

ON LAMINAR FLOW THROUGH A CHANNEL OR TUBE WITH INJECTION: APPLICATION OF METHOD OF AVERAGES*

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Introduction. The equations for the two-dimensional incompressible laminar flow through a channel with a uniform normal fluid velocity v_w at the walls have been recently [1] reduced to an ordinary nonlinear differential equation with mixed boundary conditions and with a normal-velocity parameter which, in practice, may be large, but whose reciprocal is the coefficient of the highest-order derivative in this equation. A small-perturbation solution of this equation was obtained in [1], valid for small normal velocities at the walls. The corresponding problem for flow through a circular tube has also been treated quite recently [2]. Here, the ordinary differential equation which was obtained was again solved by a small-perturbation procedure for small values of the normal fluid velocity at the wall. In addition, however, an asymptotic solution, to first powers of the reciprocal of the normal-velocity Reynolds Number, and hence valid for large values of the injection velocity at the wall, was developed.

The purpose of this paper is to present a simple approximate closed-form solution for the flow through a channel and through a circular tube with porous walls valid for the entire range of normal fluid injection velocities from zero to indefinitely large. The method of analysis will be based on the method of averages. Although this method is fairly well known in curve-fitting, it will be seen that it is particularly fruitful here in solving the ordinary differential equations under the given boundary conditions. The method of averages serves here, in fact, as a relatively simple alternative to the method of least squares (see [3]). The method of averages, moreover, will be applied here in conjunction with auxiliary boundary conditions derived from the governing *ordinary* differential equation, and hence it is to some extent** analogous to the relatively well-known integral methods, such as the Kármán-Pohlhausen method, of solving the the *partial* differential equations of the laminar boundary layer. The solutions thus obtained will be shown to reduce exactly to the small-perturbation solutions for small values of the injection (or suction) velocity at the wall, and to reduce approximately to the exact asymptotic solutions for infinite values of the injection velocity. The problem of normal fluid injection is of practical interest in connection with the transpiration- or sweat-cooling of heated surfaces such as turbine blades, rocket walls, or wing surfaces in high-speed flight.

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**One of the chief differences is that in the ordinary differential equations considered here the range of the independent variable is prescribed and finite, while in the boundary-layer equations the upper limit (which in an exact solution is infinite) of the independent variable is usually treated as an unknown in the integral methods. A study of integral methods for the laminar boundary-layer equations can be found in [4].

Basic equations and method of solution. The velocity distribution for the two-dimensional incompressible flow through a channel with uniformly porous walls has been shown [1] to be:

$$u(x, \lambda) = [\bar{u}(0) - v_w x/h]f'(\lambda), \quad (1)$$

where $f(\lambda)$ satisfies the ordinary differential equation:

$$f^{IV} + R(f'^2 - ff'')' = 0 \quad (2)$$

under the boundary conditions

$$f(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \quad (3)$$

$$f''(0) = 0. \quad (4)$$

Here, x and y are coordinates parallel and normal to the channel walls, respectively, with origin at the center of the channel, while $\lambda \equiv y/h$. The distance between the walls is $2h$. u and v are the velocity components in the x and y direction, respectively, and R is the normal-velocity Reynolds Number, $R \equiv v_w h/\nu$. R is positive for suction and negative for injection. $\bar{u}(0)$ is a constant denoting the average axial velocity at the entrance section ($x = 0$) of the channel, while the physical significance of $f(\lambda)$ is given by $f(\lambda) = v(\lambda)/v_w$.

An approximate solution of Eq. (2) for the region $0 \leq \lambda \leq 1$ can be obtained by requiring that Eq. (2) be satisfied only in the average over this region. Thus, integrating the left side of Eq. (2) with respect to λ over the region (0, 1) yields the (averaging) condition:

$$[f''' + R(f'^2 - ff'')]_0^1 = 0. \quad (5)$$

As may often be the case in such averaging, or integral, conditions, Eq. (5) has a simple physical interpretation. The differential equation (2) is essentially the condition that the axial pressure gradient $\partial p/\partial x$ be constant throughout any cross-section of the channel, i.e., $\partial^2 p/\partial y \partial x = 0$ for all values of λ in the region (0, 1) at any given x . Equation (5) expresses the weaker condition that $\partial p/\partial x$ remain the same at either end of the interval (0, 1).

