

determined. The method which essentially corresponds to the method of power series, used for ordinary differential equations, may be applied to more general kernels than those of the homogeneous polynomial type considered here. It may also be extended to other systems of linearly independent functions, such as trigonometric functions, depending on the construction of the kernel which in that case would have to involve a finite number of trigonometric terms.

The method should prove of great value in the treatment of vibration problems of systems with polynomial mass distributions and concentrated masses. Two of the simplest problems of this type, whose exact solutions have been known for a large number of years, have been solved by a closely related method in [1].

**5. Acknowledgment.** The author is indebted to Prof. A. Pfluger at the Eidgenössische Technische Hochschule, Zürich, Switzerland, for some valuable suggestions which greatly assisted the further development of the method.

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## ASYMMETRICAL FINITE DIFFERENCE NETWORK FOR TENSOR CONDUCTIVITIES\*

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In a paper recently published in this journal [1] the problem dealt with was described by its author in the following way: "Given a region in which Eq. (4) holds and a large number of points in the region chosen at random, in what way should the points be interconnected with 'physically realizable' electrical resistors in order that the voltages at the nodes shall be as nearly as possible the correct solutions of the boundary value problem characterized by Eq. (4) and appropriate boundary conditions?"

The Eq. (4) quoted is the special case of the differential equation

$$\nabla \cdot \{\sigma[\epsilon] \cdot \nabla \Phi\} + \tau = 0 \quad (1)$$

for  $[\epsilon]$  being the unit tensor, i.e., for isotropically conducting material. In Eq. (1)  $\Phi$  is the electrical potential in a two-dimensional continuum,  $\tau$  the current density of a distributed transverse external source,  $\sigma$  the conductivity of the continuum in a certain direction, and  $[\epsilon]$  the non-uniformity tensor of the conductivity. The scalars  $\tau$ ,  $\sigma$ , and the tensor  $[\epsilon]$  may be functions of the position.

The author of the present Note had been interested in a practically identical problem for which he had coined the expression "triangulation of a two-dimensional continuum" and which included the general case of tensor conductivities [2]. He replaced the interior

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of a triangle of any suitable shape and small size by a delta of resistors (see Fig. 1) so that, with the exception of the boundary, two resistors in parallel always connected a node pair in the final equivalent network. If the principal (maximum or minimum)

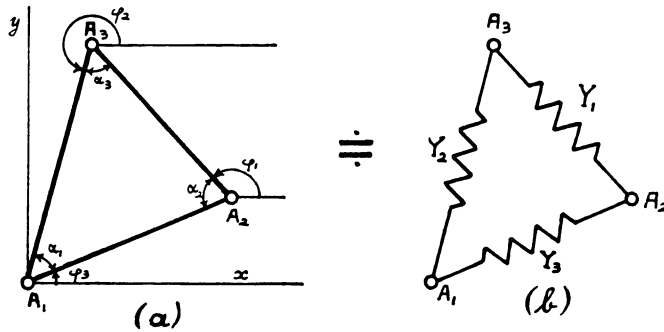


FIG. 1. Replacement of the interior of a triangle by a delta of resistors

conductivities are denoted by  $\sigma = \sigma_x$  in the  $x$ -direction and  $\sigma_y = k\sigma_x$  in the  $y$ -direction, so that

$$[\epsilon] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix},$$

the formula

$$2Y_3/\sigma_x = \frac{1+k}{2} \cdot \cot \alpha_3 + \frac{1-k}{2} \cdot \frac{\cos(\varphi_1 + \varphi_2)}{\sin \alpha_3} \tag{2}$$

(and similarly for  $Y_1$  and  $Y_2$ ) resulted, which for  $k = 1$ , i.e., isotropic materials, is identical with MacNeal's Eq. (10). With a proper lay-out of the nodes it can, for tensor conductivities also, be easily arranged that sides of triangles coincide with the boundaries and that all resulting conductances connecting two nodes are positive, i.e., physically realizable.

MacNeal's paper embodies two remarkable advances in the solution of the problem in question. One is finding that an infinite multitude of simulating networks exists for given node points, and the other is the precise definition of the part of the continuum to be lumped at a node as far as the distributed external current sources are concerned. These advances can be extended to tensor conductivities, quite frequently met with in engineering, along the line of MacNeal's method.

In Fig. 2, which corresponds to a part of Fig. 4 in MacNeal's paper, let  $A$  and  $B$  be two nodes of the simulating network and 1 and 2 the corresponding points of the dual network so that the current  $I_{AB}$  crossing the line 12 within the two-dimensional continuum divided by the potential difference  $V_{AB}$  between the points  $A$  and  $B$  determines the conductance  $Y_{AB}$  of the resistor joining  $A$  and  $B$ . Let the direction 12 be found from the direction  $AB$  in the following way: Draw the line 34 perpendicular to  $AB$ . Multiply the distance  $N4$  from the  $y$ -axis of a suitable point 4 of this line by  $1/k$ . Make  $Nx'$  parallel to  $Ox$  and  $N2$  equal to  $1/k \cdot N4$ . Using MacNeal's notation we obtain for  $I_{AB}$

$$I_{AB} = - \int_1^2 \sigma \, dr \, \mathbf{n} \cdot \{[\epsilon] \nabla \phi\} = - \int_1^2 \sigma \, dr \{ \mathbf{n}[\epsilon] \} \cdot \nabla \phi \cong - \sigma \cdot r_{34} \mid \nabla \phi \mid_0 \cos \alpha \tag{3}$$

so that

$$Y_{AB} = \sigma \frac{r_{34}}{l_{AB}} = \sigma \frac{r_{30}}{l_{AB}} + \sigma \frac{r_{04}}{l_{AB}}. \tag{4}$$

