QUARTERLY OF APPLIED MATHEMATICS

Vol. XIV JANUARY, 1957 No. 4

SOME ASPECTS OF UNSTEADY LAMINAR BOUNDARY LAYER FLOWS*

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Introduction. The problems of unsteady laminar boundary layer flow have not received nearly so much attention as the steady flow problems. Both Goldstein [1] and Schlicting [2] summarized the analyses of the boundary layer flow when the solid body starts to move impulsively or with uniform acceleration and of the boundary layer over an oscillating solid surface. Under the boundary layer approximation, the differential equations determining the flow field of an incompressible fluid are mass continuity $U_x + V_y = 0$, momentum $U_t + UU_x + VU_y = U_e U_{ex} + \nu U_{yy}$, subjected to the boundary conditions

$$U(x, \infty, t) = V(x, 0, t) = 0,$$

$$U(x, y, t) = V(x, y, t) = 0 \quad \text{for} \qquad t \leq 0,$$

$$U(x, y, t) = 0 \qquad \text{for} \quad x < -\int_0^t U_0(t)\, dt, \qquad t > 0,$$

$$U(x, 0, t) = -U_0(t) \qquad \text{for} \quad x \geq -\int_0^t U_0(t)\, dt, \qquad t > 0.$$

Here t indicates time. X and Y are the rectangular cartesian coordinates of any point in the field with respect to a frame of reference fixed in space, X-axis being in the direction of motion of the body with origin of the axes located at the position of the leading edge of the plate at the instant $t = 0$. U and V are the velocity components of fluid elements in the X and Y directions respectively. U_e is the velocity of the fluid just outside the viscous layer, i.e., the boundary layer. ν is the kinematic viscosity coefficient of the fluid. The body is assumed to move in the negative X direction with a velocity $U_0(t)$. The method of solution of various specific problems as outlined in Refs. [1] and [2] are based upon the following physical argument. Initially the boundary layer has zero thickness; and at the beginning of the motion, the effect of diffusion far outweighs the effects of convection and pressure gradient. Thus, for the first approximation, the flow field is essentially determined by the diffusion of vorticity in the Y direction, i.e., the flow field is determined by

$$U_t^{(0)} = \nu U_{yy}^{(0)},$$

*Received September 28, 1955. This paper is a condensed version of Aeronaut. Eng. Rept. No. 311, under the same title, July 1, 1955, to which the reader is referred for detailed information. This work is completed under contract no. AF 18(600)-498 United States Air Force Office of Scientific Research.

where superscript $^{(0)}$ indicates the zeroth approximation. $U^{(0)}$ is independent of X and can also be considered as the exact solution for the flow field over a doubly infinite plate moving in its own plane with velocity $U_0(t)$. This may be referred to as the Rayleigh type solution which also represents a class of exact solutions of the Navier-Stokes equations. To take into account the fact that the body is finite in length or to determine the flow field at later instants based upon the boundary layer equations, the Rayleigh type solution is perturbed and the boundary layer equation is linearized with respect to the perturbation. The perturbation is then determined as a first correction to the Rayleigh type solutions and the same procedure is employed to obtain successive corrections as far as one wants.

However, this approach fails to work if there is no pressure gradient along the solid surface; for example, the unsteady flow over a semi-infinite plate moving in its own plane. By semi-infinite flat plate, we mean a flat plate with a leading edge either facing an oncoming stream or moving into air at rest with a trailing edge at downstream infinity. In this case, the successive perturbations over the Rayleigh type solution vanish identically whether the plate is started to move impulsively, with acceleration, or the plate begins to oscillate. This seems to indicate that there might be a region in the downstream side of the plate, in which the fluid does not "know" the existence of a leading edge in the upstream end, and the unsteady flow in the region would be given by the Rayleigh type solution. On the other hand, it is physically clear that in the upstream region near the leading edge, the flow field must depend upon the distance from the leading edge.

