SCATTERING OF ELECTROMAGNETIC WAVES FROM A RANDOM SURFACE*

BY

WILLIAM C. HOFFMAN

U. S. Navy Electronics Laboratory

Abstract. Suppose $Z(x, y)$ is a real random process, continuous in the mean over a finite region D, and with mean value zero and covariance function $r(x, y; x', y')$. As such $Z(x, y)$ defines a random surface (provided certain differentiability conditions are satisfied). Then, by Karhunen's theorem on the representation of a random function, $Z(x, y)$ has an expansion in terms of an orthogonal process and the eigenfunctions and eigenvalues of the covariance function. Introducing this expression into the far-zone form of the Stratton-Chu solution of the electromagnetic field equations then leads to an approximate expression for the radiation scattered from the random surface, from which mean and covariance of the scattered field can be determined.

1. Introduction. The problem of determining the radiation scattered from a surface which varies randomly about some mean surface, for example a plane, leads to some interesting mathematical problems. The theoretical treatment is complicated by the fact that the boundary conditions are functionals of the random function describing the boundary, and a general theory of such boundary value problems appears to be lacking. Even in the relatively simple case of a perfectly conducting surface the boundary conditions themselves are subject to an integral equation[1] of apparently new type. The formidable difficulties in relating the field to the random boundary are, however, eased somewhat by assuming the medium below the random surface to be perfectly conducting, and this will be supposed in what follows. One can then avail himself of an approximate form of the Stratton-Chu solution, called the "current distribution" method, based on the "Kirchoff approximation", i.e., the assumption that the incident electromagnetic wave is reflected at every point as though an infinite plane wave were incident upon the infinite tangent plane. This requires that the curvature of the surface everywhere be small, and further that the incident plane wave not be at grazing incidence.

The spirit of the present paper is matched most closely by some work of Isakovich[2], who also employed the current distribution method to study scattering from a random surface. Isakovich's treatment, however, ignored the edge effects, and the probabilistic development was purely formal. Previous investigations[3,4,5] have in general not gone very deeply into the probabilistic aspects of the problem. A recent paper on a related subject, the solution of Laplace's equation in a half-plane with boundary conditions consisting of random functions, by Kampé de Feriet[6] is, however, a notable exception.

*Received August 23, 1954.

[1]S. Silver, *Microwave antenna theory and design*, McGraw-Hill Book Co., 1949, p. 132, Eq. (9).

[2]M. A. Isakovich, *The scattering of waves from a statistically rough surface*, *Zhurnal Eksp. Teort. Fiziki* **23**, 305-314 (1952).

[3]S. O. Rice, *Reflection of electromagnetic waves from slightly rough surfaces*, *Communs. on Pure and Appl. Math.* **4**, 351-378 (1951).

[4]W. S. Ament, *Toward a theory of reflection by a rough surface*, *Proc. I. R. E.* **41**, 142-146 (1953).

[5]J. Feinstein, *Some stochastic problems in wave propagation I*, *Trans. I. R. E. Prof. Group on Antennas and Propagation* AP-2, 23-30 (1954).

[6]J. Kampé de Feriet, *Fonctions aléatoires harmoniques dans un demi-plan*, *C. R. Acad. Sci. Paris*, **237**, No. 25, 1632-34, 21 Dec., 1953.

2. Definition of the random surface. Suppose that $Z(x, y)$ is a separable random process, real and continuous in the mean over a finite region D, with mean zero and covariance function $r(x, y; x', y')$. Since $Z(x, y)$ is to represent a surface, we require it to be three times mean square differentiable. This is assured if the covariance function has partial derivatives up to and including the fourth order[7,8]. Further, since the curvature of the surface must be small if the current distribution method is to apply, it will be supposed that this is true for almost all realizations[9].

The surface $Z(x, y)$ has been supposed to be a separable random function. Now this is the case if and only if[10] there exists the bilinear representation:

$$r(x, y; x', y') = \sum_{m,n=1}^{\infty} (1/\lambda_{mn})\varphi_{mn}(x, y)\varphi_{mn}(x', y'), \tag{1}$$

where the $\varphi_{mn}(x, y)$ and λ_{mn} are the eigenfunctions and eigenvalues of the integral equation:

$$\varphi(x, y) = \lambda \iint_D r(x, y; x', y')\varphi(x', y')\, dx'\, dy'. \tag{2}$$

Then according to Karhunen's representation theorem[10] we have for every $(x, y) \in D$ the expansion

$$Z(x, y) = \text{l.i.m.} \sum_{m,n} \lambda_{mn}^{-1/2}\varphi_{mn}(x, y)z_{mn}\ , \tag{3}$$

in terms of the orthogonal process $\{z_{mn}\}$ with[11]

$$\begin{aligned} &\mathcal{E}z_{mn} = 0, \\ &\mathcal{E}\{z_{mn}z_{pq}\} = \begin{cases} 0, & \text{if} \quad m \neq n \quad \text{and/or} \quad p \neq q; \\ 1, & \text{if} \quad m = n \quad \text{and} \quad p = q. \end{cases} \end{aligned} \tag{4}$$

At this point we shall assemble certain relations which will be used repeatedly in the sequel. Let the characteristic function of the function $f_{mn}(x, y)$ with respect to the random variable z_{mn} be denoted by

$$\chi[f_{mn}(x, y)] = \mathcal{E}\{\exp\,[if_{mn}(x, y)z_{mn}]\}, \tag{5}$$

and introduce the abbreviated notations:

$$B_{mn} = B_{mn}(x, y) = kB\lambda_{mn}^{-1/2}\varphi_{mn}(x, y) \tag{6}$$

and

$$C_{mn} = C_{mn}(x, y; x', y') = k\lambda_{mn}^{-1/2}[B_1\varphi_{mn}(x, y) - B_2\varphi_{mn}(x', y')], \tag{7}$$

---

[7] J. E. Moyal, *Stochastic processes and statistical physics*, *J. Roy. Stat. Soc.* (**B**) **11**, 167 (1949).

[8] M. Loeve, *Fonctions aléatoires du second ordre*, in *Processus Stochastiques* by P. Lévy, Gauthier-Villars, 1948, esp. p. 316.

[9] It seems to be a difficult matter to establish satisfactory sufficient conditions equivalent to this hypothesis.

[10] K. Karhunen, *Über lineare Methoden in der Wahrscheinlichkeitsrechnung*, *Ann. Acad. Scient. Fennicae* (**A**) **37**, 1-79 (1947).

