THE MECHANICS OF THE RIJKE TUBE*

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1. Introduction. An interesting phenomenon, the analysis of which has received only scattered attention, is the thermal-acoustic oscillation discovered by Rijke in 1859. It was noted, under certain circumstances, that superimposed on the anticipated steady

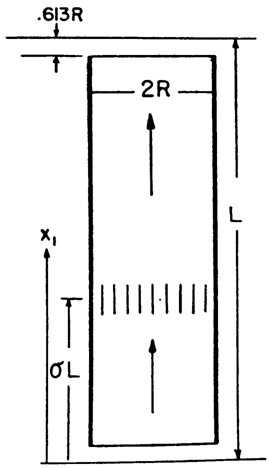

FIG. 1.1. Open ended cylindrical tube containing heated ribbon at $x_1 = \sigma L$.

convective flow in the apparatus indicated in Fig. (1.1) was an acoustic oscillation. The frequency of the phenomenon is essentially that of the first free oscillation mode of the pipe (i.e. the wave length is $2L'$),[1] but the occurrence (or lack thereof) and the intensity depend on a number of parameters.

The analysis of the effect of these parameters which is presented here proceeds from the fundamental conservation laws with an investigation of the response of the heater to a fluctuating velocity, an analysis of the non-isentropic wave propagation in the pipe with its attendant viscous losses and end radiation, and, from the interaction of these, the deduction of the complex eigenfrequency of the tube. In a more qualitative manner, the consistent appearance of harmonics is explained and the role these waves play in determining the sustained intensity of the sound is discussed.

The results are in excellent agreement with the observed physical facts, and it is believed that they, as well as the details of the analysis, may be useful in the future investigation of various combustion oscillation phenomena.

2. The wave propagation in the tube. It is convenient to divide the analysis of this composite problem into various pieces. The description of the waves in the tube for

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[1] L' is the length of the pipe corrected for end effects and non-uniform temperatures.

$x < \sigma$ and those for $x > \sigma$ are treated individually.[2] The boundary conditions at σ "joining" these waves are treated as though a disk shaped heat source, whose response to local velocity fluctuations is known, were located at σ. However, in the detailed investigation of the heater response, a more realistic description of the flow past the heater will be adopted.

In analyzing the oscillating flow downstream of the heater we must anticipate that it is non-isentropic and, in particular, that important viscous losses occur at the wall and that decaying temperature fluctuations are convected from the heater toward the exit. We assume that the steady flow upon which the oscillation is to be superimposed is known and introduce a small perturbation (in velocity, pressure, density, etc.) to represent the oscillatory phenomenon. The conservation laws (mass, momentum, and energy) are

$$(\rho' u_i')_{,i} + \rho'_{,t} = 0, \tag{2.1}$$

$$\rho'(u'_{i,t} + u'_j u'_{i,j}) + p'_{,i} = \mu(u'_{i,jj} + \tfrac{1}{3}u'_{j,ij}) + F_i , \tag{2.2}$$

$$\rho' c_v(T'_{,t} + u'_j T'_{,j}) - (p'/\rho')(\rho'_t + u'_j \rho'_{,j}) - k\Delta T' = 0, \tag{2.3}$$

and we adopt the state equation

$$p' = \rho' R' T'. \tag{2.4}$$

Here, p', ρ', T', u_i' are the usual thermodynamic variables and the velocity, and F_i is the gravitational body force. It is convenient and appropriate in this analysis to treat c_p , c_v , μ, k, as though they were independent of the thermodynamic state. It is also convenient to introduce the notation[3]

$$\begin{aligned}
u_i' &= v_i(x, r) + a[\operatorname{grad} \varphi(x, r, \tau) + \operatorname{curl} \mathbf{k}\psi(x, r, \tau)],\\
p' &= p_0(x, r)[1 + p(x, r, \tau)],\\
\rho' &= \rho_0(x, r)[1 + \eta(x, r, \tau)],\\
T' &= T_0(x, r)[1 + \theta(x, r, \tau)],\\
x &= x_1/L, \qquad r^2 = (x_2^2 + x_3^2)/L^2, \qquad a^2 = \gamma p_0/\rho_0 , \qquad \tau = at/L,
\end{aligned} \tag{2.5}$$

where $\mathbf{k}$ is the unit vector perpendicular to both the x and r directions. The tensor notation differentiations of Eqs. (2.1) to (2.3) are taken with regard to the physical coordinates (the x_i), but the vector notation differentiations of Eq. (2.5) pertain to the variables x, r. In the definitions of x, r, τ, the constant length parameter L may be thought of as the length of the tube. It will be defined more precisely later.

Equations (2.5) are now substituted into Eqs. (2.1) to (2.4), the terms containing no perturbation contributions are removed (since the steady terms are themselves solutions of the conservation equations), and the remaining equations for the perturbation quantities are linearized. Furthermore, the following simplification is adopted:

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[2]See Fig. (1.1).