The question naturally arises how the X-dependent solution in the upstream region can be connected with the X-independent solution assumed to be valid in the downstream region. In other words, how can we perturb the Rayleigh type solution so as to introduce the X-dependence under the influence of the leading edge. Moreover, in the flat plate problem, the fact that the regular perturbation fails to introduce any X-dependence into the Rayleigh type solution may be interpreted as an indication that the effect of the leading edge has not been taken into account. With this in mind, one would question the validity of the method of analyzing non-flat-plate problems as summarized in Refs. [1] and [2]. Because the X-dependence which is introduced primarily by the pressure gradient into the successive perturbations over the Rayleigh type solution could not include the effect of the leading edge; that is, the finite length of the body.

This question was first mentioned by Stewartson [3] who analyzed the problem of a semi-infinite flat plate started to move impulsively from rest. An essential singularity is found at the outer edge of the viscous layer at the initial position of the leading edge ($X = 0$), and moves upstream ($X < 0$) and toward the plate ($Y = 0$). This essential singularity serves to connect the downstream region ($X > 0$) where the flow field is independent of X (the Rayleigh type solution) and the upstream region near the leading edge at any instant $t(X = U_0t)$ where the flow field is independent of t (the steady state Blasius solution.) Attempt to investigate the nature of the flow field in the vicinity of the essential singularity fails to clarify how the joining can be accomplished. The occurrence of the type of essential singularity mentioned above is certainly not common in physical problems. To make sure that the development of the essential singularity has little to do with the initial discontinuity of the velocity $U_0(t)$ at the instant $t = 0$ under the impulsive start, it is, therefore, necessary to investigate the case in which the plate starts to move from rest continuously, in a prescribed manner. The result shows that the same essential singularity occurs at the same location. From physical considerations, the occurrence of the non-isolated essential singularity at $X = 0$ and $Y = \infty$, could be so interpreted that the flow field, as determined from the boundary layer equations, is not analytic at $X = 0$ and that singular perturbation of the Rayleigh type solution would be the only possible way of introducing X-dependence. The situation is further obscured by the fact that (Sec. 3), under the assumption of no discontinuity of the velocity u between the upstream (X-dependent) region and that in the downstream (X-independent) region at the point of joining, the singular perturbation about the Rayleigh type solution with physically conceivable singularities fails to give any non-vanishing perturbations. It is therefore of great interest to see how the solutions in the two regions could ever be joined together and how a solution valid approximately in both regions could be constructed.

2. Formulation. Consider the two-dimensional incompressible flow induced by the motion of a semi-infinite flat plate in its own plane. Both the air and the plate were at rest for $t = 0$ and the plate started to move with velocity $U_0(t)$ in the negative direction of the X-axis when $t > 0$.

The pressure is uniform throughout the entire flow field under the boundary layer approximation. If the moving cartesian coordinate with the origin fixed at the leading edge of the plate is used, the air would appear to flow in a positive x direction with velocity $U_0(t)$ and an effective local pressure gradient $-U_{0t}$. The boundary layer equation in terms of the stream function $\psi(x, y, t)$ becomes

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} = U_{0t} + \nu \psi_{yyy} . \tag{1}$$

It is easily seen that the number of independent variables can be reduced if $U_0(t) = At^n$ with the two new independent variables

$$\xi = x \cdot (n + 1)/At^{n+1}; \qquad \eta = y(n/\nu t)^{1/2} \tag{2}$$

and the new dependent variable

$$f(\xi, \eta) = \psi(x, y, t) \cdot (\nu t/n)^{-1/2} U_0(t)^{-1}. \tag{3}$$

The differential equation to be satisfied by $f(\xi, \eta)$ is

$$f_\eta - \frac{n+1}{n} \xi f_{\xi\eta} - \frac{1}{2n} \eta f_{\eta\eta} + \frac{n+1}{n} [f_\eta f_{\xi\eta} - f_\xi f_{\eta\eta}] = 1 + f_{\eta\eta\eta} \tag{4}$$

with the following boundary conditions:

$$f_\xi(\xi, 0) = f_\eta(\xi, 0) = 0. \qquad f_\eta(\xi, \infty) = 1 \quad \text{for} \quad \xi \geq 0 \tag{5}$$

and

$$f_\eta(\xi, \eta) = 1 \quad \text{for} \quad \xi < 0$$

where

$$f_\eta = u(x, y, t)/U_0(t), \qquad f_\xi = -v(x, y, t) \cdot (nt/\nu)^{1/2} (n + 1)^{-1}. \tag{6}$$

The present problem is to construct a solution of Eq. (4) satisfying the boundary conditions Eq. (5) and valid throughout the vicinity of the plate.

