

If a systematic method of obtaining these factors existed, the problem of determining stability criteria directly in terms of the elements of A , rather than in terms of the coefficients of the characteristic polynomial of A , would be resolved.

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**NOTES ON CONTROL PROCESSES—I.
ON THE MINIMUM OF MAXIMUM DEVIATION***

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Summary. The functional equation technique of the theory of dynamic programming is applied to the problem of determining the minimum of the maximum deviation of a system from a preassigned state.

Mathematically this reduces, in certain cases, to choosing a vector y , subject to certain constraints, so as to minimize

$$J(y) = \text{Max}_{0 \leq t \leq T} \|x - z\|,$$

where

$$\frac{dx}{dt} = \phi(x, y), \quad x(0) = c.$$

1. Introduction. Suppose that we have a physical system S whose state at any time t is specified by an n -dimensional vector $x(t)$. Suppose further that $x(t)$ is determined for $t \geq 0$ as the solution of the vector differential equation

$$\frac{dx}{dt} = \phi(x, y), \quad x(0) = c, \tag{1.1}$$

where y is a vector to be chosen so as to control the behavior of $x(t)$. Many versions of this problem occur in variational analysis and in applied fields such as mathematical economics and servomechanism engineering.

The problem we wish to discuss here is that of choosing y , a vector function of t , so as to minimize the maximum deviation of $x(t)$ from a given vector $z(t)$, over a fixed interval $0 \leq t \leq T$. Although this criterion frequently corresponds most closely to the physical criterion, it is seldom used as a mathematical criterion because of the analytic intractability of the maximum functional. It is interesting to note, however, that Chevychev's research on the minimum of the maximum deviation of polynomials over an interval arose from similar problems arising in the theory of linkages, a study arising from the Watt steam engine.

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Using the techniques of the theory of dynamic programming, [1], we wish to show that this problem may be reduced to the problem of solving a certain functional equation, or recurrence relation, depending upon whether the control process is of continuous or discrete type.

For the two-dimensional case, where

$$\frac{d^2u}{dt^2} = g\left(u, \frac{du}{dt}, y\right), \quad u(0) = c_1, \quad u'(0) = c_2, \tag{1.2}$$

this leads to a simple and practical method of numerical computation of the solution. Naturally, the problem is even simpler in the one-dimensional case.

2. Formulation. Consider the differential equation of (1.1) and suppose that it possesses a unique solution for all functions y belonging to certain class C . In many applications, (1.1) will have the form

$$\frac{dx}{dt} = Ax + y, \quad x(0) = c, \tag{2.1}$$

with the components of y satisfying restrictions of the form

$$-m_i \leq y_i \leq m_i, \quad i = 1, 2, \dots, n, \tag{2.2}$$

see [3].

Let $\|x\|$ denote a measure of the magnitude of the vector x . Possible norms are

$$\begin{aligned} \text{a. } \|x\| &= \sum_{i=1}^n |x_i|, \\ \text{b. } \|x\| &= \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \\ \text{c. } \|x\| &= \text{Max}_{1 \leq i \leq n} |x_i|. \end{aligned} \tag{2.3}$$

Fixing upon one of these norms, set

$$J(y) = \text{Max}_{0 \leq t \leq T} \|x - z\|, \tag{2.4}$$

where $z = z(t)$ is a given function of t . We wish to minimize $J(y)$ over all admissible y ; i.e., over all $y \in C$.

Assuming, as we shall, that the minimum exists, and that, to begin with, z and A are constant, the minimum value will be a function only of c , the initial value, and T , the duration of the process.

Let us then write

$$f(c, T) = \text{Min}_{y \in C} \text{Max}_{0 \leq t \leq T} \|x - z\|. \tag{2.5}$$

We now wish to determine a functional equation for $f(c, T)$.

