ON THE EVALUATION OF INDEFINITE INTEGRALS INVOLVING THE SPECIAL FUNCTIONS: APPLICATION OF METHOD*

BY
L. C. MAXIMON
Brown University

I. Introduction. As indicated at the end of the paper immediately preceding this one in this issue¹, the procedure developed there for the evaluation of indefinite integrals will be extended here by permitting a more flexible choice of the functions Y_1 and y_0 and by making use of the recurrence relations satisfied by these functions to derive recurrence relations for integrals, from which these integrals may then be evaluated.

II. Associated Legendre functions. Let us choose $Y_1 = P_{\nu_1}^{\mu}(x)$, a solution of Legendre's equation:

$$(1 - x^2)Y'' - 2xY' + \left[\nu_1(\nu_1 + 1) - \frac{\mu^2}{1 - x^2} \right]Y = 0 \quad (1)$$

and let $y_0 = x^{\lambda}P_{\nu_2}^{\mu}(x)$. Then from Eq. (30) of [A] we have, making use of the differential equation satisfied by $P_{\nu_2}^{\mu}(x)$,

$$\int P_{\nu_1}^{\mu} \{ 2\lambda(1 - x^2)x^{\lambda-1}(P_{\nu_2}^{\mu})' + \lambda(\lambda - 1)x^{\lambda-2}P_{\nu_2}^{\mu} + [\nu_1(\nu_1 + 1) - \nu_2(\nu_2 + 1) - \lambda(\lambda + 1)]x^{\lambda}P_{\nu_2}^{\mu} \} dx = (1 - x^2) [P_{\nu_1}^{\mu}(x^{\lambda}P_{\nu_2}^{\mu})' - x^{\lambda}P_{\nu_1}^{\mu}(P_{\nu_2}^{\mu})']. \quad (2)$$

Using the recurrence relation

$$(1 - x^2)(P_{\nu}^{\mu})' = -\nu x P_{\nu}^{\mu} + (\nu + \mu)P_{\nu-1}^{\mu}, \quad (3)$$

Eq. (2) may be written in the form of a recurrence relation between integrals:

$$\begin{aligned} & [\nu_1(\nu_1 + 1) - (\nu_2 + \lambda)(\nu_2 + \lambda + 1)] \int x^{\lambda} P_{\nu_1}^{\mu} P_{\nu_2}^{\mu} dx \\ & + 2\lambda(\nu_2 + \mu) \int x^{\lambda-1} P_{\nu_1}^{\mu} P_{\nu_2-1}^{\mu} dx + \lambda(\lambda - 1) \int x^{\lambda-2} P_{\nu_1}^{\mu} P_{\nu_2}^{\mu} dx \\ & = x^{\lambda} [(\nu_1 - \nu_2 - \lambda)x P_{\nu_1}^{\mu} P_{\nu_2}^{\mu} + (\nu_2 + \mu)P_{\nu_1}^{\mu} P_{\nu_2-1}^{\mu} - (\nu_1 + \mu)P_{\nu_1-1}^{\mu} P_{\nu_2}^{\mu} \\ & + \lambda x^{-1} P_{\nu_1}^{\mu} P_{\nu_2}^{\mu}]. \end{aligned} \quad (4)$$

In Eq. (4) we may let $\lambda = 0, 1, 2, \dots$ successively and thus obtain

$$\int x^n P_{\nu_1}^{\mu} P_{\nu_2}^{\mu} dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (5)$$

provided that none of the denominators in the resulting expressions vanishes. This can be shown to be the case for arbitrary μ if

$$\begin{aligned} \nu_1 - \nu_2 \neq n, \quad n - 2, \quad n - 4, \dots - (n - 2), -n \\ \nu_1 + \nu_2 \neq n - 1, \quad n - 3, \dots - (n - 1), -(n + 1). \end{aligned} \quad (6)$$

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¹L. C. Maximon and G. W. Morgan, *On the evaluation of indefinite integrals involving the special functions: development of method*, quoted as [A] in the following.

