ON A CLASS OF VARIATIONAL PROBLEMS*

BY

RICHARD BELLMAN

The Rand Corporation, Santa Monica, California

1. Introduction. A class of interesting analytic problems which arise in connection with various averaging or smoothing processes in applied mathematics, from the study of the control of economic or engineering systems, involves the minimization of functionals of the form

$$J(x) = \int_0^T [F_1(x - b_1(t), t) + F_2(x' - b_2(t), t) + \cdots + F_K(x^{(K-1)} - b_K(t), t)] dt \quad (1)$$

over all functions $y(t)$ subject to the constraints

$$x(0) = c_1, \quad x'(0) = c_2, \quad \cdots, \quad x^{(K-1)}(0) = c_K, \quad (2)$$

and for which the integrals exist. Here the functions $b_i(t)$ are given functions of t .

The standard approach to this problem, employing the classical variational techniques, leads to differential equations of order $2K$, in general nonlinear, with K conditions at $t = 0$ and K additional conditions at $t = T$. These last conditions are derived from the variation. Problems of this nature are called two-point boundary problems, and are notably difficult to treat, both computationally and theoretically. For the case $K = 1$, the computational problem posed by the variational equation is fairly simple to resolve. For $K \geq 2$, however, the computational problem becomes difficult since we are faced with the problem of solving K equations in K unknowns with each trial solution involving the numerical solution of a differential equation of order $2K$.

If $J(y)$ has the form

$$\int_0^T [a_1(t)(x - b_1(t))^2 + a_2(t)(x' - b_2(t))^2 + \cdots + a_K(t)(x^{(K-1)} - b_K(t))^2] dt, \quad (3)$$

the variational equation is linear, and the computational problem is relatively simple. However, the discrete version of the problem, involving the minimization of the quadratic form

$$\sum_{n=0}^T [a_1(n)(x_n - b_1(n))^2 + a_2(n)(\Delta x_n - b_2(n))^2 + \cdots]^2, \quad (4)$$

if approached in a direct way leads to difficult evaluations of determinants.

The corresponding discrete version of (1) is

$$\sum_{n=0}^T [F_1(x_n - b_1(n), n) + F_2(\Delta x_n - b_2(n), n) + \cdots + F_K(\Delta^{K-1} x_n - b_K(n), n)], \quad (5)$$

where

$$\Delta x_n = x_n - x_{n-1}, \quad (6)$$

$$\Delta^2 x_n = x_n - 2x_{n-1} + x_{n-2}$$

and so on.

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A particular version of this discrete problem arises in mathematical economics in connection with some scheduling problems in "optimal inventory" theory; see Holt, Modigliani and Simon [6]; Arrow, Harris, and Marschak [1]; and Bellman, Glicksberg and Gross [4].

In this paper we wish to apply the functional equation approach of the theory of dynamic programming [2], first to the discrete case, and then to the continuous case. Other applications to the calculus of variations may be found in [2, 3].

2. The discrete case. Let us consider the problem of determining the minimum of the function

$$\sum_{K=1}^N [\phi_K(x_K) + \psi_K(x_K - x_{K-1})] \quad (1)$$

over all possible values of the x_K , $K = 1, 2, \dots, N$, with $x_0 = x$.

To simplify the analytic details, we shall consider only this case where first differences occur. The general case may be handled along similar lines, see Sec. 4.

Let us assume that $\phi_K(x)$ and $\psi_K(x)$ are continuous functions of x for all x , possessing appropriate properties at $x = \infty$ so that the minimum is actually assumed.

Define the sequence of functions

$$f_R(x) = \underset{x_i, i=R, \dots, N}{\text{Min}} \sum_{K=R}^N [\phi_K(x_K) + \psi_K(x_K - x_{K-1})], \quad (2)$$

with $x_{R-1} = x$. We have

$$f_N(x) = \underset{x_N}{\text{Min}} [\phi_N(x_N) + \psi_N(x_N - x)], \quad (3)$$

and the recurrence relation

$$f_R(x) = \underset{x_R}{\text{Min}} [\phi_R(x_R) + \psi_R(x_R - x) + f_{R+1}(x_R)], \quad R = 1, 2, \dots, N-1. \quad (4)$$

The original N -dimensional minimization problem has thus been reduced to a set of N one-dimensional minimization problems. This approach is particularly well suited to the computational solution of problems in which the functions ϕ and ψ are non-analytic. In reference [5] there is a discussion of a problem involving the function $\text{Max}(x, 0)$, or equivalently $|x|$.

