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## Correction to my paper

## A NEW SINGULARITY OF TRANSONIC PLANE FLOWS*

Quarterly of Applied Mathematics, XII, 343-349 (1955)

By A. R. MANWELL (*University College, Swansea*)

A much more detailed study of the singular solution discussed rather briefly in the note of the above title has shown that several statements in Sec. 4 are incorrect. Briefly, the expansions (4.4) and so also (4.5), (4.6) are valid only *locally* for either $\Theta = 0$ or for $\Theta = \pi$, but not necessarily for both. We may not infer from these expansions the existence of solutions in the whole interval $(0, \pi)$. (In particular, on account of the pole at $Z = 1$, we may not replace in (4.3) a contour for which $Z - 1 = 2i \exp (i\Theta) \sin \Theta$ is very small by the unit circle $Z = 1$).

A correct discussion shows that (4.3) and (4.4) yield only *two* independent solutions. As a consequence, the singular solution can be smoothly continued across the sonic line for $\Theta > 0$ but, unless we admit further singularities in the supersonic region, the flow would not join up smoothly for $\theta < 0$. Since we are seeking possible criteria for the breakdown of flow solutions, this correction leads to a slight strengthening of our original conclusion.

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## A MINIMUM PRINCIPLE OF PLASTICITY*

By D. TRIFAN (*University of Arizona*)

This note is concerned with the removal of a certain restriction imposed by a proof[1] [Sec. 5] of a minimum principle of an isotropic, incompressible, strain-hardening material exhibiting a gradual transition from the elastic to the plastic state. The governing stress-strain relation for loading is given by

$$s_{ij}^* = 2G_0\epsilon_{ij}^* - p(E)\epsilon_{ij}E^*, \tag{1}$$

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[1]Section numbers enclosed in brackets refer to the following paper: D. Trifan, *A new theory of plastic flow*, Q. Appl. Math. **7**, pp. 201-211 (1949).

and for unloading by the differentiated form of Hooke's law

$$s_{ij}^* = 2G_0\epsilon_{ij}^* . \tag{2}$$

Variables with asterisks indicate rates of change with respect to any monotonically increasing function such as, for example, time. The stress deviation rate $s_{ij}^*$ is defined in terms of the stress rate $\sigma_{ij}^*$ by $s_{ij}^* = \sigma_{ij}^* - \frac{1}{3}\delta_{ij}\sigma_{kk}^*$—where repeated indices indicate summation in accordance with the summation convention of tensor calculus, and $\delta_{ij}$ the Kronecker delta—$\epsilon_{ij}$ is the strain tensor, $E = \frac{2}{3}\epsilon_{ij}\epsilon_{ij}$ , $G_0$ is the shear modulus in the elastic range, and $p = p(E)$, a positive definite function depending on the material and satisfying the inequality $Ep(E) < G_0$ [Sec. 2].

Let a set of primed strain rates $\epsilon_{ij}^{*\prime}$ be called admissible strain rates if they are derivable from a set of velocities $u_i^{*\prime}$ by the relation $\epsilon_{ij}^{*\prime} = \frac{1}{2}(u_{i,j}^{*\prime} + u_{j,i}^{*\prime})$ where $u_{i,j}^{*\prime} = \partial u_i^{*\prime}/\partial x_j$ , and the velocities in turn satisfy the incompressibility condition, given by $u_{i,i}^{*\prime} = 0$, and the boundary condition, given by surface velocities $u_i^*$ , i.e., $u_i^{*\prime} = u_i^*$ on the surface. The corresponding admissible stress deviation rates $s_{ij}^{*\prime}$ can be determined, for a given state $\epsilon_{ij}$ throughout the body, by Eq. (1) when $E^{*\prime} = \frac{4}{3}\epsilon_{ij}\epsilon_{ij}^{*\prime} > 0$, and by Eq. (2) when unloading occurs, $E^{*\prime} < 0$ [Sec. 2]. The actual strain rate $\epsilon_{ij}^*$ existing in the body is an admissible strain rate whose corresponding stress rate $\sigma_{ij}^*$ satisfies the equations of equilibrium $\sigma_{ij,j}^* = 0$.

Minimum principle. For any admissible set of strain rates and corresponding stress rates, the integral $J(\epsilon_{ij}^{*\prime}) = \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime}\, dv$ is an absolute minimum for the actual strain and stress rates occurring in the deformed body, i.e.,

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^*) \geq 0.$$

Equality holds only if $\epsilon_{ij}^{*\prime} = \epsilon_{ij}^*$ .

*Proof*. Due to the incompressibility condition $\epsilon_{ii}^{*\prime} = 0$, the following contracted tensor products are equivalent $\epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime} = \epsilon_{ij}^{*\prime}s_{ij}^{*\prime}$ , and thus by Eqs. (1) and (2) the integrals $J(\epsilon_{ij}^{*\prime})$ and $J(\epsilon_{ij}^*)$ become

$$J(\epsilon_{ij}^{*\prime}) = 2G_0 \int_V \epsilon_{ij}^{*\prime}\epsilon_{ij}^{*\prime}\, dv - \frac{3}{4}\int_{V_+'} p(E)\{E^{*\prime}\}^2\, dv, \tag{3}$$

$$J(\epsilon_{ij}^*) = 2G_0 \int_V \epsilon_{ij}^*\epsilon_{ij}^*\, dv - \frac{3}{4}\int_{V_+} p(E)\{E^*\}^2\, dv. \tag{4}$$

The sub-domains of volume V where loading and unloading occur are indicated by plus and minus subscripts respectively. In general the actual sub-domains $V_+$ and $V_-$ are not known; however, for admissible strain rates $V_+'$ indicates the sub-domain where $E^{*\prime} > 0$, and $V_-'$ the sub-domain where $E^{*\prime} < 0$. It is evident that the second integrals in (3) and (4) are to be integrated over the loading domains only. It is necessary to consider the portion of the volume where actual loading occurs while admissible strains indicate unloading; this we denote by $V_{+-}$ . The sub-domain $V_{-+}$ indicates actual unloading while admissible strains assume loading.