[3]This notation is appropriate for either the region upstream of σ or that downstream of σ. We shall distinguish these in Sec. 4 by subscripts 2 and 1 respectively.

p_0 , ρ_0 , T_0 , v_0 are replaced by appropriate constant "average" values (in particular $v_1 = Ma$) and v_2 , v_3 are taken to vanish. The resulting perturbation equations are:

$$\eta_{,\tau} + \Delta\varphi = 0, \tag{2.6}$$

$$\varphi_{,\tau} + \gamma^{-1}p = \tfrac{4}{3}\epsilon\,\Delta\varphi, \tag{2.7}$$

$$\theta_{,\tau} - (\gamma - 1)\eta_{,\tau} = \tfrac{4}{3}\gamma\epsilon\,\Delta\theta, \tag{2.8}$$

$$\psi_{,\tau} = \epsilon\,\Delta\psi, \tag{2.9}$$

$$p = \eta + \theta. \tag{2.10}$$

Here we have taken the Prandtl number to be 3/4 for algebraic simplicity; ϵ is defined by $\mu/\rho a L$ and $\gamma = c_p/c_v$. The operator denoted by $(\ )_{,\tau}$ implies $\partial(\ )/\partial\tau + M\partial(\ )/\partial x$. That is, it is the convective time derivative.[4]

These equations must be solved subject to the boundary conditions that, at the wall, the temperature and velocity fluctuations vanish, and at the plane $x = \sigma$, the velocity, temperature, etc. are consistent with the heater behavior. At the exit, and at the inlet, a reflection coefficient must be adopted but we shall discuss that later. Equations (2.6) to (2.10) may be integrated in detail subject to such boundary conditions. The simplest presentation of the solution is obtained by eliminating η, p, and θ, and by anticipating that the form of the solutions is exponential in x and τ. For example, $\psi(x, r, \tau) = \psi(r)\exp[i(\alpha\tau - kx)]$. Actually we must anticipate that there will be two modes of propagation; one which is essentially an acoustic wave and one which drifts with the stream. The former should consist principally of a contribution from φ where $k^2 \simeq \alpha^2$ (this implies that it propagates with a phase velocity which is essentially the acoustic speed), whereas the latter requires a less elementary description. The acoustic wave will consist of two parts; one which propagates in the downstream direction and a reflection from the tube end which propagates back upstream.

Using the same letters to denote the functions of interest with the exponential dependence factored off, we obtain the following ordinary differential equations for φ and ψ after eliminating η, p, and θ.

$$L_1L_2(\varphi) = 0, \qquad L_3(\psi) = 0, \tag{2.11}$$

where $L_1 = \Delta + \beta^2/(1 + 4i\beta\gamma\epsilon/3)$, $L_2 = \Delta - 3i\beta/4\epsilon$, $L_3 = \Delta - i\beta/\epsilon$, $\Delta = r^{-2}\partial^2/\partial r^2 + r^{-1}\partial/\partial r - k^2$, and $\beta = \alpha - kM$.

For the upstream section analysis, it follows from our earlier remarks that $\beta = \nu\alpha - kM$. The propagation modes are found by applying the homogeneous boundary conditions at the cylindrical wall and finding the eigenvalues k. If we anticipate that $\varphi = \varphi_1 + \varphi_2$ with $L_1(\varphi_1) = L_2(\varphi_2) = 0$ for the acoustic mode, and further anticipate that in this mode φ_2 and ψ will each display a boundary layer character[5] whereas φ_1 will essentially be a plane acoustic wave, the motion is described to terms of order ϵ by

$$\varphi(r) = AJ_0[(\beta^2 - k^2)^{1/2}r] + B\exp[-(r_1 - r)(3i\beta/4\epsilon)^{1/2}] \tag{2.12}$$

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[4]When we consider the waves in the upstream region, it will again be convenient to define $\tau = at/L$ giving a downstream value. The definition of $(\ )_{,\tau}$ then will become $\nu\partial/\partial\tau + M\partial/\partial x$ where $\nu = a_1/a_2$ and M is evaluated for the upstream quantities.

[5]The term boundary layer is used here in the more general sense as in [1].

and

$$\psi(r) = C \exp [-(r_1 - r)(i\beta/\epsilon)^{1/2}], \tag{2.13}$$

where $r_1 = R/L$. Since $\theta(r) = p - \eta = [-i\gamma\beta + (i\beta)^{-1}\Delta]\varphi(r)$, the characteristic equation for k becomes [using $\varphi_r(r_1) + \psi_x(r_1) = \psi_r(r_1) - \varphi_x(r_1) = \theta(r_1) = 0$],

$$\begin{vmatrix} -ik & -ik & (i\beta/\epsilon)^{1/2} \\ r_1(k^2 - \beta^2)/2 & (3i\beta/4\epsilon)^{1/2} & ik \\ i(\gamma - 1)\beta & -3/4\epsilon & 0 \end{vmatrix} = 0, \tag{2.14}$$

or (again to order ϵ)

$$r_1(k^2 - \beta^2)(i\beta/\epsilon)^{1/2} = 2[k^2 + (\gamma - 1)(4/3)^{1/2}\beta^2]$$

and

$$k = \pm\alpha\left[1 + \frac{(\gamma - 1)(4/3)^{1/2}}{r_1}(\epsilon/i\beta)^{1/2}\right]. \tag{2.15}$$

The imaginary part of this number defines the decay rate of the wave which is associated with the dissipation and heat conduction near the wall. Its order of magnitude for many experiments (in particular, ours), renders it an important source of energy loss.