In addition to Eq. (5) it is possible to satisfy other averaging conditions by integrating Eq. (2) over smaller intervals (0, λ_1), with λ_1 having values between 0 and 1. It will be seen, however, that it is simpler here, and sufficiently accurate, to satisfy, instead, conditions of symmetry and additional boundary conditions which an exact solution of Eq. (2) would necessarily satisfy. First of all, the symmetry of the problem requires that $f(\lambda)$ be odd. [This could also be derived mathematically by successive differentiation of Eq. (2). From the boundary conditions $f(0) = f''(0) = 0$ it then follows that all even derivatives of $f(\lambda)$ vanish at $\lambda = 0$.] Moreover, Eq. (2), in conjunction with conditions (3) implies

$$f^{(4)}(1) - Rf'''(1) = 0. \quad (6)$$

By differentiating Eq. (2) with respect to λ , taking values at the center ($\lambda = 0$) and taking the boundary conditions (3) and (4) into account, it further follows that

$$f^{(5)}(0) = 0. \quad (7)$$

Additional conditions at $\lambda = 0$ and at $\lambda = 1$ can, if desired, be similarly obtained by successive differentiation of Eq. (2).

Since the solution $f(\lambda)$ will be well behaved (by, for example, physical considerations) it should be possible to represent $f(\lambda)$ approximately by a polynomial. The most general odd polynomial satisfying the original boundary conditions (3) and (4), and the simple condition (7), is readily found to be:

$$f(\lambda) = \left(\frac{3}{2} \lambda - \frac{1}{2} \lambda^3 \right) + \sum_{n=7,9,\dots}^{2k+1} \left[\frac{(n-3)}{2} \lambda - \frac{(n-1)}{2} \lambda^3 + \lambda^n \right] a_n. \quad (8)$$

By substituting the assumed form for $f(\lambda)$ into conditions (5) and (6), and any such additional conditions, a set of algebraic equations for the coefficients a_n is obtained. In the present case, it will be found that the two conditions (5) and (6) are adequate to obtain a sufficiently accurate approximate solution of the differential equation (2) for the parameter range $-\infty \leq R \leq 0$. Hence, in accordance with Eq. (8), $f(\lambda)$ will be assumed to have the form:

$$f(\lambda) = \left(\frac{3}{2} \lambda - \frac{1}{2} \lambda^3 \right) + a_7(2\lambda - 3\lambda^3 + \lambda^7) + a_9(3\lambda - 4\lambda^3 + \lambda^9). \quad (9)$$

Conditions (5) and (6) then yield the following values for a_7 and a_9 :

$$a_7 = \alpha + \beta a_9 \quad (10a)$$

a_9 is the smaller root, in absolute value, of the quadratic equation:

$$(3 + 2\beta)^2 R a_9^2 - (3\gamma) a_9 - \delta = 0, \quad (10b)$$

where

$$\begin{aligned} \alpha &= \frac{R}{8(8R - 35)}, & \beta &= \frac{2(63 - 10R)}{8R - 35}, \\ \gamma &= 168 + 70\beta - R[19 + 10\beta + 4\alpha + (8/3)\alpha\beta], \\ \delta &= R[(3/4) - 30\alpha - 4\alpha^2] + 210\alpha. \end{aligned} \quad (10c)$$

For most values of R , and in particular for the entire range $-\infty \leq R \leq 0$, it will be found that $9\gamma^2 \gg |(3 + 2\beta)^2 R\delta|$. Under this condition, a very good approximation for the smaller root of Eq. (10b) is:

$$a_9 \approx -\frac{\delta}{3\gamma} \left[1 - \frac{(3 + 2\beta)^2}{3\gamma} R \left(\frac{\delta}{3\gamma} \right) \right]. \quad (10d)$$

Equations (9) and (10) constitute an approximate solution of Eq. (2) under the boundary conditions (3) and (4) for general values of the injection parameter R , i.e. for $-\infty \leq R \leq 0^*$.

Comparison of approximate solution with limiting exact solutions. Although it may appear difficult to estimate precisely the accuracy of the above approximate solution for general values of R , an indication of its accuracy can be obtained by a comparison with an exact solution valid for the limit $|R| \rightarrow 0$ and an exact solution valid in the opposite extreme limit $R \rightarrow -\infty$.