[11] The mathematical expectation of the random variable ξ will be denoted by $\mathcal{E}\xi$. The letter E will be reserved for the electric field.

where

$$B = B(\alpha); \quad B_1 = B(\alpha_1); \quad B_2 = B(\alpha_2),$$

and α, α_1, α_2 are certain angles introduced in Sec. 3 below.

Then the following expressions hold in the sense of mean covergence:

$$\mathcal{E}\{\exp [ikBZ(x, y)]\} = \prod_{m,n} \chi(B_{mn}) = \chi(kB; x, y). \tag{8}$$

$$\mathcal{E}\{z_{mn} \exp [ikB \sum_{p,q} \lambda_{pq}^{-1/2}\varphi_{pq}z_{pq}]\} = -i \frac{d}{dB_{mn}} [\prod_{p,q} \chi(B_{pq})]. \tag{9}$$

and

$$\mathcal{E}\{\exp (ik[B_1Z(x, y) - B_2Z(x', y')])\} = \prod_{m,n} \chi(C_{mn}). \tag{10}$$

$$\mathcal{E}\{z_{mn} \exp [i \sum_{p,q} C_{pq}z_{pq}]\} = -i \frac{d}{dC_{mn}} (\prod_{p,q} \chi(C_{pq})). \tag{11}$$

$$\mathcal{E}\{z_{mn}z_{pq} \exp [i \sum_{s,t} C_{st}z_{st}]\} = \begin{cases} -\dfrac{d^2\chi(C_{mn})}{dC_{mn}^2} \prod' \chi(C_{st}), & m = p \text{ and } n = q; \\ -\dfrac{d\chi(C_{mn})}{dC_{mn}} \dfrac{d\chi(C_{pq})}{dC_{pq}} \prod'' \chi(S_{st}), & \\ & \text{when } p \neq m \text{ and/or } q \neq n. \end{cases} \tag{12}$$

The notation Π' is to indicate that the product is to be taken over $s \neq m$ and/or $t \neq n$; similarly Π'' is to mean a product over $s \neq m, p$ and $t \neq n, q$. It is of course understood that the derivatives indicated above are purely formal only. The expressions (8)-(12) are easily obtained from (3) and (4) and the notations (5)-(7).

If $Z(x, y)$ is a Gaussian process with mean zero and variance $\sigma^2(x, y)$, the above expressions reduce to

$$\begin{aligned} \mathcal{E}\{\exp [ikBZ(x, y)]\} &= \exp \{-\tfrac{1}{2}k^2B^2 \sum \lambda_{mn}^{-1}\varphi_{mn}^2(x, y)\} \\ &= \exp \{-\tfrac{1}{2}k^2B^2\sigma^2(x, y)\}. \end{aligned} \tag{13}$$

$$\mathcal{E}\{z_{mn} \exp [ikB \sum_{p,q} \lambda_{pq}^{-1/2}\varphi_{pq}(x, y)z_{pq}]\} = ikB\lambda_{mn}^{-1/2}\varphi_{mn}(x, y) \exp \{-\tfrac{1}{2}k^2B^2\sigma^2(x, y)\} \tag{14}$$

and

$$\mathcal{E}\{\exp (ik[B_1Z(x, y) - B_2Z(x', y')])\} = \exp (-\tfrac{1}{2}k^2Q) \tag{15}$$

$$\mathcal{E}\{z_{mn} \exp [i \sum_{p,q} C_{pq}z_{pq}]\} = iC_{mn} \exp (-\tfrac{1}{2} \sum_{p,q} C_{pq}^2) \tag{16}$$

$$\mathcal{E}\{z_{mn}z_{pq} \exp (i \sum C_{st}z_{st})\} = \exp (-\tfrac{1}{2}k^2Q)\begin{cases} M_{mn}(x, y; x', y'), \\ \quad \text{provided } m = p \text{ and } n = q; \\ M'_{mnpq}(x, y; x', y'), \\ \quad \text{if } p \neq m \text{ and/or } q \neq n, \end{cases} \tag{17}$$

where

$$Q = Q(x, y; x', y') = B_1^2\sigma^2(x, y) + B_2^2\sigma^2(x', y') - 2B_1B_2r(x, y; x', y'); \tag{18}$$

$$M_{mn}(x, y; x', y') = 1 - (k^2/\lambda_{mn})[B_1^2\varphi_{mn}^2(x, y) + B_2^2\varphi_{mn}^2(x', y') - 2B_1B_2\varphi_{mn}(x, y)\varphi_{mn}(x', y')]; \tag{19}$$

$$M'_{mnpq}(x, y; x', y') = -\frac{k^2}{(\lambda_{mn}\lambda_{pq})^{1/2}} \{B_1^2\varphi_{mn}(x, y)\varphi_{pq}(x, y) + B_2^2\varphi_{mn}(x', y')\varphi_{pq}(x', y') - B_1B_2[\varphi_{mn}(x, y)\varphi_{pq}(x', y') + \varphi_{mn}(x', y')\varphi_{pq}(x, y)]\}. \tag{20}$$

3. The scattered field. The far-zone electromagnetic field at point P of the radiation scattered from the surface $\Sigma = \{Z(x, y) \mid (x, y) \in D\}$ is given by [12].

$$\mathbf{E}_s(P) = -\frac{i\omega\mu}{2\pi R} \exp(-ikR) \int_\Sigma \{\mathbf{n} \times \mathbf{H}_0 - [(\mathbf{n} \times \mathbf{H}_0)\cdot\mathbf{R}]\mathbf{R}\} \exp(ik\mathbf{d}\cdot\mathbf{R})\, dS, \tag{1}$$

where R is the distance from the origin of coordinates on Σ to the field point P; $\mathbf{R}$ is a unit vector in the direction of R; $\mathbf{n}$ is the unit normal vector to the surface Σ, with the same sense as the z-axis; $\mathbf{d}$ is as shown in Fig. 1; and $\mathbf{H}_0$ is the magnetic vector of the

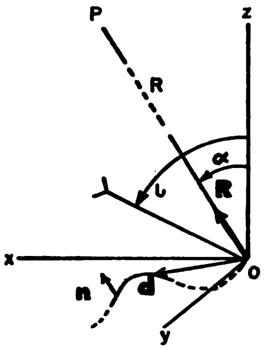

FIG. 1. Coordinate system for far-zone scattered field.

incident electromagnetic field. Expression (1) represents the Stratton-Chu solution[13] of the electromagnetic field equations in the far-zone after the introduction of a line distribution of charge on the boundary. Thus the edge effect of the shadowed area of Σ has been removed.