The propagation of the other "drifting" wave cannot be expressed in as elementary a form. However, we may obtain a reasonably simple representation of this wave if we anticipate the nature of the result. It is clear that the wave would be a plane wave drifting with the stream if it were not for the cylindrical wall boundary conditions. The effect of the walls, however, is an encroachment on this wave of such a nature that one cannot expect a simple product type representation corresponding to the acoustic wave. If one attempts the product type representation, an infinite series of such products is required. For an efficient representation, one should formulate an initial value problem in x such that the temperature has a uniform value (no dependence on r) at $x = \sigma$ but such that **v** and θ vanish at the wall. The solution to such a problem to the order of accuracy we require is given by $\psi = 0$ and

$$\varphi^{(3)}(x, r) = \frac{\partial}{\partial x}\int_0^{r_1 - r} e^{qx}K_0[\zeta(x^2 + S^2)^{1/2}]\, dS, \tag{2.16}$$

where $q = 3M/8\epsilon$ and $\zeta = (q^2 + 3i\alpha/4\epsilon)^{1/2}$. A more accurate but no more useful (corrections of order $\epsilon^{1/2}$) solution would again involve boundary layer solutions in ψ and φ in addition to a slightly modified $\varphi^{(3)}$.

Equation (2.16) is valid only for small enough x so that $\varphi^{(3)}(x, 0)$ takes essentially the value $\varphi^{(3)}(x, -\infty)$. However, for those x violating this condition, the wave has decayed so far as to be of no further interest. The critical observation concerning these eigenmodes of the tube is that the mode associated with $\varphi^{(3)}$ provides velocity and pressure contributions which are very small compared with its temperature and density contributions. Conversely, the k_1, k_2, modes give contributions of the same order in each of these four (dimensionless) state variables. This implies that the velocity and pressure fluctuations are almost entirely contributed by the acoustic modes while the final mode merely picks up any discrepancy in the temperature condition at $x = \sigma+$. We shall see this more clearly in Sec. 4.

The wave in the upstream portion of the tube contains only the acoustic modes. The other possible mode is again a downstream drifting wave which has zero "input" in the upstream section of the tube and is therefore not present. The formulas and k values for this section are identical with those of the downstream section except that the acoustic velocity, Mach number, M, etc. must be the values for the upstream state instead of the downstream state. In particular α is replaced everywhere by $\nu\alpha$ except in the exponential exp $(\alpha\tau)$.

Before closing this section, we must discuss the relation between the outgoing (away from the heater) waves and the reflected returning waves. The reflection coefficient for an acoustic wave in an unflanged tube has been investigated by Levine and Schwinger [2] for the case $M = 0$. We have shown, and shall report in detail elsewhere, that the following is true. If the plane axially directed wave incident on the exit end of a tube (from inside) has acoustic pressure $p_i = P \exp(i\omega t)$ at some point x, the reflected wave has pressure $p_r = PN' \exp(i\omega t)$ where $N'(\omega R/a, M) \equiv N[\omega R/a(1 - M^2)^{1/2}]$. The formula for the inlet end is $p_r/p_i = (1 - M/1 + M)N[\omega R/a(1 - M^2)^{1/2}]$. Here N is the reflection coefficient computed in [2] for $M = 0$. However, for $S \ll 1$, $N(S) = 1 - S^2/2 + \cdots$, and arg N is such that the wave is apparently reflected with reflection coefficient $|N|$ from a point $.613R$ beyond the end of the tube.[6] Since M is of order 10^{-3} for our problem, the moving gas correction is negligible and we use the above results with $S = wr/a$ (but with differing values of "a" in the two sections of the tube). We now define L more precisely. In fact $x_1 = 0$ is the point $.613R$ below the upstream end of the tube and L is the point $.613R$ beyond the other end. We shall return to, and use these results when we have determined the conditions at $x = \sigma$ which are imposed by the heater. To facilitate this, we record, respectively, the downstream (of σ), and upstream, acoustic wave forms for use in Sec. 4, replacing J_0 of argument whose order is 10^{-3} by unity.