Seven additional recurrence formulas for the integrals (5), each also requiring condition (6), are obtained directly from Eq. (4), from the possible combinations of (a) interchanging the subscripts ν_1 and ν_2 in the Legendre functions in Eq. (4), and (b) substituting $-\nu_i - 1$ for ν_i and noting that $P_{\nu_i}^{\mu} = P_{-\nu_i-1}^{\mu}$, $i = 1, 2$. It may be noted that if any one of the numbers ν_1 , $-\nu_1 - 1$, ν_2 , $-\nu_2 - 1$, is equal to $-\mu$, then in two of these formulas the second integral in Eq. (4) will not appear and a particularly simple recurrence formula follows. In this case, if $-\mu = \nu_1$ or $-\nu_1 - 1$ then none of the denominators will vanish, provided

$$\mu + \nu_2, \quad \mu - \nu_2 - 1 \neq n, \quad n = 2, \dots,$$

and if $-\mu = \nu_2$ or $-\nu_2 - 1$, none of the denominators vanishes provided

$$\mu + \nu_1, \quad \mu - \nu_1 - 1 \neq n, \quad n = 2, \dots,$$

the above series terminating in each case with 0 or 1 depending on whether n is even or odd.

A more symmetric recurrence relation for the integrals (5), involving only x^{λ} and $x^{\lambda-1}$ in the integrand, may be obtained by interchanging ν_1 and ν_2 in Eq. (4) and subtracting the result from Eq. (4), giving

$$\begin{aligned} & [\nu_1(\nu_1 + \lambda + 1) - \nu_2(\nu_2 + \lambda + 1)] \int x^{\lambda} P_{\nu_1}^{\mu} P_{\nu_2}^{\mu} dx \\ & + \lambda(\nu_2 + \mu) \int x^{\lambda-1} P_{\nu_1}^{\mu} P_{\nu_2-1}^{\mu} dx - \lambda(\nu_1 + \mu) \int x^{\lambda-1} P_{\nu_2}^{\mu} P_{\nu_1-1}^{\mu} dx \\ & = x^{\lambda} [x(\nu_1 - \nu_2) P_{\nu_1}^{\mu} P_{\nu_2}^{\mu} + (\nu_2 + \mu) P_{\nu_1}^{\mu} P_{\nu_2-1}^{\mu} - (\nu_1 + \mu) P_{\nu_2}^{\mu} P_{\nu_1-1}^{\mu}]. \end{aligned} \quad (7)$$

A similar equation follows upon substituting $-\nu_i - 1$ for ν_i and noting $P_{\nu_i}^{\mu} = P_{-\nu_i-1}^{\mu}$ ($i = 1, 2$), in each term in Eq. (7). In this latter equation as well as in Eq. (7), however, the condition that the denominators do not vanish is, for arbitrary μ ,

$$\begin{aligned} \nu_1 - \nu_2 & \neq n, \quad n = 1, \quad n = 2, \dots, -n \\ \nu_1 + \nu_2 & \neq n - 1, \quad n = 3, \dots, -(n - 1), -(n + 1) \end{aligned} \quad (8)$$

rather than (6).

III. Confluent hypergeometric function. As a second example of the type of integral which may be evaluated by this procedure we may choose $Y_1 = F(a_1; c; x)$, a solution of

$$xY'' + (c - x)Y' - a_1Y = 0 \quad (9)$$

and let $y_0 = x^{\lambda}F(a_2; c; x)$. Then from Eq. (30) of [A] we have, using the differential equation satisfied by $F(a_2; c; x)$,

$$\begin{aligned} & \int e^{-x} x^{c-1} F(a_1; c; x) [2\lambda x^{\lambda} F'(a_2; c; x) + \lambda(\lambda + c - 1)x^{\lambda-1} F(a_2; c; x) \\ & + (a_2 - a_1 - \lambda)x^{\lambda} F(a_2; c; x)] dx \\ & = e^{-x} x^c \{ F(a_1; c; x) [x^{\lambda} F(a_2; c; x)]' - x^{\lambda} F(a_2; c; x) F'(a_1; c; x) \}. \end{aligned} \quad (10)$$