3. Solution from downstream end. The straightforward power series expansion

$$f(\xi, \eta) = f^{(0)}(\eta) + f^{(1)}(\eta)\xi^{-1} + f^{(2)}(\eta)\xi^{-2} + \cdots \tag{7}$$

will be considered first. The zeroth approximation $f^{(0)}(\eta)$ is expected to be valid at downstream infinity and has been referred to as the Rayleigh type solution for corresponding flow over a doubly infinite flat plate. $f^{(0)}(\eta)$ is determined by the following equation

$$f^{(0)}_{\eta\eta\eta} + \frac{1}{2n}\eta f^{(0)}_{\eta\eta} - f^{(0)}_{\eta} + 1 = 0. \tag{8}$$

With the boundary conditions $f^{(0)}(0) = f^{(0)}_\eta(0) = 0$ and $f^{(0)}_\eta(\infty) = 1$. It is found as

$$\begin{aligned} f^{(0)}_\eta = 1 - \exp(-\eta^2/4n)\cdot[{}_1F_1(n + \tfrac{1}{2}, \tfrac{1}{2}, \eta^2/4n) \\ + n^{-1/2}\cdot n!/(n - \tfrac{1}{2})!\cdot \eta\, {}_1F_1(n + 1, \tfrac{3}{2}, \eta^2/4n)], \end{aligned} \tag{9a}$$

where, ${}_1F_1$ indicates the confluent hypergeometric function. The solution of Eq. (8) with a double zero at $\eta = 0$ may also be given as the following series

$$\begin{aligned} f^{(0)}_\eta = a_1\eta + a_1 \sum_{j=1}^{\infty}\left[\frac{\eta^{2j+1}}{(2j+1)!}\prod_{k=1}^{k=j}\left(\frac{2n-2k+1}{2n}\right)\right] \\ - \sum_{j=1}^{\infty}\left[\frac{\eta^{2j}}{(2j)!}\prod_{k=1}^{k=j}\left(\frac{n-k+1}{n}\right)\right], \end{aligned} \tag{9b}$$

where $a_1 = 2/\pi^{1/2}$ when $n = 1$ and can be easily determined from (9a) when $n \neq 1$.

The equations for $f^{(j)}_\eta$ with $j \neq 0$ are of second order and of Weber type. Thus neither of the two fundamental solutions can be present in the general solution under the boundary condition $f^{(j)}_\eta(0) = 0$ and $f^{(j)}_\eta(\infty) = 0$. As the equation for $f^{(1)}_\eta$ is homogeneous $f^{(1)}_\eta(\eta)$ must be identically zero. With $f^{(1)}_\eta(\eta) = 0$ all the higher perturbations $f^{(j)}_\eta(\eta)$ with $j > 1$ vanish. Accordingly, it is often conjectured that the Rayleigh type solution, $f^{(0)}_\eta$ could be valid in the region extending from the downstream infinity to some upstream stations at finite distance from the leading edge and then the Rayleigh type solution would be connected to the ξ-dependent upstream solution with or without some singularity at the junction. This conjecture is supported by the following well known physical argument.

The air which was originally sitting on the flat plate does not know the existence of the leading edge under the Prandtl boundary layer approximation where the vorticity does not diffuse lengthwise along the plate. The flow of the air originally on the plate is induced only by the diffusion of vorticity from the plate normal to the plate. Thus the flow of the air is the same at any station where $\xi > 1$ or $x > \int_0^t U_0(t)dt$, and can be expected to be described by the Rayleigh type solution $f^{(0)}_\eta$. At $\xi = 1$, some singularity may occur due to the presence of the leading edge at the initial instant $t = 0$ and will not diffuse sidewise. The flow field with $\xi < 1$ will therefore be expected to depend on ξ.