If ϕ and ψ are quadratic, we can go much further and find more explicit recurrence relations. In a subsequent paper we will discuss the application of these ideas to eigenvalue problems.

3. Explicit recurrence relation for quadratic case. Let us now consider the case where ϕ_K and ψ_K are quadratic in x , with $\phi_K = b_K(x - d_K)^2$, $\psi_K = (c_K x^2)$.

Since

$$f_N(x) = \underset{x_N}{\text{Min}} [b_N(x_N - d_N)^2 + c_N(x_N - x)^2], \quad (1)$$

we see that $f_N(x)$ is a quadratic in x ,

$$f_N(x) = u_N + v_N x + w_N x^2, \quad (2)$$

where u_N , v_N , and w_N are readily determined explicitly as functions of b_N , c_N and d_N , and $w_N > 0$.

Turn now to the relation for $f_{N-1}(x)$.

$$f_{N-1}(x) = \underset{x_{N-1}}{\text{Min}} [b_{N-1}(x_{N-1} - d_{N-1})^2 + e_{N-1}(x_{N-1} - x)^2 + f_N(x_{N-1})]. \quad (3)$$

Substituting the expression for f_N found above, we find that the minimum over x_{N-1} is attained at the point,

$$x_{N-1} = \frac{xe_{N-1} + d_{N-1}b_{N-1} - v_N/2}{b_{N-1} + e_{N-1} + w_N} \quad (4)$$

and the value of $f_{N-1}(x)$ is $u_{N-1} + v_{N-1}x + w_{N-1}x^2$, where

$$\begin{aligned} u_{N-1} &= \left[b_{N-1}d_{N-1}^2 + u_N - \left(d_{N-1}b_{N-1} - \frac{v_N}{2} \right)^2 \right] / \left[b_{N-1} + e_{N-1} + w_N \right], \\ v_{N-1} &= \frac{-2e_{N-1}(d_{N-1}b_{N-1} - v_N/2)}{b_{N-1} + e_{N-1} + w_N}, \\ w_{N-1} &= \frac{e_{N-1}(b_N + w_N)}{b_{N-1} + e_{N-1} + w_N}. \end{aligned} \quad (5)$$

This is a recurrence relation that connects the triple $(u_{N-1}, v_{N-1}, w_{N-1})$ with the triple (u_N, v_N, w_N) . Iterating this relation, we obtain the sequence (u_2, v_2, w_2) .

To determine x_1 , we use the relation

$$f_1(x_0) = \underset{x_1}{\text{Min}} [b_1(x_1 - d_1)^2 + e_1(x_1 - x)^2 + f_2(x_1)], \quad (6)$$

which yields

$$x_1 = \frac{xe_1 + d_1b_1 - v_2/2}{b_1 + e_1 + w_2}. \quad (7)$$

Similarly, the optimal choice of x_K is given by

$$x_K = \frac{x_{K-1} + d_{K-1}b_{K-1} - v_{K+1}/2}{b_K + e_K + w_K}. \quad (8)$$

This permits the sequence $x_1, x_2, \dots, x_N$ to be computed recurrently starting with the value of x_1 .