Using the recurrence relation

$$xF'(a; c; x) = aF(a + 1; c; x) - aF(a; c; x), \quad (11)$$

Eq. (10) may be written in the form

$$\begin{aligned}
 & (a_2 - a_1 - \lambda) \int e^{-x} x^{c+\lambda-1} F(a_1; c; x) F(a_2; c; x) dx \\
 & + \lambda(\lambda - 2a_2 + c - 1) \int e^{-x} x^{c+\lambda-2} F(a_1; c; x) F(a_2; c; x) dx \\
 & + 2\lambda a_2 \int e^{-x} x^{c+\lambda-2} F(a_1; c; x) F(a_2 + 1; c; x) dx \\
 & = e^{-x} x^{c+\lambda-1} \left[\begin{aligned} & (a_1 - a_2 - \lambda) F(a_1; c; x) F(a_2; c; x) \\ & + a_2 F(a_1; c; x) F(a_2 + 1; c; x) \\ & - a_1 F(a_2; c; x) F(a_1 + 1; c; x) \end{aligned} \right].
 \end{aligned} \tag{12}$$

In Eq. (12) we may let $\lambda = 0, 1, 2, \dots$ successively and thus obtain

$$\int e^{-x} x^{c+n-1} F(a_1; c; x) F(a_2; c; x) dx \quad \text{for } n = 0, 1, 2, \dots \tag{13}$$

provided that none of the resulting denominators is zero, which will be the case if

$$|a_2 - a_1| \neq 0, 1, 2, \dots n. \tag{14}$$

Here also, a more symmetric recurrence relation for the integrals (13) may be obtained by interchanging v_1 and v_2 in Eq. (12) and subtracting the result from Eq. (12), giving:

$$\begin{aligned}
 & (a_2 - a_1) \int e^{-x} x^{c+\lambda-1} F(a_1; c; x) F(a_2; c; x) dx \\
 & + \lambda(a_1 - a_2) \int e^{-x} x^{c+\lambda-2} F(a_1; c; x) F(a_2; c; x) dx \\
 & + \lambda a_2 \int e^{-x} x^{c+\lambda-2} F(a_1; c; x) F(a_2 + 1; c; x) dx \\
 & - \lambda a_1 \int e^{-x} x^{c+\lambda-2} F(a_2; c; x) F(a_1 + 1; c; x) dx \\
 & = e^{-x} x^{c+\lambda-1} \left[\begin{aligned} & (a_1 - a_2) F(a_1; c; x) F(a_2; c; x) \\ & + a_2 F(a_1; c; x) F(a_2 + 1; c; x) \\ & - a_1 F(a_2; c; x) F(a_1 + 1; c; x) \end{aligned} \right]
 \end{aligned} \tag{15}$$

from which the integrals (13) may be obtained by setting $\lambda = 0, 1, 2, \dots$ provided the condition given in (14) is satisfied.

Two particular cases which may be obtained immediately from Eqs. (10) and (12) may be noted:

1. If we set $a_2 = 0, a_1 = a$ in Eq. (12), then, since $F(0; c; x) = 1$ ($c \neq 0, -1, -2, \dots$),

we have

$$(a + \lambda) \int e^{-x} x^{c+\lambda-1} F(a; c; x) dx - \lambda(\lambda + c - 1) \int e^{-x} x^{c+\lambda-2} F(a; c; x) dx \quad (16)$$

$$= -e^{-x} x^{c+\lambda-1} [(a + \lambda) F(a; c; x) - a F(a + 1; c; x)]$$

(a). Setting $\lambda = 0, 1, 2, \dots$ successively in Eq. (16) we may obtain

$$\int e^{-x} x^{c+n-1} F(a; c; x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (17)$$

provided that $a \neq 0, -1, -2, \dots -n$. (It should be noted that for $a = 0, -1, -2, \dots -n$, $F(a; c; x)$ is a polynomial and hence the integrals (17) are elementary.)