Our question is then how the Rayleigh type solution, assumed to be valid in the region $\xi > 1$, can be continued through the singularity into the region $\xi < 1$ where the flow field will vary with ξ. Stewartson [3] noticed the existence of an essential singularity at $y = \infty$ and $X = U_0t$ in his analysis of the impulsively started flat plate; and he investigated the nature of the solution in the neighborhood of this essential singularity. The result of his analysis is however, not very fruitful. The same situation is encountered in the present problem, even though no velocity discontinuity is introduced at the instant $t = 0$. If one observes Eq. 4 with η held constant, one sees that the coefficient of $f_{\eta\xi} = \partial/\partial\xi(u/u_0)$ is proportional to $(\xi - f_\eta)$. Now at any station $\xi \leq 1$ this factor $\xi - f_\eta$ can vanish at some value of η because f_η varies from 0 to 1. Thus the equation has some singular behavior along the line extending from the leading edge $\xi = 0$, $\eta = 0$, to the point $\xi = 1$, $\eta = \infty$. The singularity at $\xi = 1$ and $\eta = \infty$ where f_η approaches unity exponentially is an essential one and it may not be isolated. A mathematical treatment of the flow field in the vicinity of such a singular point seems to be rather hopeless; moreover, from the engineering point of view, this is not quite the right thing to do. The singularity is primarily a trace of the effect of the leading edge as reflected in this particular method of solution. In the first place, a correct analysis of the flow field in the vicinity of the leading edge should not be made on the boundary layer equation. In the second place, if one considers the time-wise development of the flow in the vicinity of $\xi = 1$, the leading edge singularity, was originated at $\eta = 0$ and diffused to large η; and will therefore have modified the complete profile $f_\eta(1, \eta)$ from the Rayleigh type solution for all values of η, not just in the vicinity of the apparent singularity. If one intends to investigate carefully the details of the flow in this region $\xi \sim 1$, it seems that the complete Navier-Stokes equation must be considered. What one could do within the framework of the Prandtl boundary layer approximation is to see whether the boundary layer approximation could provide a smooth, consistent joining of the solution in the region $\xi > 1$ and that in the region $\xi < 1$ by allowing for certain singular behavior as $\xi \rightarrow 1$. The continuity of the velocity $u(\xi, \eta)$ is a necessary requirement because no velocity discontinuity has been introduced into the flow field (this argument may not be valid for impulsive start) and no velocity discontinuity is permitted to develop itself anywhere in the flow field. Therefore, our problem is essentially to investigate how the Rayleigh type solution, assumed to be valid (under the boundary layer approximation) throughout the region $\xi > 1$ could be joined continuously with any ξ dependent solution in region $\xi < 1$. Accordingly, an attempt is made to continue the Rayleigh type solution with the boundary layer equation reduced to the form of Eq. 4, by assuming singularities of the type $(1 - \xi)^{1/s}$.

The condition of continuity of u with the Rayleigh type solution as $\xi' = 1 - \xi \rightarrow 0$ requires that s must be 2 with convective effects entering as higher order corrections on the balance between the transient and the viscous effects if the upstream perturbations are non-trivial. Solutions of successive orders as a series of ξ' can be obtained as quadrature integrals involving confluent hypergeometric function. However, when the series is summed up, it is found that the final result is independent of $\xi' = 1 - \xi$ and is identical with the Rayleigh type solutions. This result indicates that even if one requires only the velocity u to be continuous, (the discontinuity of the spatial derivatives has already been admitted in this attempt) the downstream Rayleigh type solution cannot be joined at $\xi = 1$, with any upstream solution of velocity having a ξ dependence in terms of the fractional power of $1 - \xi$. It can also be shown that singularities of the type $\xi'^s \ln \xi'$ also lead to identically zero perturbations over Rayleigh type solution. Physically, it is not easily conceivable that the upstream solution should have a dependence on ξ of the exponential type or even more unconventional ones. Moreover, in the next section it is shown that if one proceeds from the leading edge $\xi = 0$, the dependence of ξ appears as $\xi^{1/2}$. As a result, one has to conclude that either the Rayleigh type solution cannot be exact throughout the region $\xi > 1$ or some unusual behavior of the velocity at $\xi = 1$ should be present or tolerated under the Prandtl boundary layer approximation.

