exponentially growing term as well as a damping term. With the exception of very special initial conditions, both terms will be present and the tube will be unstable.

4. Further remarks. It appears that the behavior of the higher order perturbation terms cannot be obtained as simply as those discussed above. These may require explicit determination of the function F_0 , F_1 , F_2 , $\cdots$. It should be noted, however, that if F_0 , $\cdots$, F_{n-1} have been found, D_n can be determined by quadratures. Furthermore, the differential equation for F_n will be of the form

$$F_n^{\mathrm{IV}} - D_0^2 F_n = \text{previously determined functions.}$$

The Green's function for this equation, in the case of simply supported ends is known [5], and can be determined for the other boundary conditions by standard methods. Thus F_n can be found by integration.

These comments indicate that the perturbation terms can be computed step-by-step by quadratures. Furthermore, in the supported end cases the critical velocity can be determined beforehand and perturbation from this point in powers of D will serve as a check on the perturbation solution in terms of powers of u.

References

[1] R. H. Long, Jr., *Experimental and theoretical study of transverse vibration of a tube containing flowing fluid*, A.S.M.E., Preprint 54-A-22
[2] F. I. N. Niordson, *Vibrations of a cylindrical tube containing flowing fluid*, Trans. Roy. Inst. Tech., Stockholm, Nr. 73, (1953)
[3] Lothar Collatz, *Eigenwertprobleme*, Chelsea Publishing Co., New York, 1948, p. 305 ff.
[4] L. Collatz, *Ibid*, p. 54
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ON AN OSCILLATION CRITERION OF DE LA VALLÉE POUSSIN*

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An oscillation criterion of de la Vallée Poussin[1] on homogeneous, linear differential equations of order n, when particularized[2] to $n = 2$, runs as follows: Let both coefficient functions of

$$x'' + g(t)x' + f(t)x = 0 \tag{1}$$

be real-valued and continuous on a t-interval and suppose that (1) has a solution $x(t) \not\equiv 0$ which vanishes for at least two points of that t-interval, say at $t = 0$ and at $t = h > 0$ (so that

$$x(0) = 0, \qquad x(h) = 0, \tag{2}$$

where, without loss of generality, $x(t) \neq 0$ when $0 < t < h$). Then

$$1 < M_1 h + M_2 h^2/2, \tag{3}$$

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[1]G. Sansone, *Equazioni differenziali nel campo reale*, vol. 1, 1948, p. 183.

[2]F. Tricomi, *Equazioni differenziali*, 1948, p. 110.

where

$$M_1 = \max_{0 \leqq t \leqq h} | g(t) |, \qquad M_2 = \max_{0 \leqq t \leqq h} | f(t) |. \tag{4}$$

It will be shown in this note that (3) can be improved to

$$1 < M_1 h/2 + M_2 h^2/6. \tag{5}$$

Incidentally, it will also follow that the assumption that $g(t)$ and $f(t)$ are real-valued, an assumption used in de la Vallée Poussin's proof, can be omitted.

It will also follow that, instead of assuming (5) for the numbers (4), it is sufficient to assume

$$1 < \int_0^h | g(t) | \, dt + \frac{h}{4} \int_0^h | f(t) | \, dt \tag{6}$$

for the coefficient functions g, f of (1).

In order that (1) has a solution $x(t) \not\equiv 0$ satisfying (2), a necessary condition for (1), containing both (5) and (6), can be formulated as follows:

$$h < \max \left\{ \int_0^h t \, | g(t) | \, dt, \quad \int_0^h (h - t) \, | g(t) | \, dt \right\} + \int_0^h t(h - t) \, | f(t) | \, dt. \tag{7}$$

If $g(t) \equiv 0$, then (1) reduces to

$$x'' + f(t)x = 0 \tag{8}$$

and (6) to

$$4 < h \int_0^h | f(t) | \, dt. \tag{9}$$

Hence the criterion (7) to be proved generalizes the following well-known fact, which goes back to Liapounoff[3]: If $f(t)$ is continuous for $0 \leqq t \leqq h$, then a necessary condition for (8) to have a solution ($\not\equiv 0$) satisfying (2) is (9).