The fact that (1) represents the far-zone field is usually taken to mean that its applicability is limited by the condition $R > 2\delta^2(D)/\lambda$, where $\delta(D)$ is the diameter of the set D and λ is the wave length of the radiation.

Further progress is dependent upon specification of the magnetic vector $\mathbf{H}_0$ of the incident field. An incident *plane wave* will be assumed, so that for *vertical polarization*,

$$\mathbf{H}_0 = \mathbf{j}(E_0/\eta) \exp\{i[\omega t + k(x \sin\iota + Z\cos\iota)]\}, \qquad Z \in \Sigma, \tag{2}$$

and for *horizontal polarization*,

$$\mathbf{H}_0 = (\mathbf{i}\cos\iota - \mathbf{k}\sin\iota)(E_0/\eta) \exp\{i[\omega t + k(x \sin\iota + Z\cos\iota)]\}, \qquad Z \in \Sigma \tag{3}$$

[12] S. Silver, *Microwave antenna theory and design*, McGraw-Hill Book Co., 1949, p. 149.

[13] J. A. Stratton, *Electromagnetic theory*, McGraw-Hill Book Co.,1941, pp. 464-470.

where $k = 2\pi/\lambda$, E_0 is the amplitude of the incident electric field, η is the impedance of the medium in the upper half-space, and ι is the angle of incidence, as shown in Fig. 1.

Let $\mathbf{I}$ denote the stochastic integral in (1):

$$\mathbf{I} = \int_\Sigma \{\mathbf{n} \times \mathbf{H}_0 - [(\mathbf{n} \times \mathbf{H}_0)\cdot\mathbf{R}]\mathbf{R}\} \exp(ik\mathbf{d}\cdot\mathbf{R})\, dS, \tag{4}$$

and affix a subscript v or h to $\mathbf{I}$ according as the polarization is vertical or horizontal. The existence of the stochastic integral follows immediately from a theorem of Doob[14].

Now introduce (2) or (3) and the formulas

$$dS = (EG - F^2)^{1/2}\, dx\, dy;$$

$$\mathbf{n} = \left(-\frac{\partial Z}{\partial x}\mathbf{i} - \frac{\partial Z}{\partial y}\mathbf{j} + \mathbf{k}\right)\Big/ (EG - F^2)^{1/2}; \tag{5}$$

$$\mathbf{R} = \sin\alpha\mathbf{i} + \cos\alpha\mathbf{k}; \tag{6}$$

$$\mathbf{d} = x\mathbf{i} + y\mathbf{j} + Z(x, y)\mathbf{k}, \tag{7}$$

into (4), and let[15]

$$A(\alpha) = \sin\iota + \sin\alpha; \qquad B(\alpha) = \cos\iota + \cos\alpha. \tag{8}$$

Then, apart from the time factor $\exp(i\omega t)$,

$$\mathbf{I}_v(\alpha) = (E_0/\eta)(-\mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha)J(\alpha;\alpha); \tag{9}$$

$$\mathbf{I}_h(\alpha) = (E_0/\eta)\Big\{\sin(\iota - \alpha)(\mathbf{i}\cos\alpha - \mathbf{k}\sin\alpha) \cdot \iint_D \frac{\partial Z}{\partial y} \exp(ik[Ax + BZ(x, y)])\, dx\, dy + \mathbf{j}J(\alpha;\iota)\Big\}, \tag{10}$$

where

$$J(\alpha;\beta) = \iint_D \exp\{ik[A(\alpha)x + B(\alpha)Z(x, y)]\}\left(\cos\beta - \frac{\partial Z}{\partial x}\sin\beta\right) dx\, dy. \tag{11}$$

The energy flow at the field point P is given by the Poynting vector $\mathbf{S}(P)$:

$$\mathbf{S}(P) = \tfrac{1}{2}\eta^{-1}(\mathbf{E}_s\cdot\mathbf{E}_s^*)\mathbf{R} = \tfrac{1}{2}\eta^{-1}\omega^2\mu^2(2\pi R)^{-2}\mathbf{I}(\alpha)\cdot\mathbf{I}^*(\alpha)\mathbf{R}. \tag{12}$$

We thus have from (9) and (10) as the energy flows corresponding to each polarization:

$$\mathbf{S}_v(P) = \tfrac{1}{2}\eta^{-1}k^2(2\pi R)^{-2}E_0^2 J(\alpha;\alpha)J^*(\alpha;\alpha)\mathbf{R}; \tag{13}$$

$$\mathbf{S}_h(P) = \frac{1}{2}\eta^{-1}k^2E_0^2(2\pi R)^{-2}\Big\{\sin^2(\iota - \alpha)\iiiint_{D\,D} \exp\{ik[A(x - x') + B(Z(x, y) - Z(x', y'))]\}\frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y'}\, dx\, dy\, dx'\, dy' + J(\alpha;\iota)J^*(\alpha;\iota)\Big\}\mathbf{R}. \tag{14}$$

---

[14] J. L. Doob, *Stochastic processes*, Wiley, 1953, theorem 2.7, p. 62. The existence is also a consequence[10] of the integrability of the covariance function of the integrand over D.

[15] It is implied by the form of (6) and (8) that the expressions developed below are restricted to field points P in the plane of incidence.

4. Mean and covariance of the scattered field for vertical polarization. Referring to expressions (3.1), (3.4), and (3.9) one sees that the expected value of the scattered field is proportional to $\mathcal{E}\{J(\alpha;\alpha)\}$ for vertically-polarized incident radiation. By Fubini's theorem and the representation (2.3), we have

$$\mathcal{E}\{J(\alpha;\alpha)\} = \cos\alpha \iint_D \exp(ikAx)\mathcal{E}\{\exp[ikBZ(x, y)]\}\, dx\, dy$$
$$- \sin\alpha \iint_D \exp(ikAx)\,\mathcal{E}\left\{\sum_{m,n} \lambda_{mn}^{-1/2} \frac{\partial\varphi_{mn}}{\partial x} z_{mn} \cdot \exp\left[ikB \sum_{p,q} \lambda_{pq}^{-1/2}\varphi_{pq}(x, y)z_{pq}\right]\right\} dx\, dy. \tag{1}$$