Stewartson [3] mentioned the possibility that an essential singularity might occur at some point $\xi > 1$ but he conceded that it is unlikely. According to the boundary layer approximation, the extent of diffusion of vorticity in the lengthwise direction is of the order of boundary layer thickness and is, therefore, neglected as a higher order small quantity. But the leading edge is the region where the boundary layer approximaton is invalid; and its extent of influence may very well be much larger than the boundary layer thickness. If this possibility is admitted, it may seem that the Rayleigh type solution might prevail from downstream infinity to some value of $\xi_a > 1$ and the solution would gradually transform into a ξ-dependent one for $\xi < \xi_a$. However, an investigation into the possibility of continuing the Rayleigh type solution at $\xi = \xi_a$ leads to the same conclusion as one encountered at $\xi = 1$.

If, furthermore, one makes the transformation $\xi = \tau^{-1}$ and assumes the Rayleigh type solution to be valid at $\tau = 0$ and tries to continue the solution into the region of positive τ with some fractional power of τ, one again obtains trivial results.

All these indicate strongly that the process of trying to obtain solutions in the upstream region by perturbing the Rayleigh type solution, assumed to be valid in the downstream region, may not be a justifiable process especially in view of the fact that the boundary layer equation is parabolic in X and Y.

4. Solution from upstream. The solution of the problem starting from the leading edge involves, as a first step, the determination of the singularities spacewise and timewise at the leading edge. Since the time variable has been incorporated into the spatial variables in the simplified similar solutions considered in the present analysis, one is concerned only with the singularity of $\xi \to 0$ which is primarily a space-wise singularity due to the Prandtl boundary layer approximation. Stewartson [3] considered the leading edge singularity as an essential singularity in the sense that the solution of the problem in the region upstream of the leading edge is independent of ξ while that in the region downstream is dependent on ξ. All the derivatives $\partial/\partial\xi$ of any order are thus discontinuous at $\xi = 0$; a situation which is analogous to that at $\xi = 1$, considered in the previous section. Under the boundary layer approximation, the instantaneous velocity of the flow in the region upstream of the leading edge, $\xi < 0$ is independent of X and Y so that u is a function of t only, and that $u(x, y, t)/U_0(t) = 1$ when $\xi < 0$. Let us investigate the possibility of a continuous joining (with all derivatives discontinuous) between the X-independent solution with the X-dependent solution valid in the regions upstream and downstream respectively of the leading edge, $\xi = 0$, by introducing singularities of the type $\xi^{1/s}$ where s is some unknown positive constant to be determined. The physical justification is similar to that in the vicinity of $\xi = 1$ as discussed in Sec. 3, that is, the vorticity diffuses only normally to the plate and no discontinuity of the velocity can develop. Thus define

$$\xi_0 = \xi^{1/s}, \qquad \zeta_0 = \eta/\alpha_0\xi_0 = (\eta/\xi_0)(n + 1/n)^{\frac{1}{2}} \quad \text{with} \quad \alpha_0 = (n/n + 1)^{\frac{1}{2}} \tag{10}$$

and

$$f(\xi_0 , \eta) = \sum_{l=0}^{\infty} F_l(\zeta_0)\xi_0^{l+\beta+1}\cdot\alpha_0^{l+1} \tag{11}$$

subjected to the initial condition that

$$\lim_{\xi_0\to 0,\, \eta\neq 0} f_\eta(\xi_0 , \eta) = 1. \tag{12}$$

Comparing Eqs. (11) and (12), one immediately sees that $\beta = 0$. By substituting $f(\xi, \eta)$ expansion (11) into Eq. (4), one finds that the lowest order of the transient terms is ξ_0^0, of the convective terms, ξ_0^{-s}, and of the viscous term ξ_0^{-2}. For positive values of s, the convective terms are then much more important than the transient terms; and for nontrivial results, the convective terms must be of the same order as the viscous term, that is $s = 2$. Thus $f_0(\zeta_0)$ is determined by

