ON SUPERSONIC FLOW BEHIND A CURVED SHOCK*

BY R. E. MEYER (*University of Sydney*)

1. When a sharp-edged cylindrical body is immersed in a uniform supersonic stream, a curved shock is attached to the nose, provided the body is not too thick. A solution is presented which accounts to a first approximation for the entropy gradients and vorticity downstream of the shock.

On the approximation of shock-expansion theory the flow downstream of the shock is a simple wave in which the entropy and a characteristic parameter (α, say) of homentropic theory have the values occurring just on the nose in the real flow. The prediction thus obtained for the pressure on the body was shown by Mahony [1, 2] to be remarkably accurate at all Mach numbers and for virtually all shock strengths insufficient to cause any regions of subsonic flow. Now, as long as viscosity is neglected, any streamline in the field represents a potential body contour and hence, *the parameter α is approximately constant along streamlines*, downstream of the shock. This has been noted also by Mahony [2] and conjectured for hypersonic flow by Stocker [3]. The solution on this approximation, which is given below, predicts the same pressure on the body as shock-expansion theory, but a flow with entropy gradients and vorticity; it is slightly less accurate, but also considerably simpler, than Mahony's solution, from whose numerical results [1, 2] the error can be estimated.

2. Assume the gas to be inviscid and perfect, so that the pressure p, density ρ, and specific entropy S are related by $(\partial p/\partial S)_\rho = p/c_v$ and $(\partial p/\partial \rho)_S \equiv a^2 = \gamma p/\rho$, where $\gamma = c_p/c_v$. Since the flow is homenergetic,

$$q^2/2 + a^2/(\gamma - 1) = \text{constant}, \tag{1}$$

where q denotes the velocity magnitude, and if the Mach number M, Mach angle μ and Prandtl angle t are introduced by

$$M = 1/\sin \mu = q/a \quad \text{and} \quad dt = (1/q) \cot \mu \, dq, \tag{2}$$

it follows that

$$dt + (\lambda/c_v)\, dS + (\gamma - 1)(\lambda/p)\, dp = 0, \tag{3}$$

where $\lambda = \sin \mu \cos \mu/[\gamma(\gamma - 1)]$. If x, y and θ denote respectively the cartesian coordinates in the flow plane and the stream direction (measured from the direction of x increasing) and if

$$\theta + t = \alpha, \qquad \theta - t = \beta, \tag{4}$$

the characteristic equations of steady, two-dimensional, supersonic flow may be written as follows [4].

$$\left.\begin{aligned} & & dy/dx &= \tan(\theta - \mu) & (5a)\\ &\text{and} & d\theta &= (\gamma - 1)(\lambda/p)dp, & (5b)\\ &\text{or by (3), (4),} & d\alpha &= -(\lambda/c_v)dS, & (5c)\end{aligned}\right\} \text{on the 'plus' Mach lines}$$

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$$\left.\begin{aligned} dy/dx &= \tan(\theta + \mu) \\ \text{and} \quad d\theta &= -(\gamma - 1)(\lambda/p)dp, \\ \text{or by (3), (4),} \quad d\beta &= (\lambda/c_v)dS \end{aligned}\right\} \text{on the 'minus' Mach lines} \quad \begin{matrix}(6a)\\(6b)\\(6c)\end{matrix}$$

$$\left.\begin{aligned} \text{and} \quad dy/dx &= \tan\theta, \\ dS &= 0 \end{aligned}\right\} \text{on the streamlines.} \quad \begin{matrix}(7a)\\(7b)\end{matrix}$$

3. For definiteness, only the flow on the 'upper' side of a body is considered in the following, i.e. the flow downstream of a shock with slope approaching that of a minus Mach line as the shock strength tends to zero. On shock-expansion theory, such a flow is a simple wave in which α = constant, and on the present approximation, therefore,

$$\alpha = \text{constant along streamlines.} \tag{8}$$

By (7b), then, $\alpha = \alpha(S)$, and if (5c) is to be satisfied,

$$d\alpha/dS = -\lambda/c_v\ . \tag{9}$$

It follows from (4), (6c) and (6b) that

$$d\theta = 0 \text{ and } dp = 0 \text{ along minus Mach lines} \tag{10}$$

so that, as on the simpler approximations for the flow, *the minus Mach lines are isoclines and isobars*; but in contrast to the simpler approximations, they are neither isovels nor straight.

