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SUBHARMONIC OSCILLATIONS IN A NONLINEAR SYSTEM WITH POSITIVE DAMPING*

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Introduction. It is known from general theory[1,2] as well as from experiments, that the oscillations in a system with nonlinear restoring force, linear positive damping and an external driving force of period $2\pi/p$ tend to become periodic of period $k2\pi/p$ (k an integer), harmonic ($k = 1$) or subharmonic ($k > 1$). Subharmonic oscillations often have a much larger amplitude than the corresponding harmonic oscillations, and this is important in many applications, e.g. in electric circuits containing iron. Much work has been done on subharmonics[3], but the problem cannot be considered as solved. In most theories a Fourier series approximation with a few terms is used. However, the convergence of the series may be slow, the method will work only for a polynomial form of the nonlinear characteristic, it is laborious even for small k, and the stability of the solution is not easy to verify.

In this paper subharmonic oscillations are discussed for a general form of the restoring force by means of a difference method due to Cartwright and Littlewood and applied by the former to the van der Pol equation with forcing term. The method is especially useful for high orders. Preliminary experiments agree with theoretical results.

Method. For the following reasons the oscillation is considered as a free oscillation, slightly disturbed by the damping, the driving force and the nonlinearity.

1) From experiments it is known that subharmonics exist only if the damping is small.
2) The physically interesting case is that of a large subharmonic instead of a small harmonic oscillation.
3) A small nonlinearity makes it possible to work with first approximations although it does not always correspond to observed physical situations. The discussion is therefore confined to the equation

$$y'' + y = \epsilon\{F \sin (px + a) - \kappa y' - f(y)\},$$

$$\epsilon, F, \kappa > 0; \qquad f(y) + f(-y) = 0, \tag{1}$$

$$\epsilon \ll 1; \qquad f(y)/y \geq 0.$$

The derivative df/dy is assumed to exist for all values of y (weaker conditions are possible). The equation therefore has a unique solution, which is a continuous function of x and of the initial values.

For $\epsilon = 0$, Eq. (1) has solutions $b \sin x$ with b arbitrary. The solution for $\epsilon \neq 0$ can be expected to be close to one of these solutions in a finite interval[4]. The introduction

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[1]N. Levinson, Ann. Math. 45, 723, 1944.

[2]M. Cartwright, Ann. Math. Studies 20, Princeton, 1950.

[3]Ref. in Stoker, *Nonlinear vibrations in mechanical and electrical systems*, New York, 1950.

[4]H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. 1, Paris, 1892.

of the phase α makes it possible to write

$$y = b \sin x + \epsilon\eta(x) \tag{2}$$

A solution with initial values

$$y(0) = y'(0) = 0 \tag{3}$$

will cross the x-axis at $x = 2\pi + \tau$ where τ is small, with a value of the derivative $y'(2\pi + \tau) = b'$ and a phase

$$\alpha' = p(2\pi + \tau) + \alpha - k2\pi, \tag{4}$$

where k is the number of full periods of the driving force in the interval $0 \leq x \leq 2\pi + \tau$ (Fig. 1).

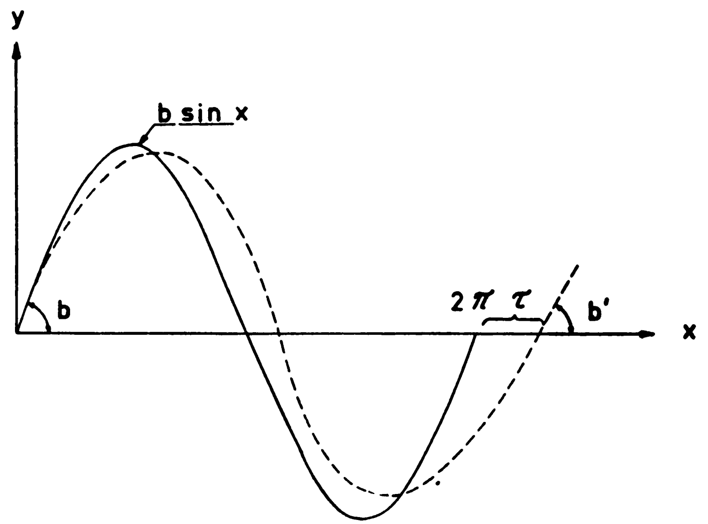

Fig. 1. Generating solution and disturbed solution (dotted)

Substituting b' for b and α' for α the process can be repeated. The solutions set up a transformation T of the point (b, α) on to $(b', \alpha') = T(b, \alpha)$. This transformation, which is (1,1) and continuous, is of interest only for $b > 0$, $0 \leq \alpha \leq 2\pi$.