$$F_0''' + \tfrac{1}{2}F_0F_0'' = 0 \tag{13}$$

subjected to the boundary conditions that $F_0(0) = F_0'(0) = 0$, $F_0'(\infty) = 1$ which is the same as the Blasius problem of steady flow over a flat plate. Thus from Ref. [1] or [2]

$$F_0(\zeta_0) = A_0\zeta_0^2/2! - A_0^2\zeta_0^5/5! + 11A_0^3\zeta_0^8/8! - 375A_0^4\zeta_0^{11}/11! + \cdots \tag{14}$$

with $A_0 = 0.332$ for ζ_0 small. The asymptotic nature of F_0 for large ζ_0 is

$$F_0 = \zeta_0 - \beta + \gamma \int_\infty^{\zeta_0} d\zeta_0' \int_\infty^{\zeta_0'} \exp\left[-(\zeta_0'' - \beta)^2/4\right] d\zeta_0''$$

with

$$\beta = 1.73 \quad \text{and} \quad \gamma = 0.231 \tag{15}$$

It is found that the solution of $F_1(\zeta_0)$ having a double zero at $\zeta_0 = 0$ is identically zero. Likewise it follows that all $F_l(\zeta)$ with odd subscripts vanish identically.

$$F_{2k+1}(\zeta_0) = 0, \qquad k = 0, 1, 2, \cdots . \tag{16}$$

The equations for $F_l(\zeta_0)$ with $l = 2, 4, 6$ etc. are given as

$$\begin{aligned} F_l''' + \tfrac{1}{2}[F_0F_l'' - lF_0'F_l' + (l+1)F_0''F_l] + \delta_{l2} &= [1 - (l-2)/2\alpha_0^2]F_{l-2}' + \tfrac{1}{2}\zeta_0F_{l-2}'' \\ &+ \frac{1}{2}\left[\sum_{k=1}^{l-1}(l-k)F_{l-k}'F_k' - \sum_{k=1}^{l-1}(l-k+1)F_{l-k}F_k''\right], \end{aligned} \tag{17}$$

where δ_{l2} is the Kronecker delta. The boundary conditions for $F_l(\zeta_0)$ are

$$F_l(0) = F_l'(0) = 0 \quad \text{and} \quad \lim_{\xi\to 0,\, \eta\neq 0} F_l'(\zeta_0)/\zeta_0^l = 0. \tag{18}$$

The last condition in Eq. (18) is found to be identical with

$$\lim_{\xi\to 0,\, \eta\neq 0} F_l'(\zeta_0) = 0 \tag{19}$$

due to the asymptotic behavior of the fundamental solutions of Eq. (17). Accordingly the usual boundary condition at large distance from the plate is automatically satisfied by the formal solution determined in the present manner from the initial condition.

The solutions of Eqs. (17) satisfying the boundary conditions Eqs. (18) and (19) are obtained numerically. The values of $A_l = F_l''(0)$ are given as follows: $A_0 = 0.33206$, $A_2 = 0.84851$,

$$\begin{aligned} A_4 &= 0.27088 - 0.49967(n-1)/n, \\ A_6 &= 0.37720 - 0.70141(n-1)/n + 0.36773(n-1)(n-2)/n^2, \\ A_8 &= 0.80413 - 1.68654(n-1)/n + 0.28896(n-1)^2/n^2 \\ &\quad + 0.84295(n-1)(n-2)/n^2 - 0.26949(n-1)(n-2)(n-3)/n^3. \end{aligned} \tag{20}$$

A comparison of the skin friction on the plate $\tau = \mu u_y(0)$ as evaluated from Eq. (20) and that τ_R as evaluated from the Rayleigh type solution Eq. (9) is shown in Fig. 1 determined as the series

$$\tau/\tau_R = \xi^{-1/2}(n+1)^{1/2}(n-\tfrac{1}{2})!/n! \sum_{k=1}^{\infty} A_{2k}(n\xi/n+1)^{2k}. \tag{21}$$