4. Discrete case: $K = 3$. Let us now consider briefly the problem discussed by Holt, Modigliani and Simon in [6]. It is required to minimize the expression

$$\sum_{K=1}^N [a_K(x_K - b_K)^2 + e_K(x_K - x_{K-1})^2 + g_K(s_K - d_K)^2], \quad (1)$$

where

$$s_K = x_1 + x_2 + \dots + x_K. \quad (2)$$

Making a change of variable

$$\begin{aligned} s_K &= y_K, \\ x_K &= y_K - y_{K-1}, \\ x_K - x_{K-1} &= y_K - 2y_{K-1} + y_{K-2}, \end{aligned} \quad (3)$$

we have the problem of minimizing the quadratic form

$$\sum_{k=1}^N [a_K(y_K - y_{K-1} - b_K)^2 + e_K(y_K + y_{K-2} - 2y_{K-1})^2 + g_K(y_K - d_K)^2], \quad (4)$$

over all $y_1, y_2, \dots, y_N$, where we assume that y_0 and y_{-1} have fixed values x and z respectively. If we define

$$f_R(y_{R-1}, y_{R-2}) = \min_{\substack{y_i \\ i=R, \dots, N}} \left\{ \sum_{k=R}^N [a_K(y_K - y_{K-1} - b_K)^2 + e_K(y_K + y_{K-2} - 2y_{K-1})^2 + g_K(y_K - d_K)^2] \right\}, \quad (5)$$

we obtain the recurrence relation

$$f_R(y_{R-1}, y_{R-2}) = \min_{y_R} [a_R(y_R - y_{R-1} - b_R)^2 + e_R(y_R + y_{R-2} - 2y_{R-1})^2 + f_{R+1}(y_R, y_{R-1})]. \quad (6)$$

The determination of the sequence $\{f_R\}$ proceeds as before, with the exception that $f_R(x, z)$ now has the form

$$f_R(x, z) = u_{1N}x^2 + 2u_{2N}xz + u_{3N}z^2 + 2u_{4N}x + 2u_{5N}z + w_{6N}. \quad (7)$$

5. Stochastic case. The same functional equation technique suffices to handle the case where the parameters a_i, b_i, e_i, g_i, d_i are taken to be random variables with a given joint distribution function, provided that we agree to minimize the expected value of the quadratic form.

6. The continuous case: $K = 2$. Let us now consider the continuous case for $K = 2$, where the analytic details are simplest. After a discussion of this case which illustrates the general method, we shall briefly indicate the extension of the method to higher values of K , using $K = 3$ as an example.

Consider the problem of minimizing the functional

$$J(y, s) = \int_s^T [a_1(t)(x - b_1(t))^2 + a_2(t)(x' - b_2(t))^2] dt, \quad 0 \leq s \leq T, \quad (1)$$

over all functions y such that

$$x(s) = c, \quad x' \in L^2(s, T). \quad (2)$$

We shall assume that all the functions which appear are continuous and that, in addition, $a_i(t) \geq 0$ for $t \geq 0$.

The introduction of the parameter s is important for our purposes. It corresponds to the R of Sec. 2.

Define the function

$$f(c, s) = \min_x J(x, s). \quad (3)$$

Let us now obtain a functional equation for $f(c, s)$ which in the limit reduces to a partial differential equation.

Write

$$f(c, s) = \int_s^{s+h} + \int_{s+h}^T, \quad 0 < h < T - s, \quad (4)$$

for an extremal $x(t)$. Choosing x' in the interval $[s, s + h]$, we see that we have a problem similar to the original with s replaced by $s + h$, and c replaced by the value of $x(t)$ at $t = s + h$. Employing what we have called the "principle of optimality", see [2], Eq. (4) gives rise to the equation

$$f(c, s) = \text{Min}_{x[s, s+h]} \left[\int_s^{s+h} [a_1(t)(x - b_1(t))^2 + a_2(t)(x' - b_2(t))^2] dt + f(x(s+h), s+h) \right]. \quad (5)$$

Let us now assume that the extremal curve is continuous in t and has a continuous derivative, and further that f possesses continuous partial derivatives with respect to c and s . These results may be established by appealing to the classical theory of the calculus of variations, or by a passage to the limit from the discrete case. We shall not, however, discuss this topic here.