In view of (2.8) and (2.9) this expression becomes

$$\mathcal{E}\{J(\alpha;\alpha)\} = \cos\alpha \iint_D \exp(ikAx)\chi(kB; x, y)\, dx\, dy$$
$$+ i\sin\alpha \iint_D \exp(ikAx) \sum_{m,n} \lambda_{mn}^{-1/2} \frac{\partial\varphi_{mn}}{\partial x} \frac{d}{dB_{mn}} \left(\prod_{p,q} \chi(B_{pq})\right) dx\, dy. \tag{2}$$

If the surface is Gaussian, so that formulas (2.13) and (2.14) apply, the second expectation on the right hand side of (1) becomes

$$\mathcal{E}\left\{\sum_{m,n} \lambda_{mn}^{-1/2} \frac{\partial\varphi_{mn}}{\partial x} z_{mn} \exp\left[ikB \sum_{p,q} \lambda_{pq}^{-1/2}\varphi_{pq}(x, y)z_{pq}\right]\right\}$$
$$= -kB \exp\left\{-\frac{1}{2} k^2B^2\sigma^2(x, y)\right\}\cdot\frac{1}{2}\frac{\partial}{\partial x} \sum_{m,n} \lambda_{mn}^{-1}\varphi_{mn}^2(x, y)$$
$$= -\frac{1}{2} kB \frac{\partial\sigma^2(x, y)}{\partial x} \exp\left\{-\frac{1}{2} k^2B^2\sigma^2(x, y)\right\}. \tag{3}$$

Thus (1) becomes in the Gaussian case,

$$\mathcal{E}\{J(\alpha;\alpha)\} = \iint_D \exp\left\{ikAx - \frac{1}{2} k^2B^2\sigma^2(x, y)\right\}\left[\cos\alpha - \frac{1}{2} ikB \sin\alpha \frac{\partial\sigma^2(x, y)}{\partial x}\right] dx\, dy, \tag{4}$$

and if further the Z-process is stationary, so that σ^2 is constant, and if $D = (-a, a) \times (-b, b)$, then the expression for the mean of $J(\alpha;\alpha)$ reduces to

$$\mathcal{E}\{J(\alpha;\alpha)\} = 4ab\cos\alpha \exp\left(-\frac{1}{2} k^2B^2\sigma^2\right) \frac{\sin(kAa)}{kAa}. \tag{5}$$

We thus obtain the mean scattered field for vertical polarization in the form

$$\mathcal{E}\{\mathbf{E}_s(P)\} = -i\omega\mu(2\pi R)^{-1} \exp(-ikR)(-\mathbf{i}\cos\alpha + \mathbf{k}\sin\alpha)\mathcal{E}\{J(\alpha;\alpha)\}, \tag{6}$$

with $\mathcal{E}\{J(\alpha;\alpha)\}$ being given by (2), (4), or (5) according as the process specifying the random surface is separable, Gaussian, or stationary Gaussian with a rectangular parameter-domain D.

We now pass to a consideration of the covariance function of the vertically-polarized scattered field, which according to (3.1) and (3.9), is related to the following expected value:

$$\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} = \iint_D\iint_D \exp\{ik(A_1x - A_2x')\}$$
$$\cdot\Big(\cos\alpha_1\cos\alpha_2\,\mathcal{E}\{\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\}$$
$$-\sin\alpha_1\cos\alpha_2\,\mathcal{E}\Big\{\frac{\partial Z}{\partial x}\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\Big\} \tag{7}$$
$$-\sin\alpha_2\cos\alpha_1\,\mathcal{E}\Big\{\frac{\partial Z}{\partial x'}\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\Big\}$$
$$+\sin\alpha_1\sin\alpha_2\,\mathcal{E}\Big\{\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial x'}\exp[ik(B_1Z(x,y) - B_2Z(x',y'))]\Big\}\Big)\,dx\,dy\,dx'\,dy'.$$

Introducing the representation (2.3) and making use of formulas (2.10)-(2.12) enables us to write (7) in the case of a separable process as

$$\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} = \iint_D\iint_D \exp\{ik(A_1x - A_2x')\}\Big(\cos\alpha_1\cos\alpha_2\prod_{p,q}\chi(C_{pq})$$
$$+\,i\sum_{m,n}\lambda_{mn}^{-1/2}\Big(\sin\alpha_1\cos\alpha_2\frac{\partial\varphi_{mn}}{\partial x} + \cos\alpha_1\sin\alpha_2\frac{\partial\varphi_{mn}}{\partial x'}\Big)\frac{d}{dC_{mn}}\Big(\prod_{p,q}\chi(C_{pq})\Big) \tag{8}$$
$$-\sin\alpha_1\sin\alpha_2\Big\{\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{mn}}{\partial x'}\frac{d^2}{dC_{mn}^2}\Big(\prod_{p,q}\chi(C_{pq})\Big) + \sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}$$
$$\cdot\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{pq}}{\partial x'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{st})\Big\}\Big)\,dx\,dy\,dx'\,dy',$$

where Σ' indicates that the summation is to extend over $p \neq m$ and/or $q \neq n$, and Π'' has the same meaning as following Eq. (2.12).

If the process is Gaussian, we have from (2.3), (2.16), and (2.18) that

$$\mathcal{E}\Big(\frac{\partial Z}{\partial x}\exp\{ik[B_1Z(x,y) - B_2Z(x',y')]\}\Big) = i\sum_{m,n}\lambda_{mn}^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}C_{mn}\exp\Big(-\frac{1}{2}\sum_{p,q}C_{pq}^2\Big)$$
$$= ik\exp\Big\{-\frac{1}{2}k^2Q(x,y;x',y')\Big\}\sum_{m,n}\lambda^{-1}\frac{\partial\varphi_{mn}}{\partial x}[B_1\varphi_{mn}(x,y) - B_2\varphi_{mn}(x',y')]$$
$$= ik\exp\Big\{-\frac{1}{2}k^2Q(x,y;x',y')\Big\}\Big(\frac{1}{2}B_1\frac{\partial\sigma^2(x,y)}{\partial x} - B_2\frac{\partial r(x,y;x',y')}{\partial x}\Big); \tag{9}$$

and similarly, from (2.3), (2.16), and (2.18),

$$\mathcal{E}\Big(\frac{\partial Z}{\partial x'}\exp\{ik[B_1Z(x,y) - B_2Z(x',y')]\}\Big) = ik\exp\Big(-\frac{1}{2}k^2Q\Big)\Big(B_1\frac{\partial r(x,y;x',y')}{\partial x'} - \frac{1}{2}B_2\frac{\partial\sigma^2(x',y')}{\partial x'}\Big). \tag{10}$$