4. The trivial case of a straight shock, which does not introduce any vorticity, will be excluded in the following. Another exceptional case is that of flow adjacent to a straight streamline. On the present approximation, by (10), (8), (4), (7b) and (3), such a flow is a parallel flow at uniform pressure throughout a strip covered by minus Mach lines. The distributions of entropy and of vorticity in such a *pure shear-flow* depend on the initial conditions.

5. If pure shear-flow is excluded, θ represents a characteristic variable labelling minus Mach lines, by (10). The streamlines may be labelled by a variable ψ defined in terms of the streamfunction Ψ by

$$d\Psi = \rho_0 a_0\, d\psi, \tag{11}$$

where a suffix 0 denotes local stagnation conditions.
Thus (6a) and (7a) may be written

$$\partial y/\partial\psi = h_\psi \sin(\theta + \mu), \qquad \partial x/\partial\psi = h_\psi \cos(\theta + \mu) \tag{12}$$

and

$$\partial y/\partial\theta = h_\theta \sin\theta, \qquad \partial x/\partial\theta = h_\theta \cos\theta, \tag{13}$$

where h_ψ and h_θ are parameters defined by these equations themselves, and from (7b), (8) and (10), $S = S(\psi)$, $\alpha = \alpha(\psi)$ and $p = p(\theta)$.

By (12), $d\Psi = \rho q \sin \mu h_\psi d\psi$, and since the motion along a streamline is isentropic and by (1), (2) and (11),

$$h_\psi(\theta, \psi) = Q(\alpha - \theta), \qquad Q(t) = (1 + (\gamma - 1)M^2/2)^b, \tag{14}$$

where $2b = (\gamma + 1)/(\gamma - 1)$. By equating the mixed second derivatives of x and y with respect to θ and ψ, one finds

$$\partial h_\theta/\partial\psi = -L(\alpha - \theta), \tag{15}$$

where $L(t) = (1 - d\mu/dt)\, Q(t) \operatorname{cosec} \mu = (\gamma + 1)\, Q(t)/(2\cos^2\mu \sin \mu)$, by (14) and (1), (2).

The boundary condition on the downstream side of the shock is, by (12) and (13),

$$\frac{h_\psi \sin(\theta + \mu)\, d\psi/d\theta + h_\theta \sin \theta}{h_\psi \cos(\theta + \mu)\, d\psi/d\theta + h_\theta \cos \theta} = \tan \epsilon,$$

where ϵ denotes the angle which the shock front makes with the direction of x increasing. Since the flow upstream is uniform, the shock equations [4] determine ϵ, μ, t and hence also $D = Q(t) \sin(\theta + \mu - \epsilon) \operatorname{cosec}(\epsilon - \theta)$ in terms of θ and the boundary condition becomes

$$h_\theta(\theta, \psi_s) = D(\theta)\, d\psi_s/d\theta,$$

where $\psi = \psi_s(\theta)$ is the equation of the shock in the characteristic plane. By (15), if $\psi = 0$ on the body,

$$D(\theta)\, d\psi_s/d\theta + \int_0^{\psi_s} L(\alpha(\psi) - \theta)\, d\psi = h_\theta(\theta, 0). \tag{16}$$

6. Equation (16) provides the solution of the 'indirect' problem of determining the shape of the body upstream of a minus Mach line on which the flow is prescribed. It may be employed, for instance, to calculate the shape of wall required to convert a uniform flow into a chosen pure shear flow for experimental purposes (provided the two flows do not differ too much for the conversion to be possible by means of an oblique shock with purely supersonic flow downstream). In that case, $\psi(\alpha)$ is prescribed on the minus Mach line forming the upstream border of the pure shear-flow, and the shape of the wall is obtained from (16) and (13), by quadrature.