The points fixed under T correspond to subharmonic solutions of period $k2\pi/p$ and the stability is determined by the properties of T in the neighborhood of a fixed point. For further information about this method the reader is referred to Reference 2.

Difference equations. The calculation of T is equivalent to finding a solution of (1) with initial values (3), valid for $0 \leq x \leq 2\pi + \tau$. This solution can be derived by successive approximations according to

$$y_{n+1}(x) = b \sin x + \epsilon \int_0^x \{F \sin (p\xi + \alpha) - \kappa y_n(\xi) - f(y_n)\} \sin (x - \xi)\, d\xi. \tag{5}$$

The functions y_n form a Cauchy-sequence converging for every finite x towards a continuous function y satisfying (1) and (3). The upper bound of $| y - y_n |$ is easily calculated. For ϵ small the first approximation is satisfactory.

Since τ is small of order ϵ

$$y(2\pi + \tau) = b\tau - \epsilon \int_0^{2\pi} \{F \sin (p\xi + \alpha) - \kappa b \cos \xi - f(b \sin \xi)\} \sin \xi \, d\xi + O(\epsilon^2) = 0, \tag{6}$$

$$y'(2\pi + \tau) = b + \epsilon \int_0^{2\pi} \{F \sin (p\xi + \alpha) - \kappa b \cos \xi - f(b \sin \xi)\} \cos \xi \, d\xi + O(\epsilon^2) = b, \tag{7}$$

and the difference equations defining T become

$$\frac{b' - b}{2\pi} = \frac{\epsilon F p}{\pi(p^2 - 1)} \sin p\pi \sin (p\pi + \alpha) - \frac{\epsilon \kappa b}{2} + O(\epsilon^2) = A(\alpha, b), \tag{8}$$

$$\frac{\alpha' - \alpha}{2\pi} = \frac{\epsilon F p}{\pi(p^2 - 1)} \sin p\pi \cos (p\pi + \alpha) - \epsilon p \frac{G(b)}{b} + p - k + O(\epsilon^2) = B(\alpha, b), \tag{9}$$

where

$$G(b) = \frac{1}{2\pi} \int_0^{2\pi} f(b \sin \xi) \sin \xi \, d\xi. \tag{10}$$

Periodic solutions. The subharmonic solutions of period $k\ 2\pi/p$ are found for $b' = b$, $\alpha' = \alpha$. If the curves $A = 0$, $B = 0$ are not nearly parallel the existence of an exactly periodic solution is obvious. To the first order it is given by

$$\frac{\epsilon F}{\pi b(p^2 - 1)} \sin p\pi \sin (p\pi + \alpha) = \frac{\epsilon \kappa}{2p}, \tag{11}$$

$$\frac{\epsilon F}{\pi b(p^2 - 1)} \sin p\pi \cos (p\pi + \alpha) = \frac{\epsilon G(b)}{b} - \frac{p - k}{p}. \tag{12}$$

These equations determine the amplitude and the phase. After squaring and adding one finds the amplitude equation

$$\left[\frac{\epsilon F \sin p\pi}{\pi b(p^2 - 1)}\right]^2 = \left[\frac{\epsilon \kappa}{2p}\right]^2 + \left[\frac{\epsilon G(b)}{b} - \frac{p - k}{p}\right]^2. \tag{13}$$

For a discussion of the amplitude as a function of the frequency it is convenient to write

$$\frac{\epsilon G(b)}{b} = \frac{p - k}{p} \pm C, \tag{14}$$

$$b = \frac{\epsilon F \mid \sin p\pi \mid}{\pi(p^2 - 1)} \left(\frac{\epsilon \kappa}{2p} + C^2\right)^{-1/2}. \tag{15}$$

The curves corresponding to (14) and (15) are easily constructed for different values of C and their points of intersection give the curves $b(p)$ (Fig. 2). In general these curves have closed parts near the curve defined by (14) for $C = 0$, and also a low branch, where the first approximation is not very good, however, since ϵF is not small compared to b. From the practical point of view, too, only the large amplitude subharmonic is of interest and so the low branch will not be discussed.