Assuming the above continuity properties, let us pass to the limit as $h \rightarrow 0$. Minimization over an interval $[s, s + h]$ reduces to minimization over values of $x'(s)$. Let us call the unknown value of $x'(s)$, v , where v is a function of c and s to be determined. Using the fact that $x(s + h) = x(s) + hx'(s) + o(h) = c + hv + o(h)$ and passing to the limit in (5) as $h \rightarrow 0$, we obtain for $f(c, s)$ the non-linear partial differential equation

$$0 = \text{Min}_v [a_1(s)(c - b_1(s))^2 + a_2(s)(v - b_2(s))^2 + f_s + vf_c]. \quad (6)$$

The minimum is assumed at

$$2a_2(s)(v - b_2(s)) + f_c = 0, \quad (7)$$

which determines v once f has been found, and the resulting equation for f_s is

$$f_s = -a_1(s)(c - b_1(s))^2 + b_2 f_c - \frac{f_c^2}{4a_2(s)}. \quad (8)$$

The initial value for f is

$$f(c, T) = 0 \quad \text{for all } c. \quad (9)$$

Let us now assume that $f(c, s)$ has the form

$$f(c, s) = u(s) + cv(s) + c^2w(s). \quad (10)$$

Equating coefficients in (8) we obtain the equations

$$\begin{aligned} (a) \quad u'(s) &= -a_1(s)b_1^2(s) + b_2(s)v(s) - \frac{v^2(s)}{4a_2(s)}, \\ (b) \quad v'(s) &= 2a_1(s)b_1(s) + 2b_2(s)w(s) - \frac{v(s)w(s)}{a_2(s)}, \\ (c) \quad w'(s) &= -a_1(s) - \frac{w^2(s)}{a_2(s)}, \end{aligned} \quad (11)$$

with the initial conditions

$$u(T) = v(T) = w(T) = 0. \quad (12)$$

Since there is a unique solution to (8) in the proper function class, this system determines it.

Equation (11c) is a Riccati equation, reducible to a second order linear differential equation, with the other functions found readily once $w(s)$ has been determined. The numerical solution of this system is quite easily obtained.

Once $f(c, s) = u(s) + cv(s) + c^2w(s)$ has been determined, we readily determine v from Eq. (7). Then x is determined by the equation

$$\frac{dx}{dt} = v(x, t), \quad x(s) = c, \quad (13)$$

an equation which we can solve explicitly since

$$v(c, s) = -\frac{v(s) + 2cw(s)}{2a_2(s)} + b_2(s) \quad (14)$$

implies

$$v(x, t) = \frac{-v(t) + 2yw(t)}{2a_2(t)} + b_2(t). \quad (15)$$

Once the functions $v(t)$ and $w(t)$ have been determined, Eq. (13) may be solved explicitly for x as a function of c and t .

7. The case $K = 2$. Let us now examine the modifications required to handle the analogous problem of minimizing the integral

$$J(y, s) = \int_s^T [a_1(t)(x - b_1(t))^2 + a_2(t)(x' - b_2(t))^2 + a_3(t)(x'' - b_3(t))^2] dt \quad (1)$$

subject to the constraints

$$x(s) = c_1, \quad x'(s) = c_2. \quad (2)$$

Setting $x''(s) = v = v(c_1, c_2, s)$, and

$$\text{Min}_x J(x, s) = f(c_1, c_2, s), \quad (3)$$

the analogue of (2.8) is

$$0 = \text{Min}_c [a_1(s)(c_1 - b_1(s))^2 + a_2(s)(c_2 - b_2(s))^2 + a_3(s)(v - b_3(s))^2 + f_s + c_2 f_{c_1} + v f_{c_2}], \quad (4)$$

with the initial value, $f(c_1, c_2 T) = 0$ for all c_1, c_2 .

Once again we can obtain a solution of the nonlinear partial differential equation for f by setting f equal to a quadratic in c_1 and c_2 ,

$$f = u_1 c_1^2 + 2u_2 c_1 c_2 + u_3 c_2^2 + u_4 c_1 + u_5 c_2 + u_6, \quad (5)$$

where the u_i 's are functions of s alone. Upon equating coefficients in (4), we obtain a system of nonlinear ordinary differential equations for the u_i of the form

$$du_i/ds = f_i(u_1, u_2, u_3, u_4, u_5, u_6), \quad u_i(T) = 0, \quad (6)$$

which determine the u_i in the range $s \leq T$.

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