(b). On the other hand, setting $\lambda = 1 - c, 2 - c, \dots$ successively in Eq. (16) we obtain

$$\int e^{-x} x^n F(a; c; x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (18)$$

provided $c - a - 1 \neq 0, 1, 2, \dots n$.

(c). Finally, setting $\lambda = -a, -a - 1, -a - 2, \dots$ successively in Eq. (16) we obtain

$$\int e^{-x} x^{c-a-2-n} F(a; c; x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (19)$$

provided $a \neq 0, -1, -2, \dots, -(n - 1)$ and $c - a - 1 \neq 0, 1, 2, \dots n$.

2. The other particular case to be noted may be obtained by setting $a_2 = c, a_1 = a$ in Eq. (10) and noting that $F(c; c; x) = e^x$. We then obtain

$$(c - a + \lambda) \int x^{c+\lambda-1} F(a; c; x) dx + \lambda(\lambda + c - 1) \int x^{c+\lambda-2} F(a; c; x) dx \quad (20)$$

$$= x^{c+\lambda-1} [(a + \lambda + x) F(a; c; x) - a F(a + 1; c; x)].$$

(a). Setting $\lambda = 0, 1, 2, \dots$ successively in Eq. (20) we may obtain

$$\int x^{c+n-1} F(a; c; x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (21)$$

provided $a - c \neq 0, 1, 2, \dots n$.

(b). Setting $\lambda = 1 - c, 2 - c, \dots$ successively we obtain

$$\int x^n F(a; c; x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (22)$$

provided $a \neq 1, 2, \dots n + 1$.

(c). Finally, setting $\lambda = a - c, a - c - 1, \dots$ successively we obtain

$$\int x^{a-2-n} F(a; c; x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (23)$$

provided $a - c + 1 \neq 1, 2, \dots n$ and $a \neq 1, 2, \dots n$.

IV. Bessel functions. As a final illustration we give a number of indefinite integrals involving the Bessel functions. Some of these do not appear in the literature; others that are to be found in the literature appear here as special cases.

Choosing $Y_1 = J_\mu(\alpha x)$, a solution of

$$x^2 Y'' + x Y' + (\alpha^2 x^2 - \mu^2) Y = 0 \quad (24)$$

and $y_0 = x^\lambda J_\nu(\beta x)$ which satisfies

$$x^2 y'' + (1 - 2\lambda)xy' + (\beta^2 x^2 + \lambda^2 - \nu^2)y = 0 \quad (25)$$

we have directly from Eq. (28) of [A],

$$\begin{aligned} \int x^{-1} J_\mu(\alpha x) \{2\lambda x [x^\lambda J_\nu(\beta x)]' + [(\alpha^2 - \beta^2)x^2 + \nu^2 - \mu^2 - \lambda^2] x^\lambda J_\nu(\beta x)\} dx \\ = x \{J_\mu(\alpha x) [x^\lambda J_\nu(\beta x)]' - x^\lambda J_\nu(\beta x) J'_\mu(\alpha x)\}. \end{aligned} \quad (26)$$

Here primes denote differentiation with respect to x , throughout. Using the recurrence relation

$$xJ'_\nu = -\nu J_\nu + xJ_{\nu-1} \quad (27)$$

Eq. (26) gives the recurrence formula

$$\begin{aligned} \int x^{\lambda-1} J_\mu(\alpha x) \{[(\alpha^2 - \beta^2)x^2 + (\nu - \lambda)^2 - \mu^2] J_\nu(\beta x) + 2\lambda \beta x J_{\nu-1}(\beta x)\} dx \\ = x^\lambda [(\lambda - \nu + \mu) J_\mu(\alpha x) J_\nu(\beta x) + \beta x J_\mu(\alpha x) J_{\nu-1}(\beta x) - \alpha x J_{\mu-1}(\alpha x) J_\nu(\beta x)]. \end{aligned} \quad (28)$$

