the Cartesian co-ordinates of A_1 , A_2 and A_3 being (x_1, y_1) , (x_2, y_2) and (x_3, y_3) respectively, intersect at L . Let the lines H_3M and H_1M be the lines derived from the lines H_3L and H_1L by the procedure by which in Fig. 2 the line 12 was derived from the line 34. Let them intersect at the point $M(x_M, y_M)$. Using basic methods of analytical geometry we obtain for y_M

$$2y_M = \frac{k(x_3 - x_2)(x_2 - x_1)(x_1 - x_3) - [y_3^2(x_2 - x_1) + y_2^2(x_1 - x_3) + y_1^2(x_3 - x_2)]}{x_3(y_2 - y_1) + x_2(y_1 - y_3) + x_1(y_3 - y_2)} \quad (5)$$

and a similar formula for x_M . These formulae are symmetrical in the co-ordinates of A_1 , A_2 and A_3 , so that the line H_2M derived from the line H_2L by the same procedure passes through M . In other words, if the points 0, $0'$ and $0''$ in Fig. 2 are the mid-points of the sides of the triangle ABC , the line 02 for the side AB and the corresponding lines $0'2$ and $0''2$ for the sides BC and CA intersect in one point 2.

If (see Fig. 3) the ratio

$$Y_3 = \sigma \frac{H_3P_3}{A_1A_2} = \sigma \frac{y_M - (y_1 + y_2)/2}{x_2 - x_1} \quad (6)$$

that corresponds to the ratio $\sigma \cdot r_{01}/l_{AB}$ of Fig. 2 is computed, we obtain after some manipulating

$$2Y_3 = \frac{\sigma}{\cot \varphi_2 - \cot \varphi_1} + \frac{k\sigma}{\tan \varphi_1 - \tan \varphi_2}. \quad (7)$$

Equation (7) is identical with Eq. (2), as can easily be shown. Hence Eq. (2) represents the special case of Eq. (4), if the point 0 (Fig. 2) and the corresponding other two points $0'$ and $0''$ are the mid-points of the sides of the triangle ABC .

Tensor conductivities can be dealt with also by a change of variables [3]. This method leads to the same results.

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IMPEDANCE SYNTHESIS WITHOUT MUTUAL COUPLING*

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In a fundamental contribution to network theory, Bott and Duffin [1] have given a method for the synthesis of an impedance which obviates the use of any mutual coupling

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as was formerly required by the Brune realization technique. The purpose of this note is to present a modification of their procedure which results in fewer elements.

The Bott-Duffin method depends upon the result [3] that if $Z(p)$ is a positive real rational function (p.r.f.) of degree n , then

$$R(p) = \frac{pZ(p) - kZ(k)}{pZ(k) - kZ(p)}, \quad k > 0 \quad (1)$$

is again a p.r.f. of degree not greater than n . Using (1), they show that $Z(p)$ is realized by a balanced bridge whose opposite pairs of arms are $Z(k)R(p)$, $Z(k)/R(p)$ and $Z(k)p/k$, $kZ(k)/p$ respectively. If now it is assumed (after well-known preliminary reductions) that at $p = i\omega_0$, $\omega_0 > 0$, $Z(i\omega_0) = i\omega_0 L$, $L > 0$,¹ then by choosing $k > 0$ such that $Z(k)/k = L$ they obtain an $R(p)$ which has a pole at $p = \pm i\omega_0$, so that

$$R(p) = R_1(p) + \frac{\alpha p}{p^2 + \omega_0^2}, \quad \alpha > 0, \quad (2)$$

where $R_1(p)$ is a p.r.f. two degrees lower than $R(p)$. Thus their algorithm requires six elements and two reduced functions.

