SIMPLE SOLUTIONS OF SAINT-VENANT TORSION PROBLEM BY USING TCHEBYCHEFF POLYNOMIALS*

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Summary. By using Tchebycheff polynomials, various forms of the stress functions associated with torsion problems have been obtained. The method can be applied for various types of cross sections. It is well known that these problems can be solved by other methods, but the method presented here gives a simple and direct solution in many cases.

Introduction. The problem of torsion of prismatical bars by couples M , applied at the ends was first studied by Barré De Saint-Venant who discussed several solutions of the equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2\mu\alpha \quad (1)$$

where ψ is a stress function, μ the modulus of rigidity and α the angle of twist per unit length of the bar

$$\psi = \text{constant along the boundary of the cross section.} \quad (2)$$

The magnitude of the torque M , on any cross section is given (See for example [1]) by

$$M = 2 \iint \psi \, dx \, dy. \quad (3)$$

The only non-vanishing stresses are $\widehat{xz}$ and $\widehat{yz}$ and these are given by

$$\widehat{xz} = \frac{\partial \psi}{\partial y}, \quad \widehat{yz} = -\frac{\partial \psi}{\partial x}. \quad (4)$$

Many attempts were made to solve Eq. (1) to satisfy the boundary condition (2) for various cross sections, and in one of these methods the solution was obtained by the aid of the complex variable.

We shall here give a simple elementary solution for Eq. (1), and the results are expressed in terms of Tchebycheff polynomials which are already tabulated. In putting $\psi/(-2\mu\alpha) = \frac{1}{4}(x^2 + y^2) + \phi$; Eq. (1) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (5)$$

Using polar coordinates, $x = r \cos \theta$, $y = r \sin \theta$, and noting that

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}, \end{aligned}$$

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Eq. (5) reduces to the well known Laplace's equation in polar coordinates viz:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2} = 0. \quad (6)$$

An elementary solution of Eq. (6) can be obtained by putting

$$\phi = F_1(\theta) \cdot F_2(r).$$

It can be easily found that

$$\begin{aligned} F_1(\theta) &= E_n \cos n\theta + H_n \sin n\theta, \\ F_2(r) &= K_n r^n + L_n r^{-n}, \end{aligned} \quad n \neq 0$$

where E_n , H_n , K_n and L_n are arbitrary constants. If $n = 0$, we have the solutions

$$\begin{aligned} F_1 &= a_0 + b_0 \theta, \\ F_2 &= c_0 \log r + d_0 \end{aligned}$$

where a_0 , b_0 , c_0 and d_0 are arbitrary constants. The general solution of Eq. (6) can be put in the form

$$\begin{aligned} \phi &= A_0 + B_0 \theta + (C_0 + D_0 \theta) \log r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta \\ &\quad + (C_n r^n + D_n r^{-n}) \sin n\theta. \end{aligned} \quad (7)$$

The stress function ψ is then given by

$$\begin{aligned} \frac{\psi}{-2\mu\alpha} &= \frac{1}{4} r^2 + A_0 + B_0 \theta + (C_0 + D_0 \theta) \log r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta \\ &\quad + (C_n r^n + D_n r^{-n}) \sin n\theta. \end{aligned} \quad (8)$$

The Tchebycheff polynomials. The Tchebycheff polynomials of the first kind $T_n(\cos \theta)$ are defined as follows:

$$T_n(\cos \theta) = \cos n\theta. \quad (9)$$

These polynomials have been treated in quite an interesting manner in [2]. We shall give here only what is needed in this paper. Evidently $T_0(\cos \theta) = 1$ and $T_1(\cos \theta) = \cos \theta$; $T_n(\cos \theta)$ for any value of n can be found by using the following recurrence relation

$$T_{n+1}(\cos \theta) - 2 \cos \theta T_n(\cos \theta) + T_{n-1}(\cos \theta) = 0 \quad (10)$$

which can be proved as follows.