On the other hand, substitution of the recurrence relation

$$xJ'_\nu = \nu J_\nu - xJ_{\nu+1} \quad (29)$$

in Eq. (26) gives²

$$\begin{aligned} \int x^{\lambda-1} J_\mu(\alpha x) \{[(\alpha^2 - \beta^2)x^2 + (\nu + \lambda)^2 - \mu^2] J_\nu(\beta x) - 2\lambda \beta x J_{\nu+1}(\beta x)\} dx \\ = x^\lambda [(\lambda + \nu - \mu) J_\mu(\alpha x) J_\nu(\beta x) - \beta x J_\mu(\alpha x) J_{\nu+1}(\beta x) + \alpha x J_{\mu+1}(\alpha x) J_\nu(\beta x)]. \end{aligned} \quad (30)$$

Several interesting particular cases follow directly from Eqs. (28) and (30)³.

1. Setting $\nu = \beta = 0$, $\alpha = 1$ in Eq. (28) we have

$$\int x^{\lambda+1} J_\mu dx + (\lambda^2 - \mu^2) \int x^{\lambda-1} J_\mu dx = x^\lambda [(\lambda + \mu) J_\mu - x J_{\mu-1}] \quad (31)$$

and setting $\lambda = \pm\mu, \pm\mu + 2, \pm\mu + 4, \dots$ successively in Eq. (31) we may evaluate

$$\int x^{\pm\mu+2n+1} J_\mu dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (32)$$

2. (a). Setting

$$\lambda - \nu = \pm\mu \quad (33)$$

in Eq. (28) gives

$$\begin{aligned} (\alpha^2 - \beta^2) \int x^{\lambda+1} J_\mu(\alpha x) J_\nu(\beta x) dx + 2\lambda \beta \int x^\lambda J_\mu(\alpha x) J_{\nu-1}(\beta x) dx \\ = x^\lambda [(\lambda - \nu + \mu) J_\mu(\alpha x) J_\nu(\beta x) + \beta x J_\mu(\alpha x) J_{\nu-1}(\beta x) - \alpha x J_{\mu-1}(\alpha x) J_\nu(\beta x)]. \end{aligned} \quad (34)$$

²Given in Ref. [1], p. 134, Eq. (7) for the particular case $\lambda = 0$.

³Ref. [1], §5.1, p. 133 Eq. (4).

If $\alpha^2 \neq \beta^2$ and we let $\lambda = 0, 1, 2, \dots$ successively in Eq. (34) we obtain

$$\int x^{n+1} J_\mu(\alpha x) J_{\mu+n}(\beta x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (35)$$

(b). If we let

$$\lambda + \nu = \pm \mu \quad (36)$$

in Eq. (30), we obtain

$$\begin{aligned} (\alpha^2 - \beta^2) \int x^{\lambda+1} J_\mu(\alpha x) J_\nu(\beta x) dx - 2\lambda\beta \int x^\lambda J_\mu(\alpha x) J_{\nu+1}(\beta x) dx \\ = x^\lambda [(\lambda + \nu - \mu) J_\mu(\alpha x) J_\nu(\beta x) - \beta x J_\mu(\alpha x) J_{\nu+1}(\beta x) + \alpha x J_{\mu+1}(\alpha x) J_\nu(\beta x)]. \end{aligned} \quad (37)$$

If $\alpha^2 \neq \beta^2$ and we let $\lambda = 0, 1, 2, \dots$ successively in Eq. (37) we obtain

$$\int x^{n+1} J_\mu(\alpha x) J_{\mu-n}(\beta x) dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (38)$$

The integrals (35) and (38) for $\alpha^2 = \beta^2$ may be found from the results for $\alpha^2 \neq \beta^2$ by the use of l'Hospital's rule. They are also expressible in terms of Bessel functions.