The proof of the statement concerning (7) depends on a device used by Nehari[4] for a similar purpose and proceeds as follows:

First, if $x(t)$, where $0 \leqq t \leqq h$, is any function possessing a continuous second derivative and satisfying (2), then it is readily verified that

$$hx(t) = (h - t) \int_0^t s x''(s) \, ds + t \int_t^h (h - s) x''(s) \, ds$$

is an identity for $0 \leqq t \leqq h$. A differentiation gives

$$hx'(t) = -\int_0^t s x''(s) \, ds + \int_t^h (h - s) x''(s) \, ds. \tag{10}$$

Next, it is clear from (2) that both μt and $\mu(h - t)$ are majorants for $| x(t) |$ if $0 \leqq t \leqq h$ and

$$\mu = \max_{0 \leqq t \leqq h} | x'(t) |. \tag{11}$$

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[3] P. Hartman and A. Wintner, *On an oscillation criterion of Liapounoff*, Amer. J. Math. **73**, 885–890 (1951), where further references will be found.

[4] Z. Nehari, *On the zeros of solutions of second order linear differential equations*, Amer. J. Math. **76**, 690 (1954).

Thus

$$| x(t) | \leqq \mu\varphi(t), \tag{12}$$

where

$$\varphi(t) = \min (t, h - t), \tag{13}$$

and it is clear that the $\leqq$ in (12) is a $<$ for some t.

If x'' is substituted from (1) into (10) (as $-gx' - fx$), it is seen from (11) and (12) that

$$h | x'(t) | \leqq \mu \int_0^t s(| g(s) | + \varphi(s) | f(s) |) \, ds + \mu \int_t^h (h - s)(| g(s) | + \varphi(s) | f(s) |) \, ds. \tag{14}$$

Let the maximum of $| x'(t)|$ on $0 \leqq t \leqq h$ be attained at $t = t_0$. Then (14) and the remark made after (13) imply that

$$h < \int_0^{t_0} s(| g(s) | + \varphi(s) | f(s) |) \, ds + \int_{t_0}^h (h - s)(| g(s) | + \varphi(s) | f(s) |) \, ds, \tag{15}$$

since $| x'(t_0)| = \mu$, by (11), and since $\mu \neq 0$, by (2), where $x(t) \not\equiv 0$.

The definition (13) of $\varphi(t)$ shows that both $s\varphi(s)$ and $(h - s)\varphi(s)$ are majorized by $s(h - s)$ for $0 \leqq s \leqq h$. Hence (15) can be written as

$$h < \int_0^{t_0} s | g(s) | \, ds + \int_{t_0}^h (h - s) | g(s) | \, ds + \int_0^h s(h - s) | f(s) | \, ds. \tag{16}$$

If the sum S of the first two integrals is considered as a function of t_0, it is seen that its derivative is

$$dS/dt_0 = t_0 | g(t_0) | - (h - t_0) | g(t_0) | = (2t_0 - h) | g(t_0) |,$$

which is non-positive or non-negative according as $t_0 \leqq \frac{1}{2}h$ or $t_0 \geqq \frac{1}{2}h$. Hence the maximum of $S = S(t_0)$ for $0 \leqq t_0 \leqq h$ is attained either at $t_0 = 0$ or at $t_0 = h$. This fact, when combined with (16), leads to the criterion (7).

Ad (6). Since neither s nor $h - s$ exceeds h for $0 \leqq s \leqq h$ and since $s(h - s) \leqq (h/2)^2$, condition (6) is contained in (7).

Ad (5). In view of (4), the inequality (7) implies that

$$h < M_1 \max \left\{ \int_0^h s \, ds, \int_0^h (h - s) \, ds \right\} + M_2 \int_0^h s(h - s) \, ds.$$

Here the factor of M_1 is $h^2/2$ and that of M_2 is $h^3/6$. Hence (5) is contained in (7).