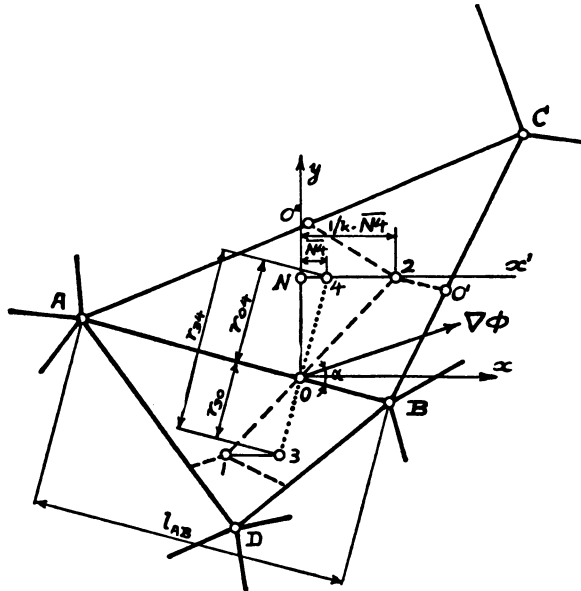


FIG. 2. Portion of the asymmetrical network of triangles.

This relation is general, whether O is the mid-point of AB or not. Equation 4 shows the splitting-up of  $Y_{AB}$  into two partial conductances corresponding to the interior of the two triangles ABC and ABD.

A closer investigation for O as the mid-point of AB is carried out in the notations of Fig. 3. The bisectors  $H_1L$ ,  $H_2L$  and  $H_3L$  of the sides of the triangle  $A_1A_2A_3$ , with

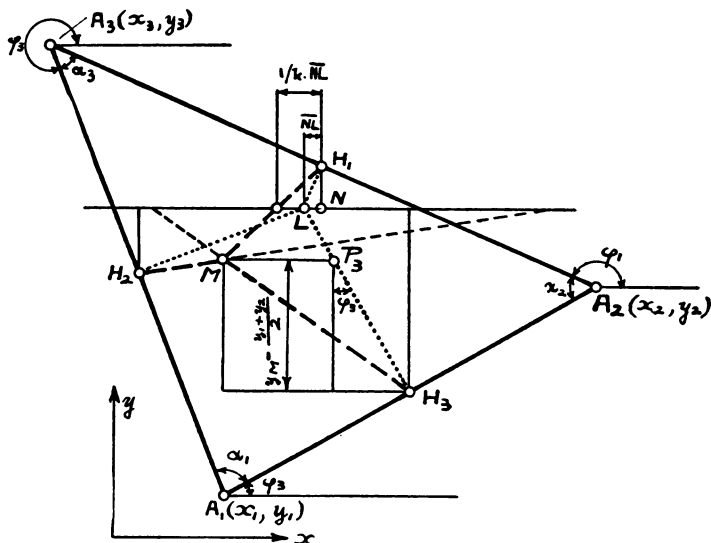


FIG. 3. Conditions for O, O', and O'' of Fig. 2 being the mid-points of AB, BC and CA respectively.

the Cartesian co-ordinates of  $A_1$ ,  $A_2$  and  $A_3$  being  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  respectively, intersect at  $L$ . Let the lines  $H_3M$  and  $H_1M$  be the lines derived from the lines  $H_3L$  and  $H_1L$  by the procedure by which in Fig. 2 the line 12 was derived from the line 34. Let them intersect at the point  $M(x_M, y_M)$ . Using basic methods of analytical geometry we obtain for  $y_M$

$$2y_M = \frac{k(x_3 - x_2)(x_2 - x_1)(x_1 - x_3) - [y_3^2(x_2 - x_1) + y_2^2(x_1 - x_3) + y_1^2(x_3 - x_2)]}{x_3(y_2 - y_1) + x_2(y_1 - y_3) + x_1(y_3 - y_2)} \quad (5)$$

and a similar formula for  $x_M$ . These formulae are symmetrical in the co-ordinates of  $A_1$ ,  $A_2$  and  $A_3$ , so that the line  $H_2M$  derived from the line  $H_2L$  by the same procedure passes through  $M$ . In other words, if the points  $0$ ,  $0'$  and  $0''$  in Fig. 2 are the mid-points of the sides of the triangle  $ABC$ , the line  $02$  for the side  $AB$  and the corresponding lines  $0'2$  and  $0''2$  for the sides  $BC$  and  $CA$  intersect in one point  $2$ .

If (see Fig. 3) the ratio

$$Y_3 = \sigma \frac{H_3P_3}{A_1A_2} = \sigma \frac{y_M - (y_1 + y_2)/2}{x_2 - x_1} \quad (6)$$

that corresponds to the ratio  $\sigma \cdot r_{01}/l_{AB}$  of Fig. 2 is computed, we obtain after some manipulating

$$2Y_3 = \frac{\sigma}{\cot \varphi_2 - \cot \varphi_1} + \frac{k\sigma}{\tan \varphi_1 - \tan \varphi_2} \quad (7)$$

Equation (7) is identical with Eq. (2), as can easily be shown. Hence Eq. (2) represents the special case of Eq. (4), if the point  $0$  (Fig. 2) and the corresponding other two points  $0'$  and  $0''$  are the mid-points of the sides of the triangle  $ABC$ .

Tensor conductivities can be dealt with also by a change of variables [3]. This method leads to the same results.

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### IMPEDANCE SYNTHESIS WITHOUT MUTUAL COUPLING\*

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In a fundamental contribution to network theory, Bott and Duffin [1] have given a method for the synthesis of an impedance which obviates the use of any mutual coupling

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