3. Setting $\alpha = \beta = 1$ in Eqs. (28) and (30) gives

$$\begin{aligned} [(\nu - \lambda)^2 - \mu^2] \int x^{\lambda-1} J_\mu J_\nu dx + 2\lambda \int x^\lambda J_\mu J_{\nu-1} dx \\ = x^\lambda [(\lambda - \nu + \mu) J_\mu J_\nu + x J_\mu J_{\nu-1} - x J_{\mu-1} J_\nu], \end{aligned} \quad (39)$$

and

$$\begin{aligned} [(\nu + \lambda)^2 - \mu^2] \int x^{\lambda-1} J_\mu J_\nu dx - 2\lambda \int x^\lambda J_\mu J_{\nu+1} dx \\ = x^\lambda [(\lambda + \nu - \mu) J_\mu J_\nu - x J_\mu J_{\nu+1} + x J_{\mu+1} J_\nu], \end{aligned} \quad (40)$$

respectively.

(a). If we let $\lambda = 0, -1, -2, \dots$ successively in Eq. (39) or (40) we obtain

$$\int x^{-n-1} J_\mu J_\nu dx \quad \text{for} \quad n = 0, 1, 2, \dots^4 \quad (41)$$

provided

$$\mu + \nu, \quad \mu - \nu \neq n, \quad n = 2, n = 4, \dots, -n. \quad (42)$$

When one of the restrictions (42) is not satisfied, the integrals in (41) may be evaluated by applying l'Hospital's rule to the results when the inequalities are satisfied and are expressed in terms of the functions

$$F_\nu(x) \equiv \frac{\partial J_\mu}{\partial \mu} \Big|_{\mu=\nu}, \quad G_\nu(x) \equiv \frac{\partial^2 J_\mu}{\partial \mu^2} \Big|_{\mu=\nu}. \quad (43)$$

⁴Evaluated in Ref. [1], p. 135, Eq. (13) for the particular case $n = 0$, and for integer μ, ν and arbitrary integer n in Ref. [5].

Actually, the integrals (41) will not depend on G , unless the factors multiplying the first integral in Eqs. (39) and (40) have a double zero, i.e. unless $\mu = 0$. If, however, $\mu = 0$ but $\nu \neq 0$ then the integrals (41) may still be expressed independently of G , since they may be determined from the recurrence relations obtained by interchanging μ and ν in Eqs. (39) and (40). Thus, only if $\mu = \nu = 0$ will the integrals (41) depend on G , and in this case they will depend only on G_0 .

If we consider the integrals (41) for integer values of μ, ν , then we may note that differentiation with respect to ν of the differential equation for J , :

$$x^2 J'' + x J' + (x^2 - \nu^2) J = 0 \quad (44)$$

gives

$$x^2 F'' + x F' + (x^2 - \nu^2) F = 2\nu J, \quad (45)$$

and therefore $F_0(x)$ may be expressed as a linear combination of the Bessel functions $J_0(x)$ and $Y_0(x)$.⁵ Further, from Eqs. (27) and (29), we have upon differentiation with respect to ν ,

$$x F'_\nu = \nu F_\nu - x F_{\nu+1} + J_\nu \quad (46)$$

and

$$x F'_\nu = -\nu F_\nu + x F_{\nu-1} - J_\nu \quad (47)$$

respectively, and thus by induction $F_n(x)$ may be expressed in terms of Bessel functions for any integer n . Thus for any integer values of μ, ν , the integrals (41) may be expressed in terms of Bessel functions, unless $\mu = \nu = 0$ (in which case G_0 is involved) and the inequalities (42) are not satisfied (i.e. $n = 0, 2, 4, \dots$)⁶. It may also be noted from all this that for arbitrary μ, ν and regardless of whether or not (42) is satisfied, the integrals (41) may be expressed in terms of the solutions of the inhomogeneous differential equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 2k J_\nu(x), \quad (48)$$

F being a solution for $k = \nu$, G_0 a solution for $\nu = 0$, $k = 1$.