7. For the 'direct' problem of determining the flow when the shape of the body is prescribed, (16) represents an integral equation for $\psi_s(\theta)$. Since the present approximation is not expected to account for terms of higher than first order in the differences of α occurring in the flow, (16) may be written

$$D(\theta)\, d\psi_s/d\theta + l(\theta)\psi_s = h(\theta) + k(\theta) \int_0^{\psi_s} (\alpha_s - \alpha)\, d\psi, \tag{17}$$

where $h(\theta) = h_\theta(\theta, 0)$, $l(\theta) = L(\alpha_s - \theta)$, $k(\theta) = dL(\alpha_s - \theta)/d(\alpha_s - \theta)$ and $\alpha_s(\theta)$ denotes the value of α on the downstream side of the shock. An approximation for ψ_s is provided by the solution of

$$D\, d\psi_1/d\theta + l\psi_1 = h, \tag{18}$$

i.e.,

$$\psi_1 = g^{-1} \int_{\theta_N} (hg/D)\, d\theta, \tag{19}$$

where $(1/g)\, dg/d\theta = l/D$ and θ_N denotes the initial value of θ on the nose. Friedrichs' approximation [5] is obtained by neglecting differences in α altogether in (19). The iteration

$$\psi_{n+1} = g^{-1} \int_{\theta_N}^{\theta} (g/D)\left[h + k \int_0^{\psi_n} (\alpha_s - \alpha(\psi'))\, d\psi' \right] d\theta, \qquad n \geq 1, \tag{20}$$

converges when θ/θ_N is bounded, and the solution is $\psi_s(\theta) \doteq \psi_2(\theta)\, [1 + 0(\alpha_m^2)]$, where α_m denotes the maximum difference in α occurring between the body and the streamline with direction θ at the shock. The shape of the shock is found from (12) by quadrature. The distributions of S and M in the characteristic plane are also determined by $\psi_s(\theta)$, since S and α are known functions of θ at the shock, and the distributions in the flow plane are again found from (12).

When the region of flow is unlimited, $l/D \sim 2/\theta$ as $\theta \to 0$, and hence $g \sim \theta^2$; but if α_1 denotes the value of α upstream of the shock, then [6] $\alpha_1 - \alpha_s(\theta) = 0(\theta^3)$ if θ is sufficiently small throughout the flow, and since [5] $\psi_s \sim \theta^{-2}$, the iteration still converges and the solution is again $\psi_s = \psi_2\, [1 + O\,(\alpha_m^2)]$.

It may be noted that the theory applies also to the flow downstream of several shocks facing in the same direction, but since D depends not only on θ, when the flow upstream of the shock is non-uniform, the determination of the second and further shocks is more laborious.

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ON RICCATI'S RESOLVENT*

By AUREL WINTNER (*The Johns Hopkins University*)

There belongs to every differential equation of the form

$$y'' + f(t)y' + g(t)y = 0 \tag{1}$$

a "Riccati resolvent,"[1] a differential equation which, in contrast to the equation (1) of second order, is not linear (but quadratic in the unknown function) but which is only of the first order. It is also known that a reduction to a differential equation of Riccati type remains possible if the single equation (1) of second order is generalized to a system of $n = 2$ equations of first order, that is, to

$$u' = a(t)u + b(t)v, \qquad v' = a(t)u + d(t)v \tag{2}$$

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[1]See, e.g., F. Klein, *Vorlesungen ueber die hypergeometrische Funktion*, Berlin, 1933, pp. 150-154.