$$\varphi^{(1)} = A_1 \exp[i\alpha\tau - ik_1(x - 1)] - |N_1| \exp[i\alpha\tau + ik_1(x - 1)], \tag{2.17}$$

$$\varphi^{(2)} = A_2 \exp(i\alpha\tau + ik_2 x) - |N_2| \exp(i\alpha\tau - ik_2 x), \tag{2.18}$$

with

$$k_1 = \alpha\left[1 + \frac{1 + (\gamma - 1)(4/3)^{1/2}}{r_1}\right](\epsilon/i\alpha)^{1/2} = k_1' - ik_1'',$$

$$k_2 = \nu\alpha\left[1 + \frac{1 + (\gamma - 1)(4/3)^{1/2}}{r_1}\right](\epsilon/i\nu\alpha)^{1/2} = k_2' - ik_2'',$$

$$N_1 = 1 - (\alpha r_1)^2/2 + \cdots,$$

$$N_2 = 1 - (\nu\alpha r_1)^2/2 + \cdots,$$

$$\nu = a_1/a_2 .$$

The k_1', k_1'', separation is such that k_1' is real for real α.

To the order of accuracy to which it is sensible to work, the downstream fluctuating velocity and pressure are given by

$$u^{(1)} = a_1\varphi_x^{(1)}, \qquad p^{(1)} = -i\alpha\gamma\varphi^{(1)} \tag{2.19}$$

[6] In our problem S is of order 0.1.

and, correspondingly,

$$u^{(2)} = a_2\varphi_x^{(2)}, \qquad p^{(2)} = i\nu\alpha\gamma\varphi^{(2)}. \tag{2.20}$$

As we shall see, the explicit formulas for the contributions of $\varphi^{(3)}$ will not be needed.

3. The heater "response". In this section[7] we shall estimate the fluctuating heat release from a heated ribbon to a stream of fluid whose velocity is fluctuating in a known manner. We say "estimate" because certain simplifying approximations will be adopted to render the problem tractable. However, the errors introduced should be of the order of a few per cent. In particular, we shall consider the problem whose simplified geometry

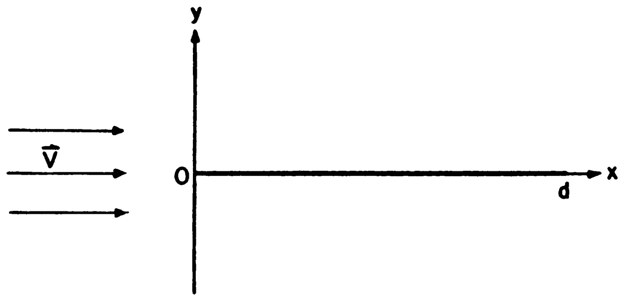

FIG. 3.1. The geometry of the simplified heater response problem. The ribbon thickness is oriented in the y direction; $0d$ is its breadth. **V** represents the fluctuating flow.

is described in Fig. 3.1. For a known flow of an incompressible fluid with constant thermal conductivity and specific heat, the energy conservation equation has the form

$$\rho c_p(T'_t + \mathbf{v}\cdot\text{grad } T') - k\,\Delta T' = 0. \tag{3.1}$$

Since the velocity distribution near the plate is rather complicated, this equation would be difficult to treat in its present form. However, the analysis of a similar problem [3], that of determining the viscous flow past an obstacle at low Reynolds number[8], has indicated that an excellent approximation to the result is obtained when one replaces $\mathbf{v}\cdot\text{grad } T'$ by cUT'_x in Eq. (3.1). Here c is a number, $0 < c < 1$, (.43 is the optimum value in the viscous flow problem), U the free stream velocity, x the coordinate oriented in the flow direction and $T'(x, y, t)$ the fluid temperature. We introduce the notation $cU = u_0 + we^{i\tau}$, $T' = \Theta(x, y) + \theta(x, y)e^{i\tau}$, and T_0 (the plate temperature) $= T + \delta e^{i\tau}$. We shall take T and δ to be constant rather than functions of distance along the plate. This is not correct in detail but two items justify this choice. A precise analysis of the surface temperature of the wire is very messy and the correction to the heat release

[7] Unfortunately, the number of symbols needed in this paper is so large that the quantities a, β, and other notations of this section are not related to those labeled with the same letters in other sections. This should lead to no confusion since the equations are never directly combined.

[8] The nature of this approximation is discussed in [3], [4], and its verification noted in [4]. The present problem is very closely related to these and it is to be anticipated that the results obtained for the macroscopic quantities should be highly accurate. The reader should note, in particular, that the predicted steady heat release is in agreement with the known information concerning the heat release from plates to fluids.

rate associated with this approximation is not an important contribution to the total release[9].

The differential equation may now be "separated" into two equations governing, respectively, the steady and fluctuating stream temperatures provided we assume that each fluctuating quantity is small compared to the corresponding steady quantity (e.g. $\theta/\Theta \ll 1$). These equations take the form

$$\Delta\Theta - a\Theta_x = 0, \tag{3.2}$$

$$\Delta\theta - a\theta_x - i\beta\theta = (aw/u_0)\Theta_x\ , \tag{3.3}$$

where the second order term in $w\theta_x$ has been omitted. In these equations $a = \rho c_p u_0/k$ and $\beta = \rho c_p \omega/k$.