Our alternative synthesis of the above case utilizes an unbalanced bridge network Γ whose pairs of opposite arms are Z_1 , Z_4 and Z_2 , Z_3 respectively, and whose bridging arm Z_5 connects the common node of Z_1 and Z_2 and the common node of Z_3 and Z_4 . We shall assume that $Z_1 = W + S$, $Z_4 = 1/W$ where W and S are both p.r.f. If $Z'(p)$ is the impedance of Γ , then (see [2, pp. 284-285])

$$\begin{aligned} Z'(p) = & \{Z_3(Z_2 + Z_5)W^2 + [Z_3S(Z_2 + Z_5) + Z_5(Z_2Z_3 + 1) + Z_2 + Z_3 + Z_5]W \\ & + S(Z_2 + Z_3 + Z_5) + Z_2(Z_3 + Z_5)\} / \{(Z_2 + Z_5)W^2 + [S(Z_2 + Z_5) \\ & + Z_2Z_3 + Z_2Z_5 + Z_3Z_5 + 1]W + S + Z_3 + Z_5\} \end{aligned} \quad (3)$$

We will now show how to choose Z_2 , Z_3 , Z_5 , S and a reduced function W so that $Z'(p) = Z(p)$. It is desirable to impose the further condition that the right member of (3) reduce to a bilinear function of W . For this to be so, the resultant² of the two quadratic polynomials which are the numerator and denominator of the fraction in (3) must be zero. After some calculation this leads to the equation

$$S = \frac{(1 - Z_2Z_3)(Z_3 + Z_5)}{Z_3(Z_2 + Z_5)}. \quad (4)$$

The simplest reactive choices of the Z 's in (4) which cause the resistors in W and $1/W$ to be eliminated at the frequency $p = \pm i\omega_0$ and which are consistent with $Z(i\omega_0)/i > 0$ are

$$Z_2 = \frac{a\omega_0^2}{p}, \quad Z_3 = bp, \quad Z_5 = ap, \quad a > 0, b > 0;$$

which gives

$$S = \frac{(1 - ab\omega_0^2)(a + b)p}{ab(p^2 + \omega_0^2)},$$

¹If $L < 0$, one considers either $1/Z(p)$ or $Z(\omega_0^2/p)$ instead of $Z(p)$.

²The resultant of $a_0W^2 + a_1W + a_2$ and $a'_0W^2 + a'_1W + a'_2$ is $(a'_1a_2 - a_1a'_2) (a'_0a_1 - a_0a'_1) - (a_0a'_2 - a'_0a_2)^2$.

where a and b must be chosen so that $1 - ab\omega_0^2 > 0$. Substituting these values of the Z 's and S in (3), dividing out the common factor

$$W + \frac{Z_3 + Z_5}{Z_3(Z_2 + Z_5)}$$

and simplifying, we obtain

$$Z'(p) = \frac{abp(p^2 + \omega_0^2)W + (a + b - ab^2\omega_0^2)p^2 + a\omega_0^2}{a(p^2 + \omega_0^2)W + p(abp^2 + 1)}. \quad (5)$$

Now it follows from (1) and (2) that $Z(p)$ may be written in terms of the reduced function $R_1(p)$ as

$$Z(p) = Z(k) \cdot \frac{p(p^2 + \omega_0^2)R_1 + (\alpha + k)p^2 + k\omega_0^2}{k(p^2 + \omega_0^2)R_1 + p(p^2 + \alpha k + \omega_0^2)}. \quad (6)$$

A comparison of (5) and (6) shows that if we let $W = R_1$, $b = 1/k$, $a = k/(\alpha k + \omega_0^2)$, we obtain $Z'(p) = Z(p)/Z(k)$. Making this impedance level change in $Z'(p)$, we conclude that $Z(p)$ may be realized as an unbalanced bridge with

$$\begin{aligned} Z_1 &= W + S, & W &= Z(k)R_1(p), & S &= \frac{Z(k)(1 - ab\omega_0^2)(a + b)p}{ab(p^2 + \omega_0^2)}; \\ Z_2 &= a\omega_0^2 Z(k)/p, & Z_3 &= bZ(k)p, & Z_4 &= Z(k)/R_1(p), & Z_5 &= aZ(k)p, \\ & & a &= k/(\alpha k + \omega_0^2), & b &= 1/k. \end{aligned}$$

Thus our algorithm requires five elements and two reduced functions.

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ON TWO METHODS OF GENERATING SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS†

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1. Definite integrals with variable limits of integration. Consider the canonical form of the linear hyperbolic differential equation in two independent variables

$$L(u) \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \quad (1)$$

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