It is quite significant that except for extremely small values of n this ratio is about $1.02 \sim 1.00$ when $\xi = 0.4 \sim 0.5$ which corresponds to the station roughly midway

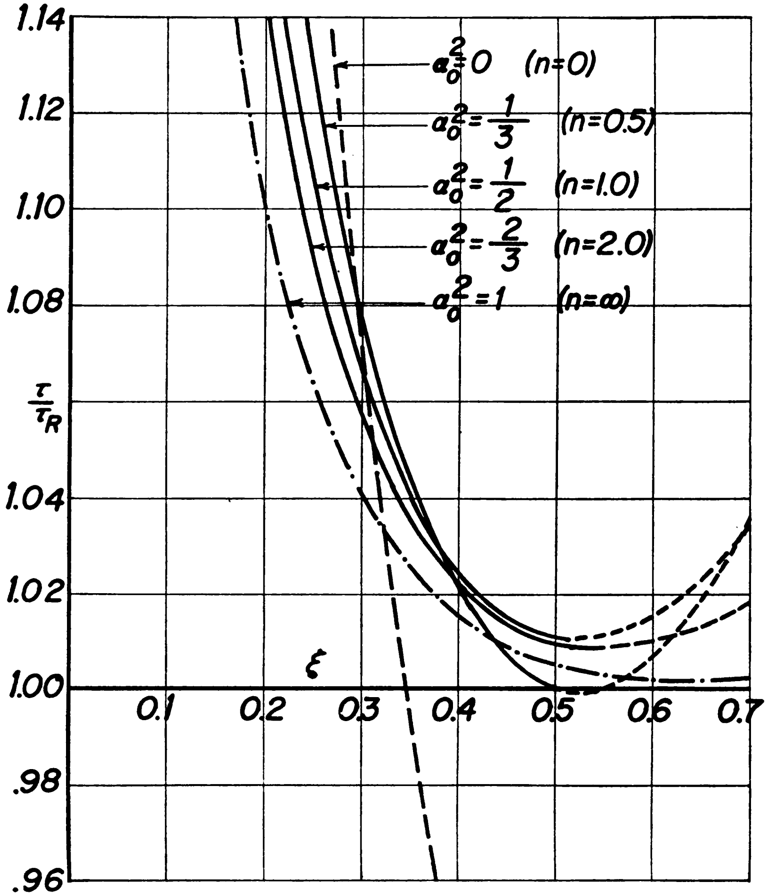

Fig. 1.

between the leading edge at any instant t and the leading edge at the initial instant $t = 0$. Thus the Rayleigh type solution gives a good engineering approximation, at least in so far as skin friction is concerned, well upstream of the leading edge position at $t = 0$ (assuming that the skin friction will eventually settle down at the value given by the Rayleigh type solution as will be shown in the next section).

In Fig. 2, a series of velocity profiles $u/u_0(t)$ at different values of ξ when $n = 1$ are shown. It is clear that the complete profile $u/u_0(t)$ as a function of η does tend to the

Rayleigh type solution. These results are calculated with six terms in Eq. (11) and seem to diverge with six terms when $\xi \geq \frac{1}{2}$.