Since $\cos(n+1)\theta + \cos(n-1)\theta = 2 \cos n\theta \cos \theta$ we have

$$T_{n+1}(\cos \theta) - 2 \cos \theta T_n(\cos \theta) + T_{n-1}(\cos \theta) = 0. \quad (11)$$

By using the recurrence relation (10), we obtain the following values of $T_n(\cos \theta)$.

$$T_0(\cos \theta) = 1,$$

$$T_1(\cos \theta) = \cos \theta,$$

$$T_2(\cos \theta) = 2 \cos^2 \theta - 1,$$

$$T_3(\cos \theta) = 4 \cos^3 \theta - 3 \cos \theta,$$

$$T_4(\cos \theta) = 8 \cos^4 \theta - 8 \cos^2 \theta + 1,$$

$$T_5(\cos \theta) = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

and so on.

The Tchebycheff polynomials of the second kind $U_n(\cos \theta)$ are defined as follows:

$$U_n(\cos \theta) = \sin n\theta.$$

Evidently $U_0(\cos \theta) = 0$, $U_1(\cos \theta) = \sin \theta$.

Since $\sin(n+1)\theta + \sin(n-1)\theta = 2 \sin n\theta \cos \theta$, we obtain the recurrence relation

$$U_{n+1}(\cos \theta) - 2 \cos \theta U_n(\cos \theta) + U_{n-1}(\cos \theta) = 0. \quad (12)$$

By using (12) we obtain the following values of $U_n(\cos \theta)$

$$U_0(\cos \theta) = 0,$$

$$U_1(\cos \theta) = \sin \theta,$$

$$U_2(\cos \theta) = 2 \sin \theta \cos \theta,$$

$$U_3(\cos \theta) = \sin \theta (4 \cos^2 \theta - 1),$$

$$U_4(\cos \theta) = \sin \theta (8 \cos^3 \theta - 4 \cos \theta),$$

$$U_5(\cos \theta) = \sin \theta (16 \cos^4 \theta - 12 \cos^2 \theta + 1)$$

and so on.

The general solution in terms of Tchebycheff polynomials. The general solution (8) can be expressed in terms of Tchebycheff polynomials as follows:

$$\begin{aligned} \frac{\psi}{-2\mu\alpha} = \frac{1}{4}r^2 + A_0 + B_0\theta + (C_0 + D_0\theta) \log r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) T_n(\cos \theta) \\ + (C_n r^n + D_n r^{-n}) U_n(\cos \theta), \end{aligned} \quad (13)$$

where $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1} y/x$.

The solutions given by $B_0\theta$, $(C_0 + D_0\theta) \log r$ and $\sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) U_n(\cos \theta)$ are in most cases not of practical importance. Putting B_0 , C_0 , D_0 , C_n and D_n all equal to zero, we get the simple solution

$$\frac{\psi}{-2\mu\alpha} = \frac{1}{4}r^2 + \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n}) T_n(\cos \theta) \quad (14)$$

which is valid for cross sections symmetrical about an axis, the axis being taken as the initial line.

If further we put $B_n = 0$, we have

$$\frac{\psi}{-2\mu\alpha} = \frac{1}{4}r^2 + \sum_{n=0}^{\infty} A_n r^n T_n(\cos \theta). \quad (15)$$

If the equation of the boundary can be put in either one of the forms

$$\frac{1}{4}r^2 + \sum_{n=0}^{\infty} A_n r^n T_n(\cos \theta) = 0 \quad (16)$$

or

$$\frac{1}{4}r^2 + \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n}) T_n(\cos \theta) = 0, \quad (17)$$

the expression on the left hand side of (16) or (17) multiplied by $-2\mu\alpha$ will give the stress function for the given cross section.

1. Circular cross section. The equation of the cross section is $x^2 + y^2 = a^2$ or $\frac{1}{4}r^2 - \frac{1}{4}a^2 = 0$ which is of the form (16) by putting $A_0 = -\frac{1}{4}a^2$ and all other constants equal to zero. The stress function ψ is given by:

$$\psi = -\frac{1}{2}\mu\alpha(r^2 - a^2)$$

or in Cartesian coordinates

$$\psi = -\frac{1}{2}\mu\alpha(x^2 + y^2 - a^2).$$

2. Elliptic cross section. The equation of the cross section is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or in polar coordinates

$$\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{r^2}$$

which can be put in the form

$$\frac{1}{4}r^2 - \frac{a^2 b^2}{2(a^2 + b^2)} - \frac{a^2 - b^2}{4(a^2 + b^2)} r^2 T_2(\cos \theta) = 0$$

which is of the form (16) when we put $A_0 = -\frac{1}{2}a^2 b^2 / (a^2 + b^2)$, $A_2 = -\frac{1}{4}(a^2 - b^2) / (a^2 + b^2)$ and all other constants equal to zero. The stress function is given by:

$$\psi = -\frac{1}{2}\mu\alpha \left[r^2 - \frac{2a^2 b^2}{a^2 + b^2} - \frac{a^2 - b^2}{a^2 + b^2} r^2 T_2(\cos \theta) \right]$$

or in Cartesian coordinates

$$\psi = -\mu\alpha \frac{a^2 b^2}{a^2 + b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right).$$