Eq. (13) is valid if $(p - k) \ll p$, which means a large interval of frequency for $k \gg 1$. The method permits a discussion of high order subharmonics, but it has been

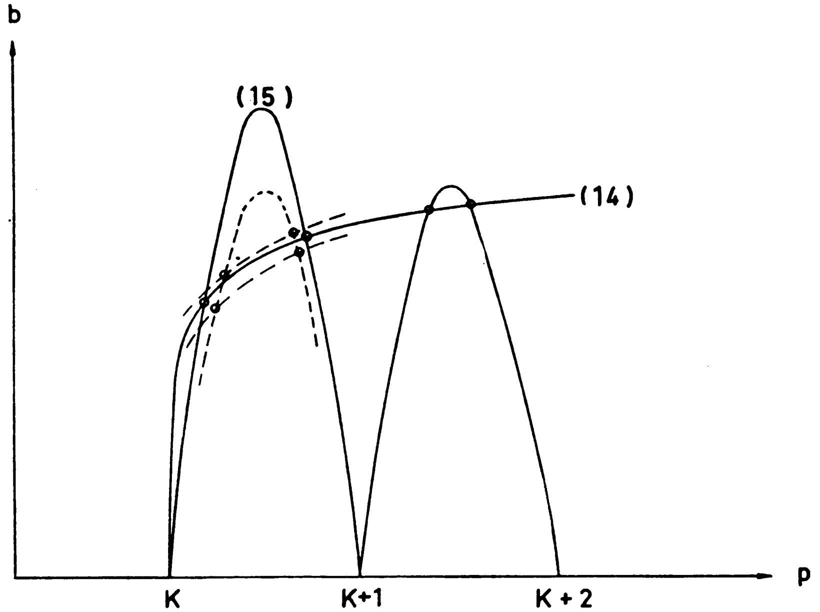

FIG. 2. Construction of response-curves according to (14) and (15). Solid curves: $C = 0$. Dotted curves: $C = C_1$.

used for $k = 2$ with good result. With $k = 1$, $(p - k) \ll 1$, Eq. (13) gives a part of the harmonic response curve, which may be used for determining the function $G(b)$.

Stability. If all points close to a fixed point P_0 move towards P_0 under T, the corresponding solution is stable, if some points move away from P_0 it is unstable. The fixed points can be classified similarly to the singular points of a first order equation[2]. The transformation vector is approximately tangential to the curves defined by $db/d\alpha = A/B$, for which the characteristic equation near a singular point (b_0, α_0) is

$$\left[\lambda + \frac{\epsilon\kappa}{2}\right]^2 + p^2\left(\frac{\epsilon G(b_0)}{b_0} - \frac{p-k}{p}\right)\left(\epsilon G'(b_0) - \frac{p-k}{p}\right) = 0. \tag{16}$$

Instability occurs when

$$\left(\frac{\epsilon G(b_0)}{b_0} - \frac{p-k}{p}\right)\left(\epsilon G'(b_0) - \frac{p-k}{p}\right) < -\left(\frac{\epsilon\kappa}{2p}\right)^2.$$

At the point where the response curve enters the unstable region $\partial p/\partial b = 0$ and $\partial F/\partial b = 0$. From this point jumps are possible (Fig. 3).

Fig. 4 gives an idea of the general form of subharmonic response curves.

Remarks. Subharmonic oscillations of a given order exist in certain frequency bands.

If several states of subharmonic oscillation are possible at a given frequency the amplitude is largest and the frequency band smallest for the lowest order.

For a given form of nonlinearity the highest order obtainable is a function of the ratio F/κ, which also determines the width of the frequency bands. The condition for the order k to exist is roughly

$$\epsilon G\left(\frac{2F}{\pi k \kappa}\right) > \frac{F}{\pi k^2 \kappa} \cdot$$

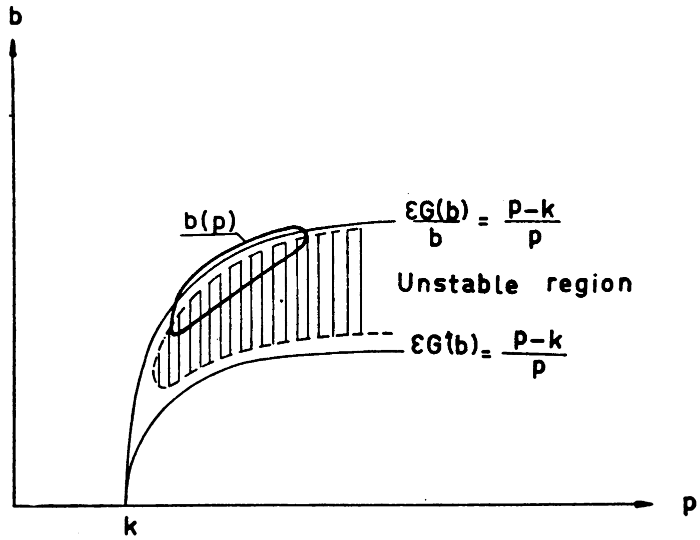

FIG. 3. Stability of subharmonic oscillations.