3. Functional equation. We have, for $S, T \geq 0$,

$$\begin{aligned} f(c, S + T) &= \text{Min}_{y \in C} \text{Max}_{0 \leq t \leq S+T} \|x - z\| \\ &= \text{Min}_{y \in C} \text{Max}_{0 \leq t \leq S} [\text{Max}_{0 \leq t \leq S} \|x - z\|, \text{Max}_{S \leq t \leq S+T} \|x - z\|]. \end{aligned} \tag{3.1}$$

The minimum over $y \in C$ may be written

$$\text{Min}_{y \in C} = \text{Min}_{y \in [0, S+T]} \text{Min}_{y \in [0, S]} \text{Min}_{y \in [S, S+T]}, \tag{3.2}$$

where $\text{Min}_{y \in [a, b]}$ signifies a minimum over functions $y(t)$ defined over $a \leq t \leq b$. Hence we have

$$\begin{aligned} f(c, S + T) &= \text{Min}_{y \in [0, S]} \text{Min}_{y \in [S, S+T]} \text{Max}_{0 \leq t \leq S} [\text{Max}_{S \leq t \leq S+T} \|x - z\|], \\ &= \text{Min}_{y \in [0, S]} \text{Max}_{0 \leq t \leq S} [\text{Max}_{y \in [S, S+T]} \text{Max}_{S \leq t \leq S+T} \|x - z\|]. \end{aligned} \tag{3.3}$$

Combining the stationarity of the process over time with the definition of $f(c, T)$, we see that

$$\text{Min}_{y \in [S, S+T]} \text{Max}_{S \leq t \leq S+T} \|x - z\| = f(c(S), T), \tag{3.4}$$

where $c(S)$ is the value of $x(S)$ obtained from (1.1) for a particular choice of $y(t)$ over $0 \leq t \leq S$.

Thus (3) reduces to

$$f(c, S + T) = \text{Min}_{y \in [0, S]} \text{Max}_{0 \leq t \leq S} [\text{Max}_{0 \leq t \leq S} \|x - z\|, f(c(S), T)]. \tag{3.5}$$

From this equation we can derive a limiting nonlinear partial differential equation for f , under suitable assumptions concerning the continuity of partial differential coefficients. However, for numerical purposes the equation derived in this way is of less utility than the recurrence relation we shall obtain in the following section.

4. The discrete process. Let us replace (1.1) by the difference equation

$$\begin{aligned} x_{n+1} &= x_n + \Delta\phi(x_n, y_n), \quad x_0 = c, \\ &= \psi(x_n, y_n). \end{aligned} \tag{4.1}$$

Set

$$J_N(\{y_k\}) = \text{Max}_{0 \leq k \leq N} \|x_k - z\|. \tag{4.2}$$

Define, as above,

$$f_N(c) = \text{Min}_y \text{Max}_{0 \leq k \leq N} \|x_k - z\|. \tag{4.3}$$

Taking $\psi(x, y)$ to be a continuous function of y , there is now no question as to the existence of a minimizing sequence $\{y_k\}$.

The function $f_0(c)$ is given by

$$f_0(c) = \|c - z\|. \tag{4.4}$$

We wish to obtain a recurrence relation connecting $f_{N+1}(c)$ and $f_N(c)$. Arguing as in Sec. 3, we readily arrive at the result

$$\begin{aligned} f_{N+1}(c) &= \text{Min}_{y_0} \text{Max} [\|x_0 - z\|, f_N(\psi(c, y_0))], \\ &= \text{Max} [\|x_0 - z\|, \text{Min}_{y_0} f_N(\psi(c, y_0))]. \end{aligned} \tag{4.5}$$

Starting with the known function $f_0(c)$, we can compute successively the other members of the sequence $\{f_k(c)\}$.

In the case where $z(t)$ and A depend upon t , we use the sequence

$$f_a(c) = \text{Min}_v \text{Max}_{0 \leq k \leq N} \|x_k - z_k\|,$$

and proceed in a similar fashion. Here N is kept fixed and a , the starting point, is an essential state variable.