(b). If we let $\lambda = \nu \pm \mu$, $\nu \pm \mu + 2$, $\nu \pm \mu + 4$, $\dots$ successively in Eq. (39) we obtain by recurrence

$$\int x^{\nu+\mu+2n} J_\mu J_{\nu-1} dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (49)$$

provided

$$\nu \pm \mu \neq -n, -n-1, -n-2, \dots -2n. \quad (50)$$

Similarly, if we let $\lambda = -\nu \pm \mu$, $-\nu \pm \mu + 2$, $-\nu \pm \mu + 4$, $\dots$ successively in Eq. (40) we obtain

$$\int x^{-\nu+\mu+2n} J_\mu J_{\nu+1} dx \quad \text{for} \quad n = 0, 1, 2, \dots \quad (51)$$

⁵In fact, $F_0 = \pi Y_0/2$ (Ref. [1], p. 59, Eq. (5) and p. 64 Eq. (2)).

⁶An evaluation for the particular case $\mu = \nu = 1, 2, 3, \dots$ and $n = 0$ is given in Ref. [1], p. 137, Eq. (6) where it is noted that "no simple formula" seems to exist for the case $\mu = \nu = n = 0$.

provided

$$-\nu \pm \mu \neq -n, -n-1, -n-2, \dots -2n. \quad (52)$$

With a slight change of notation the integrals (49) and (51) are expressible in the form

$$\int x^{\alpha+\mu+2n+1} J_{\mu} J_{\nu} dx \quad n = 0, 1, 2, \dots. \quad (53)$$

It is worth recalling that in each of the integrals evaluated thus far we may substitute any solution of the differential equation being considered for the particular one that has been chosen here, provided only that the recurrence relation for this other solution is also substituted. Further, in connection with the integrals involving Bessel functions, since the differential equations and recurrence relations satisfied by I , and K , differ trivially from those satisfied by J , we may also obtain integrals of the form given in (35), (38), (41) and (53) with I or K substituted for J . One particular case is worthy of note in view of the attention it has received in the literature⁸. Consider the integral similar to (53), with J_{μ} replaced by K_{μ} and J_{ν} replaced by either K , or I . If we set $\mu = 1/2$ and note that, apart from a constant factor, $K_{1/2} = x^{-1/2}e^{-x}$, then we obtain

$$\int x^{\alpha+\nu} e^{-x} K_{\nu}(x) dx \quad \text{for } n = 0, 1, 2, \dots \quad (54)$$

and

$$\int x^{\alpha+\nu} e^{-x} I_{\nu}(x) dx \quad \text{for } n = 0, 1, 2, \dots, \quad (55)$$

the choice $+\mu$ in (53) giving the integrals (54) and (55) with n odd, $-\mu$ giving those with n even. If, on the other hand, we had set $\alpha = -1, \beta = 1$ in obtaining Eqs. (39) and (40) from Eqs. (28) and (30), then following the procedure used in deriving (54) and (55) we would have obtained

$$\int x^{\alpha+\nu} e^x K_{\nu}(x) dx \quad \text{for } n = 0, 1, 2, \dots \quad (56)$$

and

$$\int x^{\alpha+\nu} e^x I_{\nu}(x) dx \quad \text{for } n = 0, 1, 2, \dots \quad (57)$$

Similar integrals, with the exponential replaced by sine or cosine and K and I replaced by J and Y may be obtained directly from (53) by setting $\mu = 1/2$.

V. Extension to solutions of equations of arbitrary order. The procedure which has been used to evaluate in closed form integrals involving functions which are solutions of second order linear homogeneous differential equations may be extended to functions which are solutions of n th order equations quite simply. Equations (19) and (20) of [A] now become

$$\sum_{i=0}^n \left[a_i(x) + b_i(x) \frac{\partial}{\partial t} \right] \frac{\partial^i y}{\partial x^i} = 0 \quad (58)$$

⁷Given in Ref. [1], p. 136, Eqs. (1) and (2) for $n = 0$.