*It will be seen shortly that the approximate solution is also valid for small suction (positive) values of R .

For the limit of very small $|R|$, an exact solution may be considered to be a small-perturbation solution, which has the form [1]:

$$[f(\lambda)]_{|R| \rightarrow 0} = \left(\frac{3}{2} \lambda - \frac{1}{2} \lambda^3 \right) + \frac{R}{280} (3\lambda^3 - 2\lambda - \lambda^7). \quad (11)$$

According to Eqs. (10), $a_9 \rightarrow 0$ (to first powers of R) and $a_7 \rightarrow -R/280$ as $|R| \rightarrow 0$. Hence the approximate solution reduces exactly to Eq. (11) in the limit of small $|R|$.

In the limit of $R \rightarrow -\infty$, the differential equation (2) reduces to the equation

$$(f'^2 - ff'')' = 0 \quad (12)$$

[under the assumption of a finite $f''(\lambda)$]. Equation (12) can be solved straight-forwardly under the three boundary conditions (3), with the result:

$$[f(\lambda)]_{R \rightarrow -\infty} = (-1)^{k+1} \sin \frac{(2k-1)\pi\lambda}{2}, \quad (13a)$$

where k is a positive integer. Since, physically $u \geq 0$, and hence $f'(\lambda) \geq 0$ in the region $0 \leq \lambda \leq 1$, it follows that $k = 1$. Thus,

$$[f(\lambda)]_{R \rightarrow -\infty} = \sin \frac{\pi}{2} \lambda. \quad (13b)$$

Since Eq. (13b) automatically satisfies the fourth boundary condition (4), it is an exact asymptotic solution of Eq. (2) under the boundary conditions (3) and (4)*. According to Eqs. (10), on the other hand, $\alpha \rightarrow 1/64$, $\beta \rightarrow -5/2$, $\gamma \rightarrow (145/24)R$, $16R\delta \rightarrow (287/64)R^2$ as $R \rightarrow -\infty$. Hence, Eqs. (10a)-(10c) are found to yield: $a_9 \rightarrow -0.01541$, $a_7 \rightarrow 0.05415$ as $R \rightarrow -\infty$. Substitution into Eq. (9) then yields an approximate solution for $f(\lambda)$ in very good agreement with the asymptotic solution (13b) (see Fig. 1). The skin-friction at the wall, which is proportional to $f''(1)$, is given by Eqs. (9) and (10) ($f''(1) = -2.440$) to within 1.13% of the exact value ($-\pi^2/4 = -2.468$) given by Eq. (13b).

Thus the approximate solution developed here for general R is seen to be sufficiently accurate for practical purposes at small and at large values of $-R$. At intermediate negative values of R it appears reasonable here to assume that the approximate solution will also be of comparable accuracy. It is worthwhile to note, in fact, that the function $f(\lambda)$ does not actually vary greatly with R , and that consequently the velocity profiles, as given by u/u_{\max} , where u_{\max} is the velocity at the center of the channel at any given position x from the entrance, are relatively insensitive to changes in the value of R (see Fig. 1). According to Eq. (1),

$$\frac{u}{u_{\max}} = \frac{f'(\lambda)}{f'(0)}. \quad (14)$$

Table 1 gives numerical values of a_7 and a_9 for various injection values of R .

*It may be noted that with respect to the differential equation (12), the four boundary conditions (3) and (4) are not all independent of one another, since it follows immediately from the differential equation (12) itself that condition (4) will be automatically satisfied by any solution of (12) satisfying the single boundary condition $f(0) = 0$, with $f'''(0)/f'(0)$ finite.