The boundary conditions are

$$\Theta \to 0 \quad \text{as} \quad \mathscr{I}m\,(x + iy)^{1/2} \to \infty, \qquad \Theta(x,\,0) = T \quad \text{on the plate,}$$

and

$$\theta \to 0 \quad \text{as} \quad \mathscr{I}m\,(x + iy)^{1/2} \to \infty, \qquad \theta(x,\,0) = \delta \quad \text{on the plate.}$$

We consider first the problem of determining $\Theta(x, y)$ when d(Fig. 3.1) is indefinitely large. This problem is readily solved by invoking Fourier Transforms and the Wiener-Hopf technique. Omitting the details of the analysis, the reader can readily verify that $\boldsymbol{\Theta}(\xi, y)$ is the Fourier Transform [10] with regard to x of the required $\Theta(x, y)$.

$$\boldsymbol{\Theta}/T = \frac{a}{i\xi(a - i\xi)} \exp\{-|\,y\,|\,[i\xi(a - i\xi)]^{1/2}\}. \tag{3.4}$$

The inversion integral is indented to pass beneath the origin (in the ξ plane). In particular, the transform of Θ_x/T [(which will be needed to solve Eq. (3.3)] is

$$\boldsymbol{\Theta}_x/T = [a/(a - i\xi)] \exp\{-|\,y\,|\,[i\xi(a - i\xi)]^{1/2}\}. \tag{3.5}$$

Other useful results are:

$$\Theta_y(x,\,0)/T = -(a/\pi x)^{1/2}, \qquad x > 0 \tag{3.6}$$

$$\Theta(x,\,0)/T = \operatorname{erfc}\,[(-ax)^{1/2}], \qquad x < 0 \tag{3.7}$$

and

$$Q_0 = 2kl\int_0^d \Theta_y(x,\,0)\,dx = 4klT(ad/\pi)^{1/2}. \tag{3.8}$$

This last quantity[11] will be associated with the steady heat output once we identify d with the true wire "breadth" and show that the contributions to the heat release rate of the finite plate is essentially that of the semi-infinite plate in the region $0 < x < d$. To see this we note that the heat release rate of the problem just completed, which in the physical problem should be zero for $x > d$, is given by Eq. (3.6). We can now solve the problem where we postulate that Θ_y'/T on $x > d$, $y = 0$ is $|\,y\,|\,(a/\pi x)^{1/2}/y$ and Θ'

[9]This has been analyzed crudely but will not be reported here in detail.

[10]$\boldsymbol{\Theta}$ is defined as $\int_{-\infty}^{\infty} \exp(-i\xi x)\,\Theta(x, y)\,dx$.

[11]l is the length of the wire perpendicular to the plane of Fig. 3.1.

vanishes for $x < d$. This is not quite the boundary value problem the superposition of which on the preceding one gives the correct answer, but if it turns out in this problem that Θ'_y is extremely small on $x < 0$, then the error in using the superposition of these two problems will be negligible. Actually, we anticipate that the change in heat release rate on $0 < x < d$, as well as that on $x < 0$, will be negligibly small and so we replace the boundary condition on Θ'_y in our superposition problem by the requirement that Θ'_y/T be $|y|(a\pi/d)^{1/2}/y$. If the heat release on $x < d$ associated with the solution to this problem is small, that associated with the properly formulated problem will be smaller. The solution of this problem may be treated like the previous one and the essential part of the result is:

$$(\pi d/a)^{1/2}\Theta'_y(x, 0)/T = \text{erfc}\,[a(d - x)]^{1/2} - [\pi a(d - x)]^{-1/2} \exp[-a(d - x)], \qquad x < d. \tag{3.9}$$

In our problem, a is of the order of 50 cm^{-1} and this analysis for the finite plate is justified provided d is of the order .1 cm or so. The heat release from the plate associated with Θ' is implied by

$$Q'_0 = 2kl \int_0^d \Theta'_y(x, 0)\, dx = lkT/(\pi ad)^{1/2}.$$

That is, for d of order .1 cm or more, Q'_0 is of order $.05Q_0$ and can be safely neglected. Thus Eq. (3.8) defines the steady component of the heat release rate.

We now turn to Eq. (3.3) and the boundary conditions on $\theta(x, y)$, again for d indefinitely large. Another application of the same technique gives

$$\begin{aligned}\theta(\xi, y)/T = {} & [a/(a - i\xi)]^{1/2} \exp\{-|y|[i\xi(a - i\xi)]^{1/2}\} \\ & - [a/(a_1 - i\xi)]^{1/2} \exp\{-|y|[(i\xi + a_2)(a_1 - i\xi)]^{1/2}\} iaw/\beta u_0 \\ & + [\delta a_1/i\xi(a_1 - i\xi)T] \exp\{-|y|[(i\xi + a_2)(a_1 - i\xi)]^{1/2},\end{aligned} \tag{3.10}$$