⁸The integrals (54) to (57) are given in Ref. [2] where it is noted that their closed form evaluations seems to have been overlooked in Ref. [4].

and

$$\sum_{i=0}^n [a_i(x) + sb_i(x)] \frac{d^i Y}{dx^i} = \sum_{i=0}^n b_i(x) \frac{\partial^i y(x, 0)}{\partial x^i} \equiv K(x), \quad (59)$$

where Y is defined in Eq. (21) of [A]. Again using variation of parameters we may write in place of Eq. (4) of [A]

$$Y = \frac{1}{s} \sum Y_i I_i, \quad (60)$$

where the Y_i are linearly independent solutions of

$$\sum_{i=0}^n c_i(x) \frac{d^i Y}{dx^i} = 0, \quad (61)$$

$$c_i(x) = a_i(x) + sb_i(x), \quad (62)$$

$$I_i = s \int \frac{\Delta_i K(x)}{c_n W} dx, \quad (63)$$

$$W = \begin{vmatrix} Y_1 & Y_2 & \cdots & Y_n \\ Y'_1 & Y'_2 & \cdots & Y'_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ Y_1^{(n-1)} & Y_2^{(n-2)} & \cdots & Y_n^{(n-1)} \end{vmatrix} \quad (64)$$

$$= \text{const.} \times \exp \left[- \int \frac{c_{n-1}}{c_n} dx \right],$$

where

$$\Delta_i = \text{cofactor of } Y_i^{(n-1)} \text{ in the determinant for } W \text{ in Eq. (64), for } n > 1, \text{ and} \quad (65)$$

$$\Delta_i \equiv 1 \text{ for } n = 1.$$

Again, as with the second order equation, we assume that the boundary and initial conditions are such that $y(x, t) = y(x, 0) \equiv y_0(x)$ so that

$$\sum_{i=0}^n a_i(x) \frac{d^i y_0}{dx^i} = 0 \quad (66)$$

and

$$Y = \frac{1}{s} y_0(x). \quad (67)$$

The integrals I_i may now be determined from the n simultaneous equations which are obtained by substituting Eq. (67) in Eq. (60) and differentiating:

$$\sum_{i=1}^n Y_i^{(k)} I_i = y_0^{(k)} \quad k = 0, 1, 2, \cdots, n-1. \quad (68)$$

In Eq. (68) no derivatives of I_i appear in view of the equations

$$\sum_{i=1}^n Y_i^{(k)} I_i' = 0 \quad k = 0, 1, 2, \dots, n-2, \quad (69)$$

which may be verified directly from Eqs. (63), (64), (65). From Eq. (68) we may solve for the n integrals I_i . We need consider only I_1 , however, since the solutions for the other integrals constitute merely a renumbering of the solutions Y_i . The determinant for the I_i is W , as comparison with Eq. (64) shows. We have, therefore,

$$I_1 = W^{-1} D_0, \quad (70)$$

where

$$D_0 = \text{the determinant obtained from } W, \text{ Eq. (64), by replacing } Y_1 \text{ by } y_0. \quad (71)$$

From Eq. (7) of [A] and Eqs. (59) and (66) we have

$$sK(x) = \sum_{i=0}^n (c_i - a_i) \frac{d^i y_0}{dx^i} = \sum_{i=0}^n c_i \frac{d^i y_0}{dx^i} \quad (72)$$

and hence from Eqs. (63), (64), (70) and (72) we may write

$$\int c_n^{-1} \exp \left[\int \frac{c_{n-1}}{c_n} dx \right] \Delta_1 \sum_{i=0}^n (c_i - a_i) \frac{d^i y_0}{dx^i} dx = \exp \left[\int \frac{c_{n-1}}{c_n} dx \right] D_0 \quad (73)$$

which is the generalization of Eq. (28) of [A], or a similar equation with $c_i - a_i$ replaced by c_i , which is the generalization of Eq. (30) of [A], and no longer requires that y_0 satisfy Eq. (66).

It is seen that the idea developed in [A], combined with the extension presented in this paper, constitutes a simple and unified method for the evaluation in closed form of a large number of integrals involving the special functions.

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