The bracket in Eq. (4) is positive, if the speed increases as one moves from P in the direction of the resulting velocity to a neighboring point P_1 on the stream line through P. If the images of the points P and P_1 in the hodograph plane are denoted by P'' and P_1'', the bracket in (4) is therefore positive, if P'' is closer to the origin O'' of the hodograph plane than P_1''. As the signs of u and v are readily determined, we have therefore a convenient criterion for the sign of the right-hand side of (4), that is for the sign of the plastic power. For a steady flow, in particular, this criterion requires that the particle that is instantaneously at P should be accelerating or decelerating according to whether u and v have the same signs or not.

Bibliography

1. W. Prager, Trans. Roy. Inst. Technology, Stockholm, No. 65 (1953)
2. A. P. Green, J. Mech. Phys. Solids 2, 73, 296 (1954)

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NEGATIVE SPECTRA*

By C. R. PUTNAM (*Purdue University*)

Let $f(t)$ be a real-valued continuous function on the t-interval $0 \leqq t \leqq 2T$ and let $x = x(t)$ be a real-valued solution ($\not\equiv 0$) of the differential equation

$$x'' + f(t)x = 0. \tag{1}$$

If $f^+ = f$ or 0 according as $f \geqq 0$ or $f < 0$, and if $x(a) = x(b) = 0$, where $a < b$, then there holds the inequality

$$\int_a^b f^+(t)\, dt > 4/(b - a), \tag{2}$$

due essentially to Liapounoff (see [1]; also [8]). Moreover, according to [4] (see also [1]), the constant 4 of (2) is the best possible, in the sense that (2) need not hold (for arbitrary f) if the 4 is replaced by $4 + \epsilon$, where $\epsilon > 0$. Hence, it is easy to see that the inequality

$$T \int_0^T [f^+(t) + f^+(2T - t)]\, dt \qquad \left(= T \int_0^{2T} f^+(t)\, dt\right) > 2 \tag{3}$$

is necessary, but $T \int_0^{2T} f^+(t)\, dt > 2 + \epsilon$ is not, in order that the Sturm-Liouville boundary value problem

$$L(x) + \lambda x = 0 \qquad [L(x) \equiv x'' + fx], \qquad x(0) = x(2T) = 0 \tag{4}$$

possess an eigenvalue $\lambda < 0$ (or even $\leqq 0$).

It will be shown in this note that there is a *sufficient* criterion similar to (3). In fact,

*Received June 11, 1954. This work was supported by the National Science Foundation research grant NSF-G-481.

the following will be proved: *If $f(t)$ satisfies the inequality*

$$M(f) \equiv T^{-1} \int_0^T t^2[f(t) + f(2T - t)]\, dt > 2, \tag{5}$$

then the spectrum of the boundary value problem (4) *possesses some eigenvalue* $\lambda < 0$. Furthermore, the constant 2 occurring on the right side of (5) is the best possible in the sense that for every $\epsilon > 0$, there is a continuous function $f(t)$ satisfying $M(f) > 2 - \epsilon$, and such that the values λ in the spectrum of the eigenvalue problem (4) satisfy $\lambda > 0$. (Note that the similarity between (3) and (5) is particularly apparent if $f(t) \geqq 0$, so that $f = f^+$.)

If one grants, for the present, the italicized assertion, the proof of the claim following it is easy. In fact, let a continuous function $f(t)$ be defined on $0 \leqq t \leqq 2T$ so that $f(t) \geqq 0$ and so that

$$2 - \epsilon < M(f) < 2 - \epsilon/2 \qquad \text{and} \qquad \left| M(f) - T \int_0^{2T} f(t)\, dt \right| < \epsilon/2. \tag{6}$$

Clearly this can be done; one need only define f to be 0 everywhere on $0 \leqq t \leqq 2T$ except on some sufficiently small interval about T. Then the spectrum of the boundary value problem (5), for the above constructed f, consists of positive eigenvalues only. In fact, if there were an eigenvalue $\lambda \leqq 0$, there would exist a solution $x(t) \not\equiv 0$ of (1) satisfying $x(0) = x(T_1) = 0$, $0 < T_1 \leqq 2T$. Consequently, relation (2) for $a = 0$ and $b = T_1$ would now imply (3), that is $T \int_0^{2T} f(t)\, dt > 2$, in contradiction with (6).