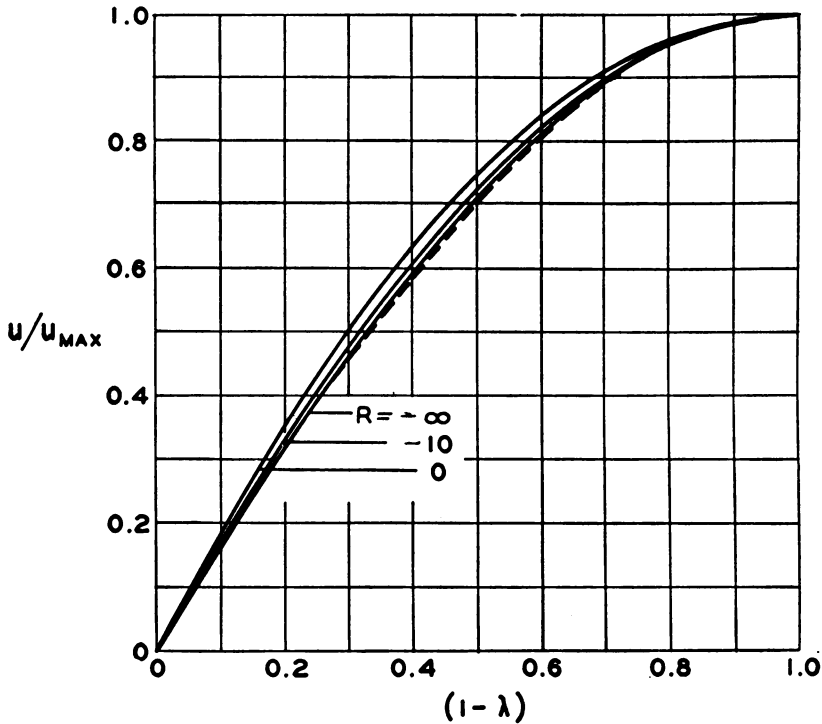


FIG. 1 Velocity profiles in channel with fluid injection at walls
 ---Exact (asymptotic) solution for $R \rightarrow -\infty$

TABLE 1.

Values of a_7 and a_9 for flow through channel with various values of the injection Reynolds Number R .

R	$-\infty$	-25	-10	-5	-2	-1	0
a_7	.0542	.0383	.0256	.0160	.00710	.00360	0
a_9	-.0154	-.00935	-.00520	-.00254	-.000678	-.000205	0

Skin-friction. It may be of physical interest to note the effect of fluid injection on the skin-friction in the channel. The local skin-friction coefficient at either wall of the channel at any point x will be:

$$c_f \equiv -\frac{(\mu \partial u / \partial y)_{\lambda=1}}{\rho \bar{u}^2(0) / 2} = -\frac{2}{Re} \left[1 - \left(\frac{R}{Re} \right) \left(\frac{x}{h} \right) \right] f''(1), \tag{15}$$

where $Re \equiv \bar{u}(0)h/\nu$. From Eq. (15) it is found that, except close to the channel entrance ($x = 0$), normal fluid injection increases the skin friction (see Fig. 2), in contrast to its effect in laminar boundary-layer flow without confining boundaries, such as the flow over a flat plate parallel to the stream. This is also slightly in contrast to the flow in a

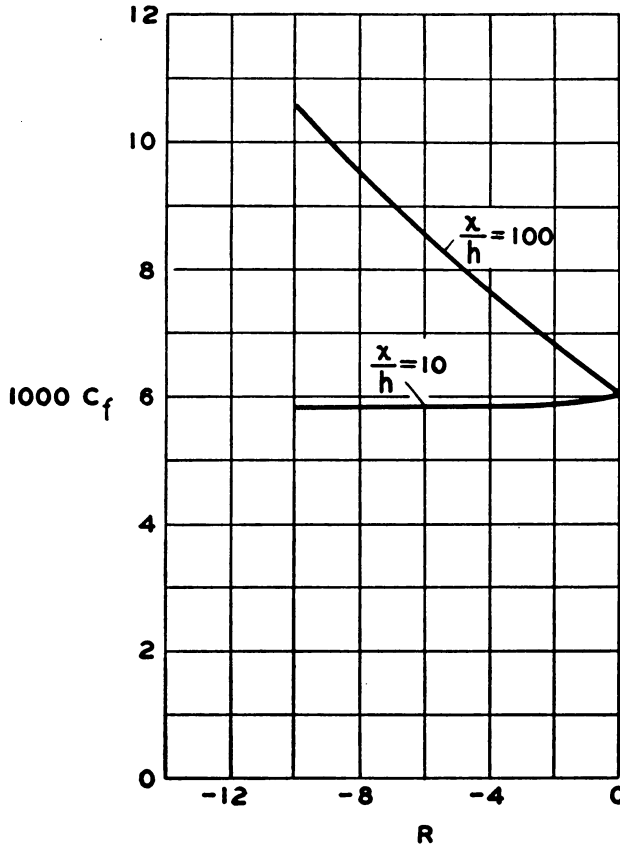


FIG. 2 Skin-friction in channel as a function of fluid-injection parameter R
($Re = 1000$)

porous circular tube [2], in which fluid injection increases the skin-friction coefficient even near the tube entrance.