3. Equilateral triangular cross section. The equation of the cross section Fig. 1 is given by

$$\left(x + \frac{1}{3}a\right)\left(x - \sqrt{3}y - \frac{2}{3}a\right)\left(x + \sqrt{3}y - \frac{2}{3}a\right) = 0,$$

i.e.

$$\frac{1}{4}(x^2 + y^2) - \frac{1}{27}a^2 - \frac{1}{4a}(x^3 - 3xy^2) = 0$$

FIG. 1.

and in polar coordinates by

$$\frac{1}{4}r^2 - \frac{1}{27}a^2 - \frac{1}{4a}r^3T_3(\cos \theta) = 0$$

which is of the form (16) when we put $A_0 = -a^2/27$, $A_3 = -1/4a$ and all other constants equal to zero.

The stress function is given by

$$\psi = -\frac{1}{2}\mu\alpha\left[r^2 - \frac{4}{27}a^2 - \frac{1}{a}r^3T_3(\cos \theta)\right]$$

or in Cartesian coordinates

$$\psi = -\frac{1}{2}\mu\alpha\left[x^2 + y^2 - \frac{4}{27}a^2 - \frac{1}{a}(x^3 - 3xy^2)\right].$$

4. The cross section is a circle of radius a with a notch whose boundary is a circle of radius b whose centre is on the circumference of the circle (as shown in Fig. 2). The

FIG. 2.

cross section is made up of the two circles $r = 2a \cos \theta$ and $r = b$, its equation can be written in the form:

$$(r^2 - b^2) \left(1 - \frac{2a}{r} \cos \theta \right) = 0$$

or

$$\frac{1}{4} r^2 - \frac{1}{4} b^2 - \frac{1}{2} ar T_1(\cos \theta) + \frac{1}{2} \frac{ab^2}{r} T_1(\cos \theta) = 0$$

which is of the form (17) when we put $A_0 = -\frac{1}{4}b^2$, $A_1 = -\frac{1}{2}a$, $B_1 = \frac{1}{2}ab^2$ and all other constants equal to zero. The stress function is given by:

$$\psi = -\frac{1}{2} \mu \alpha (r^2 - b^2) \left[1 - \frac{2a}{r} T_1(\cos \theta) \right].$$

5. Cardioid cross section. The equation of the cross section Fig. 3 is given by

$$r = \frac{1}{2a^2} (1 + \cos \theta)$$

FIG. 3.

or

$$(2a^2 r - \cos \theta)^2 = 1$$

which can be put in the form:

$$a^2 r^2 (2a^2 r - \cos \theta)^2 = \frac{1}{2} r (1 + \cos \theta) = r \cos^2 \frac{\theta}{2}$$

or

$$ar(2a^2 r - \cos \theta) = r^{1/2} \cos \frac{\theta}{2}$$

which reduces to

$$\frac{1}{4} r^2 - \frac{1}{8a^3} r^{1/2} T_{1/2}(\cos \theta) - \frac{1}{8a^2} r T_1(\cos \theta) = 0.$$

This equation is of the form (16) by putting $A_4 = -1/8a^3$, $A_1 = -1/8a^2$ and all other constants equal to zero. The stress function is given by

$$\psi = -\frac{1}{2}\mu\alpha\left[r^2 - \frac{1}{2a^3}r^{1/2}T_{1/2}(\cos\theta) - \frac{1}{2a^2}rT_1(\cos\theta)\right].$$

Other forms of cross sections can be similarly obtained.

REFERENCES

1. S. Timoshenko, *Theory of elasticity*, 1934
2. Preston R. Clement, *The Chebyshev approximation method*, Quart. Appl. Math. 11, 167 (1953)