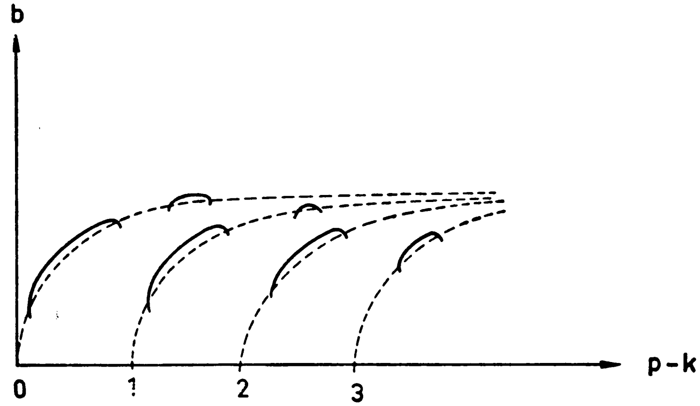

FIG. 4. Amplitude of subharmonic oscillations (solid curves). On branch starting at $p = k$ the frequency of the subharmonic oscillation is p/k.

There is no relation between the analytical form of and the possible values of k, as is sometimes stated[5]. If $f(y)$ is not odd a constant term must be added in Eq. (2).

Experiment. Experiment on subharmonics have recently been made by Messrs. Hansson and Göransson, who have kindly permitted me to quote their results[6]. They

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[5]C. Hayashi, Journ. Appl. Phys. 24, 521, 1953.

[6]K. Göransson and L. Hansson, Tekn. Högskolans Handlingar (in press).

have studied the oscillations in an electric circuit containing iron, where the nonlinearity is essentially of the seventh degree. The damping is reduced by an electronic device, so it has been possible to obtain all subharmonics from the second to the ninth order. For small values of ϵ a good agreement with this theory is obtained. The band character and the existence of an upper limit for k has been clearly demonstrated. The subharmonic state was excited by a voltage pulse of the same magnitude as b.

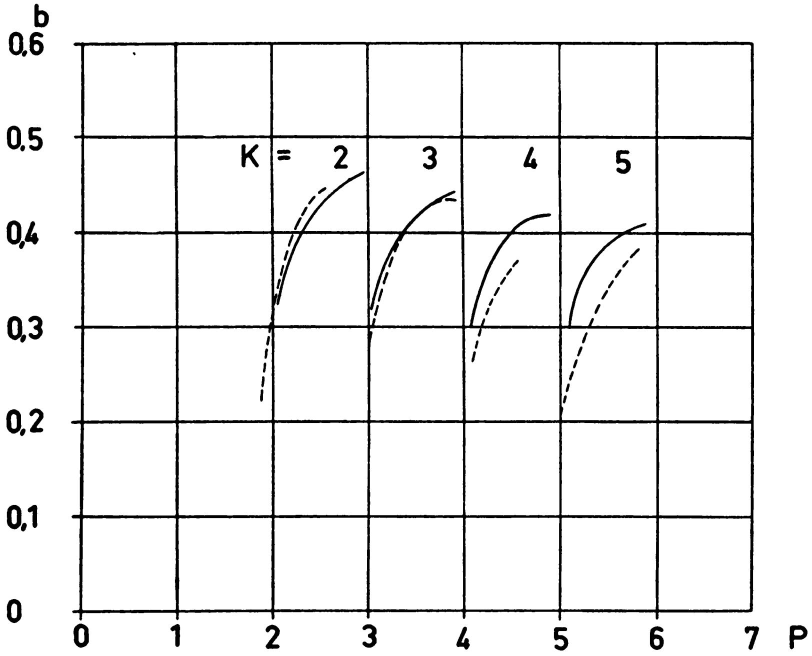

FIG. 5. Experimental curves (dotted) according to K. Göransson and L. Hansson and theoretical values from Eq. (13).

ON THE STABILITY OF THE AXIALLY SYMMETRIC LAMINAR JET*

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1. The stability of the axially symmetric laminar jet will be investigated herein subject to rotationally symmetric disturbances. We suppose the laminar jet to be issuing from a small hole with the motion symmetrical about the z axis which is aligned with the jet. The angular position of any point is given by the angle φ measured from the positive x-axis, and r the perpendicular distance from the axis. Thus a set of cylindrical polar coordinates is used.

The steady state flow can be solved in closed form and is given in Goldstein [Ref. 1].

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