Flow through a circular tube. It may be worth while here to indicate briefly results for the flow through a uniformly porous circular tube, since the method of averages is found to be equally fruitful here in the case of fluid injection at the wall. The ordinary differential equation to be solved in this case is [2]:

$$(\eta F''')'' + R(F'^2 - FF'')' = 0, \tag{16}$$

where now $R \equiv v_w a / \nu$ (negative for injection), a is the radius of the tube, $F = F(\eta)$, $\eta \equiv (r/a)^2$ and r is the radial polar coordinate measured from the center of the tube at any cross-section. The appropriate boundary conditions are:

$$F(0) = 0, \quad F'(1) = 0, \quad F(1) = \frac{1}{2}, \quad \lim_{\eta \rightarrow 0} (\eta)^3 F''(\eta) = 0. \tag{17}$$

Integrating Eq. (16) with respect to η over the region (0, 1) yields the averaging condition

$$[\eta F'''' + F''(1 - RF) + RF'^2]_0^1 = 0. \tag{18}$$

Moreover, from the original differential equation (16) and the boundary conditions (17) it follows that at the wall,

$$F^{(4)}(1) + \left(2 - \frac{R}{2}\right)F'''(1) = 0. \quad (19)$$

An approximate solution of the differential equation (16) can be obtained by assuming the solution in the fourth-degree polynomial form

$$F(\eta) = \left(\eta - \frac{\eta^2}{2}\right) + a_3(\eta - 2\eta^2 + \eta^3) + a_4(2\eta - 3\eta^2 + \eta^4). \quad (20)$$

Equation (20) satisfies identically all of the boundary conditions (17). The coefficients a_3 and a_4 can be obtained from the conditions (18) and (19). It is thus found that

$$a_3 = ba_4, \quad (21a)$$

$$(b + 2)^2Ra_4^2 - ca_4 + \frac{R}{2} = 0, \quad (21b)$$

where

$$b = -4\frac{6 - R}{4 - R}, \quad c = 12(3 + b) - R(7 + 3b). \quad (21c)$$

An approximate value, for all injection (negative) values of R , of the physically appropriate root, which is the smaller root in absolute value, of Eq. (21b) will be:

$$a_4 \approx \frac{R}{2c} \left[1 + 2(b + 2)^2 \left(\frac{R}{2c}\right)^2 \right]. \quad (21d)$$

Equations (20) and (21) constitute a general approximate solution of Eq. (16) under the boundary conditions (17) for the entire range $-\infty \leq R \leq 0$. Once again, in the limit of small $|R|$, Eqs. (20) and (21) will yield exactly the small-perturbation solution

$$[F(\eta)]_{|R| \ll 1} = \left(\eta - \frac{\eta^2}{2}\right) + R\left(\frac{\eta}{18} - \frac{\eta^2}{8} + \frac{\eta^3}{12} - \frac{\eta^4}{72}\right). \quad (22)$$

Moreover, in the other extreme limit of $R \rightarrow -\infty$, Eqs. (20) and (21) will be found to agree well with the exact asymptotic solution of Eq. (16)*, namely

$$[F(\eta)]_{R \rightarrow -\infty} = \frac{1}{2} \sin \frac{\pi}{2} \eta. \quad (23)$$

In particular, in the limit of $R \rightarrow -\infty$, $F'''(1)$, to which the skin-friction is proportional, has the value $[-9 + (17)^{1/2}]/4 = -1.219$ according to the approximate solution, in comparison with the exact (asymptotic) value $(-\pi^2/8) = -1.234$, the percentage difference being 1.2%.

*This is obtained similarly to Eq. (13b) for channel flow. See also [2].

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