From this representation the value of $E(D_n^k)$ may be obtained by retaining in the above expression only the terms that yield a non-zero value after z_{kl} has been set equal to zero.

A particularly interesting case is that where $x_{kl} = \pm 1$ with equal probability. Then

$$i^{nk}E(D_n^k) = \Theta_n^k \left[\prod_{k,l=1}^{n} (\cos z_{kl}) \right]_{z_{kl}=0}. \tag{4.3}$$

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AN INEQUALITY FOR THE FIRST EIGENVALUE OF AN ORDINARY BOUNDARY VALUE PROBLEM*

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Let both coefficient functions of the differential equation

$$y'' + g(x)y' + f(x)y = 0 \tag{1}$$

be real-valued and continuous on the interval $a \leq x \leq b$ and, unless $g'(x)$ is not involved (as it is not in (4) below), suppose that the coefficient of y' has a continuous first derivative $g'(x)$. Consider the boundary condition

$$y(a) = 0, \qquad y(b) = 0. \tag{2}$$

A solution $y(x)$ of (1) satisfying (2) is the trivial solution,

$$y(x) \equiv 0. \tag{3}$$

In what follows, conditions on the function pair (g, f) will be considered which assure that (1) has no solution $y(x)$, distinct from (3), satisfying (2).

Such a condition is known to be

$$f \leq 0 \tag{4}$$

(with an arbitrary g). Another such condition is

$$f - \tfrac{1}{2}g' \leq 0. \tag{5}$$

Still another one is

$$f - g' \leq 0. \tag{6}$$

(It is understood that each of these three conditions is required for all values of x on the interval $a \leq x \leq b$.) Actually, the sufficiency of (4), (5) and (6) is contained in the results of Paraf, Picard and Lichtenstein, respectively, on (elliptic) *partial* differential equations [1]. The method proving the sufficiency of Lichtenstein's condition (6) is quite different from that proving the sufficiency of Picard's condition (5) or of the more primitive condition (4), and it is clear that no one of the three conditions (4)-(6) need be satisfied if the other two are satisfied.

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A comparison of (5) with (6) on the one hand and with (4) on the other will suggest that the $\frac{1}{2}$ in (5) can be replaced by any constant between $\frac{1}{2}$ and 1 or between 0 and $\frac{1}{2}$. But the methods referred to do not reveal the posibility of such generalization. It turns out, however, that this generalization is correct; in other words, that it is sufficient to assume, for all x on the interval $a \leq x \leq b$ and for some $\theta =$ const., that

$$f(x) - \theta g'(x) \leq 0, \qquad \text{where} \qquad 0 \leq \theta \leq 1. \tag{7}$$

According to Picard, his criterion (5) can be generalized to the following requirement: For some real constant c, the inequality

$$(cg - \tfrac{1}{2}g)^2 + (f - cg') \leq 0 \tag{8}$$

is satisfied at every point of the interval $a \leq x \leq b$ (in fact, (8) results if the function which Picard [2] denotes by $\lambda = \lambda(x)$ is chosen to be the function $cg(x)$, where $c =$ const. is arbitrary). If (8) is written in the form

$$f(x) - cg'(x) + (c - \tfrac{1}{2})^2 g^2(x) \leq 0, \tag{9}$$

it is seen that the sufficiency of (7) cannot be concluded from it. The converse inference is also impossible (unless $0 \leq c \leq 1$).

It will, however, be shown that condition (9) can be replaced by the following requirement: For some real constant c,

$$f(x) - cg'(x) + c(c - 1)g^2(x) \leq 0 \tag{10}$$

holds at every point of the interval $a \leq x \leq b$. In other words, it will be shown that the existence of a real constant c satisfying (10) is sufficient in order that (3) be the only solution of (1) satisfying (2).

This will imply the sufficiency of (7). In fact, if $0 \leq c \leq 1$, then $c(c - 1) \leq 0$. Hence, if $c = \theta$, where $0 \leq \theta \leq 1$, then (7) is sufficient for (10).

The proof for the sufficiency of (10) proceeds as follows. Choose any real constant c and, in terms of any solution $y = y(x)$ of (1), define a function $Y = Y(x)$ by placing

$$Y(x) = y(x) \exp\left[c \int_a^x g(t)\, dt\right]. \tag{11}$$

Then (2) is equivalent to

$$Y(a) = 0, \qquad Y(b) = 0, \tag{12}$$

and (3) to

$$Y(x) \equiv 0, \tag{13}$$

finally (1) to a differential equation of the same form as (1), say

$$Y'' + G(x)Y' + F(x)Y = 0. \tag{14}$$

In fact, if (11) is written in the form $y = Y \exp(- \cdots)$ and if this y is substituted into (1), an easy calculation shows that (1) appears in the form (14), with

$$G = (1 - 2c)g \tag{15}$$

and

$$F = f - cg' + c(c - 1)g^2. \tag{16}$$

Identify (14) with (1) and apply the criterion (4). It then follows that (13) is the only solution of (14) satisfying (12) if the condition $F \leq 0$, which corresponds to (4), is satisfied. But the definition (16) shows that the condition $F \leq 0$ is identical with assumption (10). Since (12) is equivalent to (2), and (13) to (3), the proof is complete.

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A NOTE ON THE TRANSVERSE VIBRATION OF A TUBE CONTAINING FLOWING FLUID*

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1. Introduction. The problem of determining the lowest frequency of vibration of a tube containing flowing fluid has recently received considerable attention. Long [1][1] has considered the tube as a beam and has calculated the frequencies for various end conditions by a power series method. He has pointed out that care must be taken in evaluating the resulting high order determinants to avoid erroneous results. Niordson [2] has given a very elegant treatment of the problem based on shell theory and has derived the beam equation as one approximation. Furthermore, for the case of simply supported ends, he has given closed analytic solutions for two limiting cases of the parameters and numerical results for other values. It is the purpose of this note to show that it is possible to determine the nature of the frequencies for the various end conditions used by Long for two ranges of the flow velocity solely from the structure of the differential equation without determining specific solutions.

Long has written the differential equation governing the deflection of the beam in the following non-dimensional form:

$$F^{\mathrm{IV}} + ku^2F^{\mathrm{II}} + iuDF' - D^2F = 0, \tag{1}$$

where F is the deflection and u the flow velocity (both in non-dimensional terms). Roman numerals denote differentiation with respect to x $(0 \leqq x \leqq 1)$ and D is the eigenvalue found by removing the time dependent term exp (iDt) from the original partial differential equation. The boundary conditions considered are

Both ends simply supported:	$F(0) = F''(0) = F(1) = F''(1) = 0,$
Both ends fixed:	$F(0) = F'(0) = F(1) = F'(1) = 0,$
Fixed-simple ends:	$F(0) = F'(0) = F(1) = F''(1) = 0,$
Fixed-free ends:	$F(0) = F'(0) = F''(1) = F'''(1) = 0.$

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[1]Numbers in square brackets refer to the list of references given at the end of this note.