It is convenient at this point to introduce the following notation[2]:

$$u_i^{*\prime} = (u_i^{*\prime} - u_i^*) + u_i^* = \Delta u_i^* + u_i^* ,$$

$$\epsilon_{ij}^{*\prime} = (\epsilon_{ij}^{*\prime} - \epsilon_{ij}^*) + \epsilon_{ij}^* = \Delta\epsilon_{ij}^* + \epsilon_{ij}^* ,$$

$$E^{*\prime} = (E^{*\prime} - E^*) + E^* = \Delta E^* + E^*,$$

$$V_+' = V_+ + V_{-+} - V_{+-} .$$

[2]The author is indebted to H. J. Greenberg for this suggestion.

Integral (3) now becomes

$$J(\epsilon_{ij}^{*\prime}) = 2G_0 \int_V (\Delta\epsilon_{ij}^*\Delta\epsilon_{ij}^* + 2\Delta\epsilon_{ij}^*\epsilon_{ij}^* + \epsilon_{ij}^*\epsilon_{ij}^*)\, dv$$

$$- \frac{3}{4}\int_{V_+} (\{\Delta E^*\}^2 + 2\Delta E^* E^* + \{E^*\}^2)p(E)\, dv$$

$$- \frac{3}{4}\int_{V_{-+}} \{E^{*\prime}\}^2 p(E)\, dv + \frac{3}{4}\int_{V_{+-}} \{E^{*\prime}\}^2 p(E)\, dv. \qquad (5)$$

Since

$$\int_V s_{ij}^*\Delta\epsilon_{ij}^*\, dv = \int_V \sigma_{ij}^*\Delta\epsilon_{ij}^*\, dv = \int_V \sigma_{ij}^*(\Delta u_i^*)_{,j}\, dv = \int_V (\sigma_{ij}^*\Delta u_i^*)_{,j}\, dv$$

$$= \int_S \sigma_{ij}^*\Delta u_i^* n_j\, dS = 0,$$

due to the incompressibility condition, strain rate—velocity relation, symmetry of the stress rates, equations of equilibrium, divergence theorem, and the fact that $\Delta u_i^* = 0$ on the surface, it follows by Eqs. (1) and (2) that

$$2G_0 \int_V \epsilon_{ij}^*\Delta\epsilon_{ij}^*\, dv - \frac{3}{4}\int_{V_+} E^*\Delta E^* p(E)\, dv = 0. \qquad (6)$$

A simplification of integral (5) is possible with the application of (4) and (6)

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^*) = 2G_0 \int_V \Delta\epsilon_{ij}^*\Delta\epsilon_{ij}^*\, dv - \frac{3}{4}\int_{V_+} \{\Delta E^*\}^2 p(E)\, dv$$

$$- \frac{3}{4}\int_{V_{-+}} \{E^{*\prime}\}^2 p(E)\, dv + \frac{3}{4}\int_{V_{+-}} \{E^{*\prime}\}^2 p(E)\, dv. \qquad (7)$$

In $V_{-+}$ it is evident that $\{\Delta E^*\}^2 > \{E^{*\prime}\}^2$, and since $V_{-+}$ is included in $V_-$ ,

$$\int_{V_{-+}} \{E^{*\prime}\}^2 p(E)\, dv < \int_{V_{-+}} \{\Delta E^*\}^2 p(E)\, dv < \int_{V_-} \{\Delta E^*\}^2 p(E)\, dv.$$

This inequality applied to (7) in conjunction with the positive character of the integral over $V_{+-}$ reduces (7) to

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^*) > 2G_0 \int_V \Delta\epsilon_{ij}^*\Delta\epsilon_{ij}^*\, dv - \frac{3}{4}\int_V \{\Delta E^*\}^2 p(E)\, dv. \qquad (8)$$

By means of the Schwarzian inequality

$$\{\Delta E^*\}^2 = \left\{\frac{4}{3}\epsilon_{ij}\Delta\epsilon_{ij}^*\right\}^2 \le \frac{16}{9}\epsilon_{ij}\epsilon_{ij}\Delta\epsilon_{rs}^*\Delta\epsilon_{rs}^* = \frac{8}{3}E\Delta\epsilon_{ij}^*\Delta\epsilon_{ij}^* ,$$

so that

$$J(\epsilon_{ij}^{*\prime}) - J(\epsilon_{ij}^*) > 2\int_V \{G_0 - Ep(E)\}\Delta\epsilon_{ij}^*\Delta\epsilon_{ij}^* > 0.$$ [3]

It has been assumed in this proof that $\epsilon_{ij}^{*\prime} \neq \epsilon_{ij}^*$ . The case when $\epsilon_{ij}^{*\prime} = \epsilon_{ij}^*$ is trivial and leads to the equality in the minimum principle.

[3]In [Sec. 5] it was necessary to assume that $E^{*\prime} > 0$ whenever $E^* > 0$, i.e. no $V_{+-}$ domain, in order to obtain this result.