where a_1, $-a_2$, are the roots of $z^2 - az - i\beta = 0 (a_1 \to a$, as $\beta \to 0)$. In particular,

$$\begin{aligned}\theta_y(x, 0)/T = {} & [i(a^3/\pi x^3)^{1/2}\, w/2\, u_0\beta][1 - \exp(-a_2 x)] \\ & + [(a_1)^{1/2}\, \delta/T][(\pi x)^{-1/2} \exp(-a_2 x) + a_2^{1/2} \text{ erf } (a_2 x)^{1/2}],\end{aligned}$$

and

$$\begin{aligned}2kl \int_0^d \theta_y(x, 0)\, dx = {} & \frac{2ia^{3/2}klTw}{\beta u_0} [(a_2)^{1/2} \text{ erf } (a_2 d)^{1/2} + (\pi d)^{1/2}(e^{-a_2 d} - 1)] \\ & + \delta kl(a_1/a_2)^{1/2}[(a_2 d + \text{d}) \text{ erf } (a_2 d)^{1/2} + (a_2 d/\pi)^{1/2} e^{-a_2 d}].\end{aligned} \tag{3.11}$$

Again, we can estimate the correction required by the fact that the semi-infinite plate treatment is only approximate but again this correction is smaller than is consistent with the accuracy of our basic model.

In order to deduce the surface temperature fluctuation δ we must analyze the flow of heat within the wire. A very crude analysis of this item will demonstrate that the δ contribution is negligible. This rough analysis is conducted as follows. We consider an infinitely long slab of thickness $2b$ with surface temperature $\delta e^{i\tau}$. The differential equation to be solved (conservation of energy) is

$$K_w \theta_{yy}^{(w)} - \rho_w C_w \theta_t^{(w)} = 0$$

and the boundary conditions are

$$\theta^{(w)}(\pm b, \tau) = \delta e^{i\tau}.$$

Here, (w) merely denotes wire.

The solution can readily be obtained and, in particular,

$$\theta_y^{(w)}(b, \tau) = \Omega\delta \tanh (\Omega b) e^{i\tau}, \tag{3.12}$$

where

$$\Omega = (i\omega\rho_w C_v/K_w)^{1/2}.$$

The heat release rate for a wire of length l and breadth d is given by $Q' = 2K_w l d\theta_y^{(w)}(\mathrm{b})$, omitting the $e^{i\tau}$. This must be equated to the heat release to the fluid as given by Eq. (3.12) in order to determine δ. For ribbons of thickness of order .03 cm and breadth .1 cm (within rather large factors), δ is so small that "the δ term" of Eq. (3.11) is of order 10^{-2} or less times "the w term."

Thus the fluctuating heat release from the ribbon is essentially given by

$$q = -2kl \int_0^d \theta_y(x, 0)\, dx = \frac{-2ia^{3/2}klTw}{\beta u_0} [(a_2)^{1/2} \operatorname{erf} (a_2 d)^{1/2} + (\pi d)^{1/2}(e^{-a_2 d} - 1)]. \tag{3.13}$$

It should now be noted that (a_2/a) becomes imaginary and small when $\beta/a^2 \to 0$ but has angle $\pi/4$ and becomes large when $a^2/\beta \to 0$. This implies that for large β/a^2 the bracketed quantity in Eq. (3.11) is nearly a pure imaginary and the phase lag between the heat release rate fluctuation and the velocity fluctuation is small.[12] Conversely, when β/a^2 is small, the phase lag approaches $3\pi/8$. For the conventional Rijke tube experiment, the phase lag is of order $3\pi/8$.

Before we abandon the discussion of the heater behavior, we should discuss qualitatively the large amplitude behavior. To do this, consider first the extreme case where the free stream velocity is zero. In this case the response of the heater to a positive velocity fluctuation must be identical with its response to a negative velocity fluctuation. This is a direct consequence of the symmetry of the situation. Thus while the velocity fluctuation traverses one period, the heater is aware only of the absolute value of the velocity fluctuation and thus executes two cycles of heat release fluctuation. In other words, an input velocity fluctuation of frequency ω excites a heat release fluctuation of frequency 2ω. If the phenomenon is not linear, the higher harmonics will also be present. If one now estimates the expected behavior where w/u_0 is not small and where u_0 is not zero, he must conclude that the input signal at frequency ω gives rise to a heat release fluctuation with both ω, 2ω, and probably higher order components. The ratio of the 2ω content to the ω content of the output should be an increasing function of w/u_0. This information will be of use when we try to explain the harmonic content and sound level of the Rijke tube output.