By means of (2.3), (2.17), and (2.18) we find also that for a Gaussian surface,

$$\mathcal{E}\left(\frac{\partial Z}{\partial x}\frac{\partial Z}{\partial x'}\exp\{ik[B_1Z(x,y)-B_2Z(x',y')]\}\right)=\exp\left(-\frac{1}{2}k^2Q\right)\left\{\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{mn}}{\partial x'}\right.$$
$$-k^2\left[B_1^2\sum_{m,n}\sum_{p,q}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x,y)\lambda_{pq}^{-1}\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x,y)+B_2^2\sum_{m,n}\sum_{p,q}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x',y')\right.$$
$$\cdot\lambda_{pq}^{-1}\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x',y')-B_1B_2\sum_{m,n}\sum_{p,q}(\lambda_{mn}\lambda_{pq})^{-1}\left(\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x,y)\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x',y')\right.$$
$$\left.\left.\left.+\frac{\partial\varphi_{mn}}{\partial x}\varphi_{mn}(x',y')\frac{\partial\varphi_{pq}}{\partial x'}\varphi_{pq}(x,y)\right)\right]\right\}$$
$$=\exp\left(-\frac{1}{2}k^2Q\right)\left\{\frac{\partial^2r(x,y;x',y')}{\partial x\,\partial x'}-k^2\left[\frac{1}{2}B_1^2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial r}{\partial x'}\right.\right.$$
$$\left.\left.+\frac{1}{2}B_2^2\frac{\partial r}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}-\frac{1}{4}B_1B_2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}-B_1B_2\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'}\right]\right\}. \tag{11}$$

Thus (7) becomes, for a Gaussian process,

$$\mathcal{E}\{J(\alpha_1;\alpha_1)J^*(\alpha_2;\alpha_2)\}=\iiiint_{D\ D}\exp\left\{ik(A_1x-A_2x')-\frac{1}{2}k^2[B_1^2\sigma^2(x,y)+B_2^2\sigma^2(x',y')\right.$$
$$\left.-2B_1B_2r(x,y;x',y')]\right\}\left\{\cos\alpha_1\cos\alpha_2+\sin\alpha_1\sin\alpha_2\frac{\partial^2r(x,y;x',y')}{\partial x\,\partial x'}\right.$$
$$-ik\sin\alpha_1\cos\alpha_2\left[\frac{1}{2}B_1\frac{\partial\sigma^2(x,y)}{\partial x}-B_2\frac{\partial r}{\partial x}\right]-ik\cos\alpha_1\sin\alpha_2\left[B_1\frac{\partial r}{\partial x'}-\frac{1}{2}B_2\frac{\partial\sigma^2}{\partial x'}\right]$$
$$-k^2\sin\alpha_1\sin\alpha_2\left[\frac{1}{2}B_1^2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial r}{\partial x'}+\frac{1}{2}B_2^2\frac{\partial r}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}-\frac{1}{4}B_1B_2\frac{\partial\sigma^2(x,y)}{\partial x}\frac{\partial\sigma^2(x',y')}{\partial x'}\right.$$
$$\left.\left.-B_1B_2\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'}\right]\right\}dx\,dy\,dx'\,dy'. \tag{12}$$

Suppose now that the process is stationary as well as Gaussian. Then the variance $\sigma^2(x,y)$ is constant, and letting u and v denote the displacements $x'-x$ and $y'-y$, respectively, the covariance function of the surface becomes

$$r(x,y;x',y')=r(u,v); \tag{13}$$

so that

$$\partial r/\partial x'=\partial r/\partial u=-\partial r/\partial x; \tag{14}$$
$$\partial^2r/\partial x\,\partial x'=-\partial^2r/\partial u^2. \tag{15}$$

Thus (12) becomes for the stationary Gaussian case,

$$\mathcal{E}\{J(\alpha_1;\alpha_1)J^*(\alpha_2;\alpha_2)\}=\exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2+B_2^2)\}$$
$$\cdot\iiiint_{D\ D'}\exp\{ik(A_1-A_2)x-ikA_2u+k^2B_1B_2r(u,v)\}\left\{\cos\alpha_1\cos\alpha_2\right.$$
$$-\sin\alpha_1\sin\alpha_2\frac{\partial^2r}{\partial u^2}-ik\frac{\partial r}{\partial u}(B_2\sin\alpha_1\cos\alpha_2+B_1\cos\alpha_1\sin\alpha_2)$$
$$\left.-k^2B_1B_2\sin\alpha_1\sin\alpha_2\left(\frac{\partial r}{\partial u}\right)^2\right\}dx\,dy\,du\,dv.$$

If as in (5), $D = (-a, a) \times (-b, b)$, then this expression may be simplified. First write the integral as

$$\int_{-a}^{a}\int_{-a-x}^{a-x}\int_{-b}^{b}\int_{-b-y}^{b-y} \exp\,[icx - ikA_2u][g(u, v) - ih(u, v)]\,dx\,du\,dy\,dv, \tag{16}$$

where

$$c = k(A_1 - A_2), \tag{17}$$

$$g(u, v) = \exp\,[k^2B_1B_2r(u, v)]\left[\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2\frac{\partial^2 r}{\partial u^2} - k^2B_1B_2\sin\alpha_1\sin\alpha_2\left(\frac{\partial r}{\partial u}\right)^2\right], \tag{18}$$

$$h(u, v) = \exp\,[k^2B_1B_2r(u, v)]k(B_2\sin\alpha_1\cos\alpha_2 + B_1\cos\alpha_1\sin\alpha_2)\,\frac{\partial r}{\partial u}\cdot \tag{19}$$

