in which C and C' are real and positive, is the flow across a rectangular grid. The cross sections of the bars far from the intersections are roughly circular, particularly if b is large compared with C, and d is large compared with C'.

The flow corresponding to

$$F(\alpha) = C\alpha^2, \qquad G(\beta) = D\beta^2, \qquad H(\gamma) = E\gamma^2,$$

is one for which the coordinate planes are not pierced by streamlines and are therefore "streamline surfaces" perpendicular to equi-potential surfaces.

Three-dimensional flows constructed by the present method belong to a special class, of which two-dimensional flows form a sub-class. Axisymmetric flows, however, do not belong to this class.

The reviewer of the original manuscript of this paper suggested that the expressions of the velocity components in terms of the complex conjugates of f, g, and h be also given. If the latter are denoted by f', g', and h', then the Cauchy-Riemann equations are

$$f_x = f'_y, \quad f_y = -f'_x; \quad g_y = g'_z, \quad g_z = -g'_y; \quad h_z = h'_x, \quad h_x = -h'_z \tag{6}$$

and one has, from Eqs. (1), (4), and (6):

$$u = h'_z - f'_y, \quad v = f'_x - g'_z, \quad w = g'_y - h'_x. \tag{7}$$

In vector form, Eq. (6) can be written as

$$\mathbf{V} = -\operatorname{curl} \mathbf{A}$$

in which

$$\mathbf{A} = \mathbf{i}g' + \mathbf{j}h' + \mathbf{k}f'$$

($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ being unit vectors along the coordinate axes) is the vector potential for the flow for which ϕ is the scalar potential.

It may be remarked without further discussion that two-dimensional potential flows which are not parallel to mutually perpendicular planes can also be superimposed to form a three-dimensional potential flow.

A NOTE ON THE MEAN VALUE OF RANDOM DETERMINANTS*

By RICHARD BELLMAN (*The Rand Corporation*)

1. Introduction. In a recent paper, [1], Nyquist, Rice and Riordan discussed the problem of determining the expected values of powers of a random determinant. Here a random determinant, D_n, is defined to be

$$D_n = |x_{ij}|, \qquad i, j, = 1, 2, \cdots, n,$$

where the x_{ij} are independent random variables.

The purpose of the present note is to give an explicit representation for $E(D_n^k)$ in terms of the characteristic functions of the x_{ij}. These need not be identical.

At the moment we are merely interested in presenting an expression which will

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yield a systematic technique for obtaining the moments numerically. In a subsequent paper devoted to various theoretical aspects such as asymptotic behavior we shall discuss the problem in greater detail. For the case of identical distributions, the problem is closely connected with the study of invariants of the symmetric group. The operator we employ below is related to the operator of Capelli discussed in Weyl's book on the classical groups.

2. A useful operator. Let us consider the operator Θ_n defined as

$$\Theta_n = | \partial/\partial z_{kl} |, \qquad k, 1 = 1, 2, \cdots, n, \tag{2.1}$$

where the z_{kl} are independent variables. Thus

$$\Theta_1 = \partial/\partial z_{11} ,$$
$$\Theta_2 = \frac{\partial}{\partial z_{11}} \frac{\partial}{\partial z_{22}} - \frac{\partial}{\partial z_{12}} \frac{\partial}{\partial z_{21}} , \tag{2.2}$$

and so on.

Let X represent the matrix (x_{kl}) and Z the matrix (z_{kl}). Then we have

$$\exp [itr(XZ^T)] = \exp (i \sum_{k,l} x_{kl}z_{kl}) \tag{2.3}$$

(Here Z^T is the transpose of Z, and tr (Z) is the trace of Z).
The basic identity we shall employ below is

$$\Theta^k[\exp [itr(XZ^T)]] = i^{nk} D_n^k \exp [itr(XZ^T)], \tag{2.4}$$

for $k = 1, 2, \cdots$.**

3. $E(D_n^r)$. Taking the expected value of both sides in (2.4), we obtain the result

$$\Theta_n^r\left[\prod_{k,l=1}^{n} \phi_{kl}(z_{kl}) \right] = i^{nr}E(D_n^k \exp [itr(XZ^T)]), \tag{3.1}$$

where

$$\phi_{kl}(z) = \int_{-\infty}^{\infty} e^{izx}\, dG_{kl}(x), \tag{3.2}$$

is the characteristic function of the random variable x_{kl} . Setting $z_{kl} = 0$, we obtain the result

$$i^{nr}E(D_n^r) = \Theta_n^r\left[\prod_{k,l=1}^{n} \phi_{kl}(z_{kl}) \right]_{z_{kl}=0} \tag{3.3}$$

4. Identical distributions. If the variables are identically distributed and symmetric about zero, we may write

$$\phi(z_{kl}) = \exp (a_1z_{kl}^2 + a_2z_{kl}^4 + \cdots), \tag{4.1}$$

obtaining as a consequence in place of (3.3) the result

$$i^{nk}E(D_n^k) = \Theta_n^k[\exp (a_1 \sum_{k,l} z_{kl}^2 + a_2 \sum_{k,l} z_{kl}^4 + \cdots)]_{z_{kl}=0} \tag{4.2}$$

**This is a well-known device in the theory of matric automorphic functions.

From this representation the value of $E(D_n^k)$ may be obtained by retaining in the above expression only the terms that yield a non-zero value after z_{kl} has been set equal to zero.

A particularly interesting case is that where $x_{kl} = \pm 1$ with equal probability. Then

$$i^{nk}E(D_n^k) = \Theta_n^k \left[\prod_{k,l=1}^{n} (\cos z_{kl}) \right]_{z_{kl}=0}. \tag{4.3}$$

Bibliography

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AN INEQUALITY FOR THE FIRST EIGENVALUE OF AN ORDINARY BOUNDARY VALUE PROBLEM*

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Let both coefficient functions of the differential equation

$$y'' + g(x)y' + f(x)y = 0 \tag{1}$$

be real-valued and continuous on the interval $a \leq x \leq b$ and, unless $g'(x)$ is not involved (as it is not in (4) below), suppose that the coefficient of y' has a continuous first derivative $g'(x)$. Consider the boundary condition

$$y(a) = 0, \qquad y(b) = 0. \tag{2}$$

A solution $y(x)$ of (1) satisfying (2) is the trivial solution,

$$y(x) \equiv 0. \tag{3}$$

In what follows, conditions on the function pair (g, f) will be considered which assure that (1) has no solution $y(x)$, distinct from (3), satisfying (2).

Such a condition is known to be

$$f \leq 0 \tag{4}$$

(with an arbitrary g). Another such condition is

$$f - \tfrac{1}{2}g' \leq 0. \tag{5}$$

Still another one is

$$f - g' \leq 0. \tag{6}$$

(It is understood that each of these three conditions is required for all values of x on the interval $a \leq x \leq b$.) Actually, the sufficiency of (4), (5) and (6) is contained in the results of Paraf, Picard and Lichtenstein, respectively, on (elliptic) *partial* differential equations [1]. The method proving the sufficiency of Lichtenstein's condition (6) is quite different from that proving the sufficiency of Picard's condition (5) or of the more primitive condition (4), and it is clear that no one of the three conditions (4)-(6) need be satisfied if the other two are satisfied.

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