5. Example. Consider, as an example of the general technique discussed above, the problem of determining $f = f(t)$, subject to the constraint $-1 \leq f \leq 1$, so as to minimize

$$J(u) = \text{Max}_{0 \leq t \leq T} |1 - u|, \quad (5.1)$$

where

$$u'' + u = f, \quad u(0) = 0, \quad u'(0) = 1. \quad (5.2)$$

Let us begin by converting (5.2) into a 2-dimensional system with general initial conditions,

$$\begin{aligned} u' &= v, & u(0) &= c_1, \\ v' &= -u + f, & v(0) &= c_2, \end{aligned} \quad (5.3)$$

and then into an approximating difference system

$$\begin{aligned} u_{n+1} &= u_n + \Delta v_n, & u_0 &= c_1, \\ v_{n+1} &= v_n + \Delta(f_n - u_n), & v_0 &= c_2, \end{aligned} \quad (5.4)$$

where $n = 0, 1, 2, \dots, N; N\Delta = T$.

Write

$$f_n(c_1, c_2) = \text{Min}_v \text{Max}_{0 \leq k \leq n} |1 - u_k|. \quad (5.5)$$

Then

$$f_0(c_1, c_2) = |1 - u_0| = |1 - c_1|, \quad (5.6)$$

and

$$f_{n+1}(c_1, c_2) = \text{Max} [|1 - c_1|, \text{Min}_{-1 \leq f_0 \leq 1} f_n(c_1 + \Delta c_2, c_2 + \Delta(f_0 - c_1))]. \quad (5.7)$$

Using this recurrence relation, the sequence $\{f_k(c_1, c_2)\}$ can be readily obtained, using a digital computer.

If we consider the question of stability, insofar as round-off error is concerned, it is better to use the recurrence relation

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\Delta^2} + u_k = f_k, \quad u_0 = c_1, \quad u_1 = c_1 + \Delta c_2, \quad (5.8)$$

in place of (5.4).

6. Remarks. Other applications of the functional equation technique may be found in [1] and [2]. Some nonclassical variational problems involving awkward functions such as $\text{Max} |1 - u|$ and $\int_0^T |1 - u| dt$, and others, are treated in [3] and [4].

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A NETWORK PROOF OF A THEOREM ON HURWITZ POLYNOMIALS AND ITS GENERALIZATION*

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Bückner has recently given a canonical form for Hurwitz polynomials. Since network functions and Hurwitz polynomials are intimately related, we shall give an alternate proof of his theorem by network theoretic methods and a statement of a similar theorem. Finally, a general method for obtaining such theorems will be indicated.

Bückner¹ has shown that: "If the polynomial $f(p)$, normalized so that $f(0) = 1$, can be written as

$$f(p) = \begin{vmatrix} 1 + a_1p & -1 & 0 & \cdot & \cdot & 0 \\ 1 & a_2p & -1 & & & \\ 0 & 1 & a_3p & -1 & & \\ 0 & 0 & 1 & a_4p & -1 & \\ \cdot & & & & & \\ \cdot & & & & & -1 \\ 0 & \cdot & & \cdot & 0 & 1 & a_np \end{vmatrix}$$

where $a_i > 0$ ($i = 1, 2, \dots, n$), then $f(p)$ is a Hurwitz polynomial and conversely."

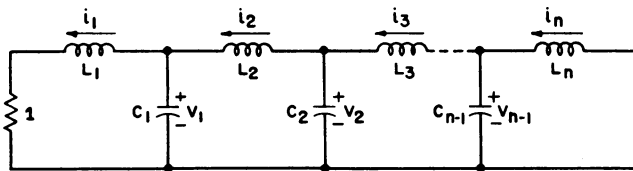


FIG. 1.

Consider the network shown on Fig. 1. If the network variables shown on the figure are used, the network equations can be written as:

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¹H. Bückner, *A formula for an integral occurring in the theory of linear servomechanisms and control systems*, Quart. Appl. Math. (3) 10, 205-213 (1952).