From the definition of covariance and the hypothesis of stationarity, we have immediately that

$$r(-u, -v) = r(u, v) \qquad \text{and} \qquad r(-u, v) = r(u, -v). \tag{20}$$

It follows that

$$\frac{\partial r(-u, -v)}{\partial u} = -\frac{\partial r(u, v)}{\partial u}; \qquad -\frac{\partial r(-u, v)}{\partial u} = \frac{\partial r(u, -v)}{\partial u}; \tag{21}$$

$$\frac{\partial r(-u, -v)}{\partial v} = -\frac{\partial r(u, v)}{\partial v}; \qquad -\frac{\partial r(-u, v)}{\partial v} = \frac{\partial r(u, -v)}{\partial v}; \tag{22}$$

and

$$\frac{\partial^2 r(-u, -v)}{\partial u^2} = \frac{\partial^2 r(u, v)}{\partial u^2}; \qquad \frac{\partial^2 r(-u, v)}{\partial u^2} = \frac{\partial^2 r(u, -v)}{\partial u^2}; \tag{23}$$

$$\frac{\partial^2 r(-u, -v)}{\partial v^2} = \frac{\partial^2 r(u, v)}{\partial v^2}; \qquad \frac{\partial^2 r(-u, v)}{\partial v^2} = \frac{\partial^2 r(u, -v)}{\partial v^2}. \tag{24}$$

Consequently

$$g(-u, -v) = g(u, v); \qquad g(-u, v) = g(u, -v); \tag{25}$$

and

$$h(-u, -v) = -h(u, v); \qquad h(-u, v) = -h(u, -v). \tag{26}$$

Next invert the order of integration in (16) and perform the x,y-integration. The result is, after some obvious changes of variable and use is made of relations (25) and (26):

$$\frac{4}{c}\int_0^{2a}\int_0^{2b}\sin\left[c\left(a - \frac{u}{2}\right)\right](2b - v)\left\{\cos\left(\frac{1}{2}c'u\right)[g(u, v) + g(u, -v)] - \sin\left(\frac{1}{2}c'u\right)[h(u, v) + h(u, -v)]\right\}du\,dv,$$

where

$$c' = k(A_1 + A_2). \tag{27}$$

Thus, for a stationary Gaussian random surface with rectangular parameter-domain[16]:

$$\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} = 4k^{-1}(A_1 - A_2)^{-1} \exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2 + B_2^2)\}$$
$$\cdot \int_0^{2a}\int_0^{2b} \sin\left[k(A_1 - A_2)\left(a - \frac{u}{2}\right)\right](2b - v) \tag{28}$$
$$\cdot \{\cos[\tfrac{1}{2}k(A_1 + A_2)u][g(u, v) + g(u, -v)]$$
$$- \sin[\tfrac{1}{2}k(A_1 + A_2)u][h(u, v) + h(u, -v)]\}\, du\, dv.$$

The covariance of the scattered field in the vertically-polarized case is then given by

$$\text{cov}\,[\mathbf{E}_s(P_1), \mathbf{E}_s(P_2)] = \mathcal{E}\{[\mathbf{E}_s(P_1) - \mathcal{E}\mathbf{E}_s(P_1)]\cdot[\mathbf{E}_s(P_2) - \mathcal{E}\mathbf{E}_s(P_2)]^*\}$$
$$= (2\pi)^{-2}k^2E_0^2(R_1R_2)^{-1}\cos(\alpha_1 - \alpha_2)\exp\{-ik(R_1 - R_2)\} \tag{29}$$
$$\cdot[\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\} - \mathcal{E}\{J(\alpha_1\,;\alpha_1)\}\mathcal{E}^*\{J(\alpha_2\,;\alpha_2)\}],$$

where $\mathcal{E}\{J(\alpha_1\,;\alpha_1)J^*(\alpha_2\,;\alpha_2)\}$ and $\mathcal{E}\{J(\alpha;\alpha)\}$ are given by (8) and (2), (12) and (4), or (28) and (5), according as the Z-process is separable, Gaussian, or stationary Gaussian with rectangular parameter-domain D.

The variance of the scattered field for vertical polarization follows at once from (29) by setting $P_1 = P_2 = P$ (whence $\alpha_1 = \alpha_2 = \alpha$). In the stationary Gaussian, rectangular parameter-domain case, for example, one obtains the following expression for the variance of the scattered field:

$$\text{var}\,[\mathbf{E}_s(P)] = \tfrac{1}{4}k^2E_0^2(\pi R)^{-2}\exp(-k^2\sigma^2B^2)$$
$$\cdot\left(\int_0^{2a}\int_0^{2b}(2a - u)(2b - v)\{\cos(kAu)[g_\alpha(u, v) + g_\alpha(u, -v)]\right. \tag{30}$$
$$\left.- \sin(kAu)[h_\alpha(u, v) + h_\alpha(u, -v)]\}\,du\,dv - 8a^2b^2\cos^2\alpha\,\frac{\sin^2(kAa)}{(kAa)^2}\right),$$

where $g_\alpha(u, v)$ and $h_\alpha(u, v)$ are given by (18) and (19) upon setting $\alpha_1 = \alpha_2 = \alpha$.

The mean square energy flow, specified by the Poynting vector (3.13), is intimately connected with the variance (30). The mean square energy flow at P is obtained by setting $P_1 = P_2 = P$ in (8), (12), or (28) according as the Z-process is separable, Gaussian, or stationary Gaussian with rectangular domain D, and multiplying the resulting expression by $1/2\eta^{-1}k^2E_0^2(2\pi R)^{-2}\mathbf{R}$.

5. Mean and covariance of the scattered field for horizontal polarization. Examination of Eq. (3.10) shows that the expressions (4.2)-(4.5) already developed can be used to determine that portion of $\mathcal{E}\{\mathbf{I}_h(\alpha)\}$ involving the expectation of $J(\alpha;\iota)$. The remaining part of $\mathcal{E}\{\mathbf{I}_h(\alpha)\}$ can be evaluated in exactly the same fashion as for the second term of $\mathcal{E}\{J(\alpha;\alpha)\}$, i.e., as in (4.1). The result, for a separable random surface $Z(x, y)$, is

$$\mathcal{E}\{\mathbf{I}_h(\alpha)\} = (E_0/\eta)\iint_D \exp(ikAx)\left\{\mathbf{j}\cos\iota\,\chi(kB; x, y)\right.$$
$$+ i\sum_{m,n}\lambda_{mn}^{-1/2}\frac{d}{dB_{mn}}\Big(\prod_{p,q}\chi(B_{pq})\Big)\left[\mathbf{j}\sin\iota\,\frac{\partial\varphi_{mn}}{\partial x}\right. \tag{1}$$
$$\left.\left.- (\mathbf{i}\cos\alpha - \mathbf{k}\sin\alpha)\sin(\iota - \alpha)\frac{\partial\varphi_{mn}}{\partial y}\right]\right\}dx\,dy.$$

---

[16]The author is indebted to the referee for pointing out the proper form of this expression.