There remains then to prove the italicized assertion. To this end, suppose, if possible, that there does not exist a negative eigenvalue, so that, in view of the Parseval equality, the relation

$$\int_0^{2T} (x'^2 - fx^2)\, dt = -\int_0^{2T} xL(x)\, dt = \sum \lambda_k c_k^2 \geqq 0 \tag{7}$$

holds for all real-valued functions x possessing continuous second derivatives on $0 \leqq t \leqq 2T$ and satisfying $x(0) = x(2T) = 0$. Here, the constants c_k are defined by $c_k = \int_0^{2T} x\phi_k\, dt$, where ϕ_k is the normalized eigenfunction belonging to the eigenvalue λ_k. (In connection with (7), see [3], p. 392 and the reference given there to Hamel [2]. For applications of an inequality similar to (7) but relating to singular boundary value problems [7] on the half-line $0 \leqq t < \infty$, see, e.g., [5], [6].) Next, define the continuous function $y(t)$ on $0 \leqq t \leqq 2T$ by putting $y(t) = t$ on $0 \leqq t \leqq T$ and $y(t) = 2T - t$ on $T \leqq t \leqq 2T$. (See [1], p. 69.) Then

$$\int_0^{2T} (y'^2 - fy^2)\, dt = 2T - \int_0^T t^2 f(t)\, dt - \int_T^{2T} (2T - t)^2 f(t)\, dt.$$

It follows from (5) that

$$\int_0^{2T} (y'^2 - fy^2)\, dt < 0.$$

Clearly, the function y can be approximated by functions x which possess continuous second derivatives on $0 \leqq t \leqq 2T$, satisfy $x(0) = x(2T) = 0$, and which violate the inequality of (7). This violation yields a contradiction and shows that the assumption that the spectrum contains no negative eigenvalues is untenable. The proof of the italicized assertion is thus complete.

References

[1] G. Borg, *On a Liapounoff criterion of stability*, Am. J. Math. **71**, 67-70 (1949)

[2] G. Hamel, *Ueber das infinitäre Verhalten der Integrale einer linearen Differentialgleichung zweiter Ordnung, wenn die characterische Gleichung zwei gleiche Wurzeln hat*, Math. Z., **1**, 220-228 (1918)

[3] P. Hartman, *On non-oscillatory differential equations of second order*, Am. J. Math. **74**, 389-400 (1952)

[4] E. R. van Kampen and A. Wintner, *On an absolute constant in the theory of variational stability*, Am. J. Math. **59**, 270-274 (1937)

[5] C. R. Putnam, *An oscillation criterion involving a minimum principle*, Duke Math. J. **16**, 633-636 (1949)

[6] C. R. Putnam, *On the least eigenvalue of Hill's equation*, Q. Appl. Math. **9**, 310-314 (1951)

[7] H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. **68**, 220-269 (1910)

[8] A. Wintner, *On the non-existence of conjugate points*, Am. J. Math. **73**, 368-380 (1951)

Correction to my paper

A NEW SINGULARITY OF TRANSONIC PLANE FLOWS*

Quarterly of Applied Mathematics, XII, 343-349 (1955)

By A. R. MANWELL (*University College, Swansea*)

A much more detailed study of the singular solution discussed rather briefly in the note of the above title has shown that several statements in Sec. 4 are incorrect. Briefly, the expansions (4.4) and so also (4.5), (4.6) are valid only *locally* for either $\Theta = 0$ or for $\Theta = \pi$, but not necessarily for both. We may not infer from these expansions the existence of solutions in the whole interval $(0, \pi)$. (In particular, on account of the pole at $Z = 1$, we may not replace in (4.3) a contour for which $Z - 1 = 2i \exp(i\Theta) \sin \Theta$ is very small by the unit circle $Z = 1$).

A correct discussion shows that (4.3) and (4.4) yield only *two* independent solutions. As a consequence, the singular solution can be smoothly continued across the sonic line for $\Theta > 0$ but, unless we admit further singularities in the supersonic region, the flow would not join up smoothly for $\theta < 0$. Since we are seeking possible criteria for the breakdown of flow solutions, this correction leads to a slight strengthening of our original conclusion.

*Received March 7, 1955.

A MINIMUM PRINCIPLE OF PLASTICITY*

By D. TRIFAN (*University of Arizona*)

This note is concerned with the removal of a certain restriction imposed by a proof[1] [Sec. 5] of a minimum principle of an isotropic, incompressible, strain-hardening material exhibiting a gradual transition from the elastic to the plastic state. The governing stress-strain relation for loading is given by

$$s_{ij}^* = 2G_0\epsilon_{ij}^* - p(E)\epsilon_{ij}E^*, \tag{1}$$

*Received Feb. 28, 1955.

[1]Section numbers enclosed in brackets refer to the following paper: D. Trifan, *A new theory of plastic flow*, Q. Appl. Math. **7**, pp. 201-211 (1949).