{ix}}{\partial n} - e^{ix} \frac{\partial u_-^*}{\partial n} - u_+ \frac{\partial e^{-ix}}{\partial n} + e^{-ix} \frac{\partial u_+}{\partial n} \right) ds \\
&= \mathcal{I}m \int \left(\varphi_-^* \frac{\partial \varphi_+}{\partial n} - \varphi_+ \frac{\partial \varphi_-^*}{\partial n} \right) ds.
\end{aligned}$$

In the last integral we are free to choose C as the path of integration, and since $\varphi_+ = 0$ on C and $\varphi_-^* = 0$ on C^*, we find

$$\begin{aligned}
\sigma^* - \sigma &= \mathcal{I}m \left\{ \int_C \varphi_-^* \frac{\partial \varphi_+}{\partial n}\, ds - \int_{C^*} \varphi_-^* \frac{\partial \varphi_+}{\partial n}\, ds \right\} \\
&= -\mathcal{I}m \iint_{D-D^*} \{ \nabla \varphi_-^* \nabla \varphi_+ - \varphi_-^* \varphi_+ \}\, dx\, dy,
\end{aligned}$$

where D^* is the exterior of C^*. We now consider only terms of the first order in the infinitesimal shift δn and derive, with the notation $\sigma^* - \sigma = \delta\sigma$, the Hadamard variational formula

$$\delta\sigma = -\mathcal{I}m \int_C \frac{\partial \varphi_-}{\partial n} \frac{\partial \varphi_+}{\partial n}\, \delta n\, ds \tag{24}$$

for the scattering cross section σ. The result is valid for small shifts δn of either sign.

Formula (24) has an obvious importance for the estimation of the scattering cross section σ of figures near those for which σ has been studied. As a special application of (24), we remark that it substantiates the conjecture that a vertical segment has the least scattering cross section among all simple closed curves which enclose its end-points. In fact, according to (24) we check that by symmetry $\delta\sigma = 0$ for any normal shift of the vertical segment carrying it into a neighboring curve with the same end-points.

Finally, we call attention to the fact that only the most elementary alterations are required in order to extend all the remarks of this section to the case in which the normal derivatives of u are prescribed in the boundary conditions along C. A generalization to space of three dimensions is likewise possible.

References

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