If the surface is Gaussian, it follows from (2.13) and (2.14) that

$$\mathcal{E}\{\mathbf{I}_h(\alpha)\} = (E_0/\eta) \iint_D \exp\left\{ikAx - \frac{1}{2}k^2B^2\sigma^2(x, y)\right\}\left\{\mathbf{j}\cos\iota + \frac{1}{2}ikB\left[(\mathbf{i}\cos\alpha - \mathbf{k}\sin\alpha)\sin(\iota - \alpha)\frac{\partial\sigma^2(x, y)}{\partial y} - \mathbf{j}\sin\iota\frac{\partial\sigma^2(x, y)}{\partial x}\right]\right\} dx\, dy. \tag{2}$$

If the surface is stationary as well as Gaussian, and if $D = (-a, a) \times (-b, b)$, then (2) reduces to

$$\mathcal{E}\{\mathbf{I}_h(\alpha)\} = \mathbf{j}4ab\cos\iota(E_0/\eta)\exp\left(-\frac{1}{2}k^2B^2\sigma^2\right)\frac{\sin(kAa)}{(kAa)}. \tag{3}$$

The mean scattered field for horizontal polarization is thus given by the expression:

$$\mathcal{E}\{\mathbf{E}_s(P)\} = -\frac{i\omega\mu}{2\pi R}\exp(-ikR)\mathcal{E}\{\mathbf{I}_h(\alpha)\}, \tag{4}$$

with $\mathcal{E}\{\mathbf{I}_h(\alpha)\}$ from (1), (2), or (3) according as the process $Z(x, y)$ representing the surface is separable, Gaussian, or stationary and Gaussian with rectangular domain D.

Next the covariance of the scattered field will be determined for horizontal polarization. From (3.10),

$$\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} = (E_0/\eta)^2\Big[\sin(\iota - \alpha_1)\sin(\iota - \alpha_2)\cos(\alpha_1 - \alpha_2) \cdot \iint_D\iint_D \exp\{ik(A_1x - A_2x')\}\mathcal{E}\left\{\frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y'}\exp(ik[B_1Z(x, y) - B_2Z(x', y')])\right\}dx\, dy\, dx'\, dy' + \mathcal{E}\{J(\alpha_1\,;\iota)J^*(\alpha_2\,;\iota)\}\Big]. \tag{5}$$

In the subsequent computations the expressions (4.8), (4.12), or (4.28) already developed can be used to find $\mathcal{E}\{J(\alpha_1\,;\iota)J^*(\alpha_2\,;\iota)\}$. From (2.12) we have that

$$\mathcal{E}\left\{\frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y'}\exp(ik[B_1Z(x, y) - B_2Z(x', y')])\right\} = \sum_{m,n}\sum_{p,q}(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{pq}}{\partial y'} \cdot\mathcal{E}\{z_{mn}z_{pq}\exp(i\sum_{s,t}C_{st}z_{st})\}$$

$$= -\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{mn}}{\partial y'}\frac{d^2}{dC_{mn}^2}[\prod_{p,q}\chi(C_{pq})] - \sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{pq}}{\partial y'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{st}), \tag{6}$$

where Σ' and Π'' have the same meanings as following expression (2.12). Substituting this expression into (5) and modifying (4.8) as required for $\mathcal{E}\{J(\alpha_1\,;\,\iota)J^*(\alpha_2\,;\,\iota)\}$, one obtains in the case of a separable process:

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= (E_0/\eta)^2\Big(-\sin(\iota-\alpha_1)\sin(\iota-\alpha_2)\cos(\alpha_1-\alpha_2)\\ &\cdot\iint_D\iint_D \exp\{ik(A_1x-A_2x')\}\Big\{\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{mn}}{\partial y'}\frac{d^2}{dC_{mn}^2}\Big[\prod_{p,q}\chi(C_{pq})\Big]\\ &+\sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial y}\frac{\partial\varphi_{pq}}{\partial y'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{st})\Big\}\,dx\,dy\,dx'\,dy'\\ &+\iint_D\iint_D \exp\{ik(A_1x-A_2x')\}\Big\{\cos^2\iota\prod_{p,q}\chi(C_{pq})+i\sum_{m,n}\lambda_{mn}^{-1/2}\sin\iota\cos\iota\\ &\cdot\Big(\frac{\partial\varphi_{mn}}{\partial x}+\frac{\partial\varphi_{mn}}{\partial x'}\Big)\frac{d}{dC_{mn}}\Big[\prod_{p,q}\chi(C_{pq})\Big]-\sin^2\iota\Big(\sum_{m,n}\lambda_{mn}^{-1}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{mn}}{\partial x'}\frac{d^2}{dC_{mn}^2}\Big[\prod_{p,q}\chi(C_{pq})\Big]\\ &+\sum_{m,n}\sum{}'(\lambda_{mn}\lambda_{pq})^{-1/2}\frac{\partial\varphi_{mn}}{\partial x}\frac{\partial\varphi_{pq}}{\partial x'}\frac{d\chi(C_{mn})}{dC_{mn}}\frac{d\chi(C_{pq})}{dC_{pq}}\prod{}''\chi(C_{st})\Big)\Big\}\,dx\,dy\,dx'\,dy'\Big).\end{aligned}\tag{7}$$

If the process representing the surface is Gaussian, Eq. (6) reduces to an expression like (4.11), the only difference being that the partial differentiations are to be performed with respect to y and y' rather than x and x'. Thus (5) becomes, for a Gaussian surface,

