—NOTES—

A GENERALIZATION OF THE FUNCTIONAL RELATION $Y(t + s) = Y(t) \cdot Y(s)$ TO PIECEWISE-LINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS*

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The functional relation mentioned in the title,

$$Y(t + s) = Y(t) \cdot Y(s), \qquad Y(t) = (y_{ij}(t)), \qquad (i, j = 1, \cdots, n) \tag{1}$$

is satisfied ([3], p. 13) by any fundamental matrix of solutions of a system of linear homogeneous differential equations with constant coefficients

$$y' = Ay, \qquad (' = d/dt; \qquad A = (a_{ik})), \tag{2}$$

where y is a real n-vector and A is a (real) constant $n \times n$ matrix. We shall consider certain inhomogeneous equations corresponding to (2) and obtain a generalization of (1) which has many useful applications in the design of relay servomechanisms [1] and on-off control systems [2].

Consider the equation

$$y' = Ay + f[\Sigma(t - t_d)], \tag{3}$$

$$\Sigma(t) \equiv b \cdot By, \tag{4}$$

where y and A are as above, b is a constant vector, B a constant matrix, and $\cdot$ represents the scalar product. The forcing term f is a (vector) step-function of the scalar Σ; it is assumed that f is constant for at least an interval of length $4t_d$ following each (discontinuous) change in its value.

A typical example is the system

$$y'' + y' + f[\Sigma(t - t_d)] = 0, \qquad \Sigma(t) = y(t), \tag{5}$$

$$f[\Sigma(t)] = \begin{cases} 0, & \text{for} \quad -\theta + (\operatorname{sgn} \Sigma'(t))\Delta\theta < \Sigma(t) < \theta + (\operatorname{sgn} \Sigma'(t))\Delta\theta, \\ \operatorname{sgn} \Sigma(t), & \text{otherwise}, \end{cases}$$

where θ, $\Delta\theta$, and t_d are parameters; in this case it can be shown [1] that the hypothesis concerning the constancy of f is valid whenever $t_d < 2\theta$ (see *Remark* at end of paper).

It is assumed that the system (3) possesses some undesirable behavior when $t_d > 0$ (e.g., a limit-cycle), whereas the corresponding undelayed system

$$y' = Ay + f[\Sigma(t)] \tag{6}$$

does not. (The behavior of many systems of type (6) is analyzed in [1] and [5].)

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Consequently one wishes to replace the relation (4) by a more general one,

$$\Sigma_1(t) \equiv b \cdot B\{Py(t) + Qf[\Sigma_1(t - t_d)]\}, \tag{7}$$

where the constant matrices P, Q are to be so chosen that

$$\Sigma_1(t) = \Sigma(t + t_d). \tag{8}$$

The delayed system

$$y' = Ay + f[\Sigma_1(t - t_d)] \tag{9}$$

will then be identical with the ideal system (6), except at points where the hypothesis concerning f fails to hold (e.g., the "end points" of [5]; this situation is discussed in more detail in [1]).

That P and Q can always be chosen so as to verify (8) is a consequence of the following result.

THEOREM. *Consider the system* (9) *during an interval* $(t_0 - 2t_d, t_0 + 2t_d)$ *in which it is known that* f *is constant. If the general solution of* (9), *for* $y(0) = y_0$, *is given by*

$$y(t) = Y(t)y_0 + Z(t)f, \tag{10}$$

then in $(t_0, t_0 + t_d)$,

$$y(t + t_d) = Y(t_d)y(t) + Z(t_d)f. \tag{11}$$

COROLLARY. *Let*

$$P = Y(t_d), \qquad Q = Z(t_d) \tag{12}$$

in (7), *and* (8) *will hold.*

Proof. As is well known ([3], [4]), the solution of (9) is given by

$$\begin{aligned} y(t) &= \exp(At)y_0 + \exp(At) \int_0^t \exp(-Au)f\,du \\ &= \exp(At)y_0 + \left(\int_0^t \exp[A(t - u)]\,du\right)f. \end{aligned} \tag{13}$$

Hence

$$\begin{aligned} y(t + t_d) &= \exp(At_d)[\exp(At)y_0] + \left(\int_0^{t+t_d} \exp[A(t + t_d - u)]\,du\right)f \\ &= \exp(At_d)y(t) - \exp(At_d)\left(\int_0^t \exp[A(t - u)]\,du\right)f \\ &\qquad + \left(\int_0^{t+t_d} \exp[A(t + t_d - u)]\,du\right)f \\ &= \exp(At_d)y(t) + \left(\int_t^{t+t_d} \exp[A(t + t_d - u)]\,du\right)f \\ &= \exp(At_d)y(t) + \left(\int_0^{t_d} \exp[A(t_d - u)]\,du\right)f. \end{aligned} \tag{14}$$

On comparing (10) and (13) one finds

$$Y(t) = \exp(At), \qquad Z(t) = \int_0^t \exp[A(t-u)]\,du; \tag{15}$$

thus (11) follows from (14) and (15).

Remark added in proof. In order that (6) and (9) be equivalent, it is not necessary that (8) hold always; rather, it is $f[\Sigma(t)] = f[\Sigma_1(t - t_d)]$ that is required. But this is possible even when f is not constant for intervals of length at least $4t_d$. In some cases (e.g. (5)) a minimum of t_d is sufficient.

References

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NOTES ON MATRIX THEORY—X
A PROBLEM IN CONTROL*

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Summary. In the theory of control processes, it is important to be able to calculate $\int_0^\infty (x, Bx)dt$ without having to solve explicitly the differential equation $dx/dt = Ax$, $x(0) = c$. A method for doing this is presented in this paper, generalizing one due to Anke for nth order linear differential equations.

1. Introduction. In a recent paper [1], Anke showed that the expression

$$J = \int_0^\infty x^2\,dt \tag{1}$$

could be computed as a rational function of the coefficients, $a_1, a_2, \cdots, a_n$ in the differential equation for x,

$$\frac{d^{(n)}x}{dt^n} = a_1 \frac{d^{(n-1)}x}{dt^{n-1}} + \cdots + a_n x, \tag{2}$$

and the initial values $x(0) = c_1$, $x'(0) = c_2$, $\cdots$, $x^{(n-1)}(0) = c_{n-1}$, without solving the equation explicitly, provided that all the solutions of (2) approached zero as $t \to \infty$. This is equivalent to the condition that all the roots of the equation

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0 \tag{3}$$

have negative real parts. Determinantal criteria for this were first given by Hurwitz.

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