4. The matching conditions at the heater. The problem of combining the results of the foregoing sections is now a relatively straightforward matter. We merely write down the requirement that mass, momentum and energy be conserved across the heater. If we denote the quantities of interest at the heater inlet by the usual symbols with

[12]This corresponds to the situation conventionally encountered in hot-wire instrumentation problems.

subscript 2 and those at the exit with subscript 1, these laws take the form

$$\rho_1 u_1 - \rho_2 u_2 = -\int_0^X \rho_t \, dx, \tag{4.1}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 - D/\pi R^2 - \int_0^X (\rho u)_t \, dx, \tag{4.2}$$

$$\rho_1 u_1 c_p T_1 - \rho_2 u_2 c_p T_2 = Q/\pi R^2 - \int_0^X (\rho c_p T)_t \, dx, \tag{4.3}$$

where Q is the heat input rate to the gas from the ribbon and D is the drag of the ribbon. It is tacitly assumed in these equations that 0 and X are in those regions where conduction is not important, and it can readily be verified that the omitted kinetic energy terms are of higher order than we have consistently retained. In limits of the integrals in these equations, 0 and X are appropriately chosen points just upstream and downstream of the heater. The integral term of Eq. (4.1) can be expected to be of the same order of magnitude as the other terms in that equation and a careful estimate of its value is required. As we shall see, the value of X is most readily chosen after we obtain the information which allows us to circumvent the necessity of finding ρ_t in $0 < x < X$. Before proceeding with this, however, we note that the integral terms in the other equations are of higher order in M than the dominating terms and a few simple estimates lead to the results

$$p_1' = p_2', \tag{4.4}$$

$$u_1' = u_2'(1 + \lambda), \tag{4.5}$$

with $\lambda = (\gamma - 1)q/\gamma\pi R^2 p_0$, when q is defined by Eq. (3.13). The primes indicate the time-dependent components of the state variables and velocities. To obtain a more useful form for Eq. (4.1), we write the conservation equations for a heat conducting gas in a passage assuming a given heat input rate[13] $Q/\pi R^2$ and a one-dimensional flow. For brevity, we accept the implication of Eq. (4.4) that $p = p(t)$ only, in the x range of interest (i.e. $0 < x < X$), rather than obtain this result again from the momentum equation in differential form. The other conservation equations (mass and energy) are [with $p = p(t)$]

$$(\rho u)_x + \rho_t = 0, \tag{4.6}$$

$$\rho\mu h_x + \rho h_t - (k/c_p)h_{xx} = Q/\pi R^2, \tag{4.7}$$

where h is the enthalpy.

Writing $\rho = \rho_0(x) + \rho'(x, t)$, $u = u_0(x) + u'(x, t)$, etc. and noting that the steady-state pressure is a constant, p_0, we have

$$\rho_0 u_0 = \text{const} = A, \qquad \rho_0 h_0 = \text{const} = [\gamma/(\gamma - 1)]p_0$$

and

$$A h_{0,x} - (k/c_p)h_{0,xx} = Q_0/\pi R^2.$$

[13] Q is the heat input per unit time per unit distance in the flow direction.

From this, $h_0(x)$ can readily be obtained when Q_0 and $h(-\infty)$ are known. The perturbation terms in the sum: [Eq. (4.7) plus h times (4.6)] lead to the equation

$$\frac{\gamma}{\gamma - 1}(p_0 u_x' + p_t') - (k/c_p)h_{xx}' = Q'/\pi R^2, \tag{4.8}$$

where the primes denote the fluctuating components.

It follows that

$$u' - u_1' = \frac{\gamma - 1}{\gamma p_0}\left[\frac{k}{c_p}h_x' + (\pi R^2)^{-1}\int_0^X Q'\,dx - \frac{\gamma}{\gamma - 1}p_t' x\right]. \tag{4.9}$$

Thus, $u_2' - u_1'$ is given by Eq. (4.5) only when X is large enough so that h_x' vanishes to our order of accuracy. That is, $kh_x'/c_p p_0 \ll u_1'$. There are, of course, two contributions to h_x' in the downstream region; that associated with the acoustic wave and that of the entropy wave. If the contribution of each wave to each side of this inequality is estimated, it is seen that only when the entropy wave has decayed essentially to extinction (this occurs in a very small fraction of an acoustic wave wave length) does the inequality hold. Thus, we must choose X downstream of the heater at such a distance that only the "isentropic" waves are still of appreciable magnitude and then Eqs. (4.6) and (4.7) constitute the boundary conditions on the upstream and downstream acoustic waves. Having determined the acoustic waves, one could, if it were of interest, use Eqs. (4.6) and (4.9) to find u' just at the rear of the heater and use this value together with u_1' to find the contribution of the entropy wave $\varphi^{(3)}$. With $\varphi^{(3)}$ so determined, the decaying density, temperature, and velocity, fields behind the heater are known.