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= (E_0/\eta)^2\iint_D\iint_D\exp\Big\{ik(A_1x-A_2x')\\ &-\frac{1}{2}k^2[B_1^2\sigma^2(x,y)+B_2^2\sigma^2(x',y')-2B_1B_2r(x,y;x',y')]\Big\}\\ &\cdot\Big\{\sin(\iota-\alpha_1)\sin(\iota-\alpha_2)\cos(\alpha_1-\alpha_2)\Big(\frac{\partial^2 r}{\partial y\,\partial y'}-k^2\Big[\frac{1}{2}B_1^2\frac{\partial\sigma^2(x,y)}{\partial y}\frac{\partial r}{\partial y'}\\ &+\frac{1}{2}B_2^2\frac{\partial r}{\partial y}\frac{\partial\sigma^2(x',y')}{\partial y'}-\frac{1}{4}B_1B_2\frac{\partial\sigma^2(x,y)}{\partial y}\frac{\partial\sigma^2(x',y')}{\partial y'}-B_1B_2\frac{\partial r}{\partial y}\frac{\partial r}{\partial y'}\Big]\Big)\\ &+\cos^2\iota+\sin^2\iota\frac{\partial^2 r}{\partial x\,\partial x'}-ik\sin\iota\cos\iota\Big[\frac{1}{2}\Big(B_1\frac{\partial\sigma^2}{\partial x}-B_2\frac{\partial\sigma^2}{\partial x'}\Big)\\ &+\Big(B_1\frac{\partial r}{\partial x'}-B_2\frac{\partial r}{\partial x}\Big)\Big]-k^2\sin^2\iota\Big[\frac{1}{2}\Big(B_1^2\frac{\partial\sigma^2}{\partial x}\frac{\partial r}{\partial x'}+B_2^2\frac{\partial r}{\partial x}\frac{\partial\sigma^2}{\partial x'}\Big)\\ &-B_1B_2\Big(\frac{\partial r}{\partial x}\frac{\partial r}{\partial x'}+\frac{1}{4}\frac{\partial\sigma^2}{\partial x}\frac{\partial\sigma^2}{\partial x'}\Big)\Big]\Big\}\,dx\,dy\,dx'\,dy'.\end{aligned}\tag{8}$$

Suppose now that the surface is stationary as well as Gaussian and that the parameter-domain is rectangular: $D = (-a, a) \times (-b, b)$. Since $u = x' - x$ and $v = y' - y$, we have in addition to formulas (4.14) and (4.15) that

$$\partial r/\partial y' = \partial r/\partial v = -\partial r/\partial y; \tag{9}$$

and

$$\partial^2 r/\partial y\, \partial y' = -\partial^2 r/\partial v^2. \tag{10}$$

On account of the stationarity, $\sigma^2(x, y) = \text{const.}$, and (8) can be written as

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= (E_0/\eta)^2 \exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2 + B_2^2)\}\\ &\cdot \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-a-x}^{a-x} du \int_{-b-y}^{b-y} dv \exp(icx - ikA_2u)[G(u, v) - iH(u, v)],\end{aligned} \tag{11}$$

where c is given by (4.17), and

$$\begin{aligned}G(u, v) = \exp[k^2B_1B_2r(u, v)]\Big\{&\cos^2 \iota - \sin^2 \iota \frac{\partial^2 r}{\partial u^2} - k^2 \sin^2\iota\, B_1B_2\left(\frac{\partial r}{\partial u}\right)^2\\ &- \sin(\iota - \alpha_1)\sin(\iota - \alpha_2)\cos(\alpha_1 - \alpha_2)\left[\frac{\partial^2 r}{\partial v^2} + k^2B_1B_2\left(\frac{\partial r}{\partial v}\right)^2\right]\Big\};\end{aligned} \tag{12}$$

$$H(u, v) = \exp[k^2B_1B_2r(u, v)]k \sin\iota \cos\iota\,(B_1 + B_2)\frac{\partial r}{\partial u}. \tag{13}$$

It follows from expressions (4.21) to (4.24) that

$$G(-u, -v) = G(u, v); \qquad G(-u, v) = G(u, -v); \tag{14}$$

$$H(-u, -v) = -H(u, v); \qquad H(-u, v) = -H(u, -v). \tag{15}$$

Formally, then, the integral in (11) resembles that in Eq. (4.16), and we can therefore write (11) at once in the form

$$\begin{aligned}\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} &= 4k^{-1}(A_1 - A_2)^{-1}(E_0/\eta)^2 \exp\{-\tfrac{1}{2}k^2\sigma^2(B_1^2 + B_2^2)\}\\ &\cdot \int_0^{2a}\int_0^{2b} \sin\left[k(A_1 - A_2)\left(a - \frac{u}{2}\right)\right](2b - v)\{\cos[\tfrac{1}{2}k(A_1 + A_2)u]\\ &\cdot [G(u, v) + G(u, -v)] - \sin[\tfrac{1}{2}k(A_1 + A_2)u][H(u, v) + H(u, -v)]\}\, du\, dv.\end{aligned} \tag{16}$$

Then the covariance of the scattered field for the case of horizontal polarization is given by

$$\begin{aligned}\operatorname{cov}[\mathbf{E}_s(P_1), \mathbf{E}_s(P_2)] &= (2\pi)^{-2}\omega^2\mu^2(R_1R_2)^{-1}\exp\{-ik(R_1 - R_2)\}\\ &\cdot [\mathcal{E}\{\mathbf{I}_h(\alpha_1)\cdot\mathbf{I}_h^*(\alpha_2)\} - \mathcal{E}\{\mathbf{I}_h(\alpha_1)\}\cdot\mathcal{E}^*\{\mathbf{I}_h(\alpha_2)\}],\end{aligned} \tag{17}$$

where the terms in the square bracket on the right are to be supplied from (7) and (1), (8) and (2), or (16) and (3) according as the process representing the random surface is separable, Gaussian, or stationary and Gaussian with rectangular domain D.

The variance of the scattered field for the case of horizontal polarization is obtained from (17) by setting $P_1 = P_2 = P$, so that $\alpha_1 = \alpha_2 = \alpha$. For instance in the stationary Gaussian case, for a rectangular parameter-domain D, one obtains in this way the following expression for the variance of the scattered field:

$$\operatorname{var}[\mathbf{E}_s(P)] = \tfrac{1}{2}k^2E_0^2(\pi R)^{-2}\exp(-k^2\sigma^2B^2)\Big(\int_0^{2a}\int_0^{2b}(2a-u)(2b-v)$$

$$\cdot\{\cos(kAu)[G_\alpha(u,v)+G_\alpha(u,-v)] \tag{18}$$

$$-\sin(kAu)[H_\alpha(u,v)+H_\alpha(u,-v)]\}\,du\,dv - 8a^2b^2\cos^2\iota\,\frac{\sin^2(kAa)}{(kAa)^2}\Big).$$

Expressions (7), (8), and (16) are of course closely related to the mean square energy flow $\mathcal{E}\{\mathbf{S}_h(P)\}$ at the field point P. As indicated by Eq. (3.14) one sets $P_1 = P_2 = P$ in (7), (8), or (16), and then multiplies the resulting expression by $1/2\eta^{-1}\omega^2\mu^2(2\pi R)^{-2}\mathbf{R}$ in order to obtain the mean value of the Poynting vector $\mathbf{S}_h(P)$.