Equations (4.4) and (4.5), applied to the contributions of $\varphi^{(1)}$ and $\varphi^{(2)}$ at $x = \sigma$ (to obtain algebraic simplicity without essential loss of accuracy), lead to the characteristic equation

$$\begin{vmatrix} \sin[k_1'(1-\sigma)] - iz_1 \exp[-ik_1'(1-\sigma)], & -\nu[\sin k_2'\sigma - iz_2 \exp(-ik_2'\sigma)] \\ \cos[k_1'(1-\sigma)] - z_1 \exp[-ik_1'(1-\sigma)], & (1+\lambda)[\cos k_2'\sigma - z_2 \exp(-ik_2'\sigma)] \end{vmatrix} = 0 \tag{4.10}$$

where

$$z_1 = 1 - |N_1| \exp[-k_1''(1-\sigma)], \qquad z_2 = 1 - |N_2| \exp(-k_2''\sigma)$$

This is a rather messy transcendental equation for α but it can be treated by noting that λ and z_i are small compared to unity (but larger than the contributions we have consistently omitted). For the case $\lambda = z_i = 0$, the characteristic equation contains only real contributions (for real α) and has a real eigenvalue, α. If $k_2 = k_1$, it reduces to $\sin k_1 = 0$ and the solution of major interest is $k_1 = \pi$. It is easy to compute (numerically) the eigenvalues for $a_2/a_1 = k_1/k_2 \neq 1$, and such a solution can be used as the basis of a perturbation calculation for α. Denoting the zero order values of k_1' and k_2' by k_{10} and k_{20} we let $k_i' = k_{i0} + k_{i1}$. The characteristic equation can now be linearized in the k_{i1} noting that $k_{11}/k_{21} = a_1/a_2 = \nu$, and the resulting equation has in general, a complex solution.

If we denote by μ_1, the imaginary part of k_{11}, we obtain

$$\begin{aligned}[(1-\sigma) + \nu^2\sigma)] \cos[k_{20}\sigma] \cos[k_{10}(1-\sigma)] - \nu \sin[k_{20}\sigma] \sin[k_{10}(1-\sigma)]\mu_1 \\ = -\mathcal{I}m(\lambda) \sin[k_{10}(1-\sigma)] \cos[k_{20}\sigma] - (z_2 + z_1) \sin[k_{10}(1-\sigma)] \sin[k_{20}\sigma] \\ - (z_1 + z_2) \cos[k_{10}(1-\sigma)] \cos[k_{20}\sigma],\end{aligned}$$

and a negative value of μ_1 implies a negative imaginary contribution to α. Thus, the tube will sing when $\mu_1 < 0$. It is consistent, of course, that $\mathcal{I}m(\lambda)$ is always negative and the z_1 are positive functions of real α. μ_1 will be less than zero for the lowest mode only when the bracket multiplying it is negative. This corresponds to having the heater in a prescribed lower portion of the tube which would be $\sigma < 1/2$ for $a_1 = a_2$ but which is $\sigma < K$ where $K < 1/2$ for $\alpha_1 > \alpha_2$. Together with this position requirement, a negative μ_1 also requires that the λ contribution to the right side of the equation be larger than the other terms. It is not a desirable task to tabulate the dependence of this criterion on the many parameters involved, but the relation of certain predictions to the observed facts can readily be discussed. For a velocity, u_0, of 1.7 ft/sec and a heater temperature of the order of 800°F, the prediction of the foregoing analysis is that $\mathcal{I}m(\lambda)$ is large enough so that the tube should sing for ribbon breadths of .08 cm or greater, but that a wire of less than this size should require a higher temperature. These facts were verified by experiments conducted by J. J. Bailey at the Harvard University Combustion Laboratory. He observed that ribbons of breadths 3/16″, 3/32″, 1/4″, etc., play well at such a temperature (in an apparatus with $L = 75$ cm, $l = 90$ cm, $R = 5$ cm), but that a 1/16″ wire needed such a large temperature that several wires were burned out during the experiments.

Many investigators (see the bibliography) have noted that the tube will sing at the fundamental only when $\sigma < L'/2$ (L' denotes a correction for temperature). Bailey, however, has successfully "played" the second harmonic by placing the ribbon in the range $1/2 < \sigma < 3/4$. The eigenvalues of the characteristic equation predict a growing wave at essentially the second harmonic frequency under these circumstances.

Another experimental fact not reported by other investigators is the following. For a given heater position and various ribbon temperatures (and, hence, various sound levels), the ratio of intensity of second and first harmonics has been recorded. This ratio increases (monotonically and in an experimentally repeatable manner) with increase in sound level. This is consistent with the qualitative discussion of the large amplitude heater response at the end of Sec. 3. There, it was noted that an increasing fraction of the heat release induced by the fluctuating velocity could be expected to appear at frequency 2ω as w/u_0 increased. This is indeed the case and may well be the major influence in deciding the sound level of the tube. The conjecture is this: as the sound level increases, more and more of the energy associated with $\mathcal{I}m\,\lambda$ is stolen by the second harmonic until the first harmonic contribution of $\mathcal{I}m(\lambda)$ is balanced by the z_i. This is a crude picture which could readily be made more precise by a detailed knowledge of the large w/u_0 heater response.

We have not presented a quantitative account of the Harvard experiments here since the instrumentation was carried only to an accuracy consistent with the foregoing description. However, this singing tube analysis would not have evolved to the present stage without the enthusiastic experimental accompaniment of J. J. Bailey.

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