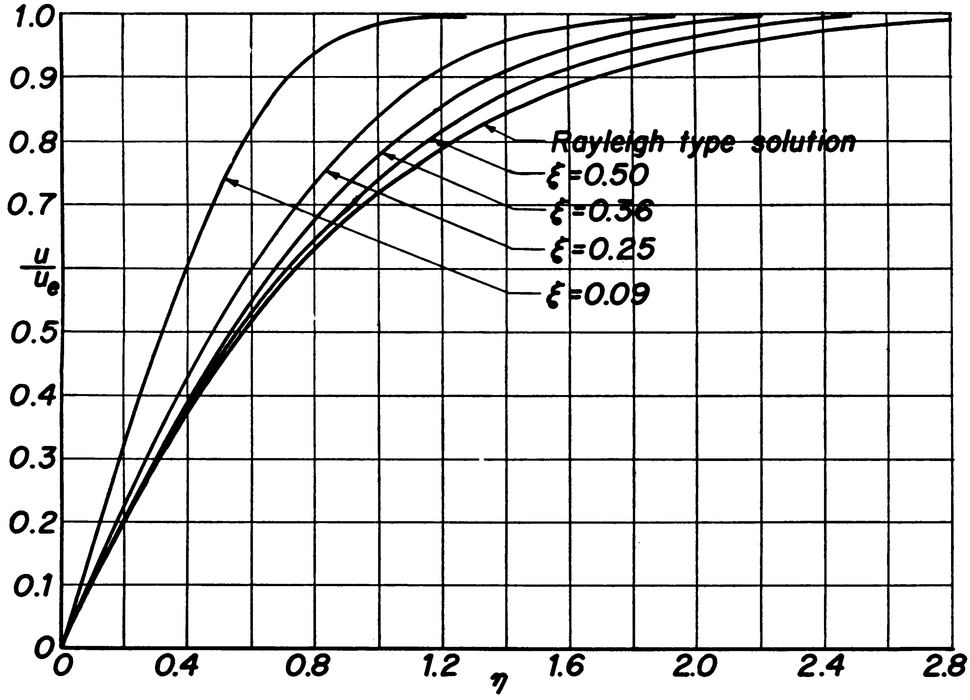

FIG. 2.

5. Solution from intermediate station. To see whether and how the series solution Eq. (11) will eventually approach the Rayleigh type solution at very large ξ, one has to continue the known profile $u/u_0(t)$ at arbitrary intermediate station ξ_i downstream. Let us assume

$$f_\eta(\xi_i\,,\,\eta) = \sum_{l=1}^{\infty} c_l(\xi_i)\eta^l. \tag{22}$$

The summation begins at $l = 1$ due to the nonslip condition on the plate $\eta = 0$. The series (22) must of course be assumed to satisfy the condition $\lim_{\eta\to\infty} f_\eta(\xi_i\,,\,\eta) = 1$. As has been previously indicated, Eq. (4) is possibly singular where $f_\eta = \xi_i \leq 1$ and this may modify the complete profile. Therefore, the possibility of a series solution in terms of $\xi_1 = (\xi - \xi_i)^{1/s}$ is considered. It is found that $s = 2$ and that convective terms always enter as higher order small quantities at successive perturbations with small ξ_1. The following series is used

$$f(\xi_1\,,\,\eta) = \sum_{l=0}^{\infty} c_{l+1}\alpha_1^{l+2}F_l(\zeta_1)\cdot\xi_1^{l+2}, \tag{23}$$

with

$$\xi_1 = (\xi - \xi_i)^{1/2} \qquad \zeta_1 = \eta/\alpha_1\xi_1 = (\eta/\xi_1)[\xi_i(n+1)/2n]^{1/2}. \tag{24}$$

The equations for $F_l'(\zeta_1)$ are again second order equations of Weber type with the boundary conditions

$$F_l(0) = F_l'(0) = 0 \quad \text{and} \quad \lim_{\xi_1 \to 0, \eta \neq 0} F_l'(\zeta_1)/\zeta_1^{l+1} = 1. \tag{25}$$

These equations for $F_l'(\zeta_1)$ will admit solutions if and only if $c_l(\xi_i)'s$ satisfy a series of relations which prescribe precisely that the known profile Eq. (22) must be a solution of Eq. (4) itself. When the solutions $f_\eta(\xi_1, \eta)$ are expanded and rearranged as power series of ξ_1, without questioning the validity, it is found that all the odd powers of ξ_1 vanish. This indicates that, except for the vicinity of certain peculiar points where such a rearranging process is invalid, the downstream solution can be obtained by regular perturbations. Accordingly, the following series solution is considered

$$f(\xi, \eta) = g_0(\eta) + g_1(\eta)\alpha_1^2(\xi - \xi_i) + \cdots \tag{26}$$

with $g_0'(\eta) = f_\eta(\xi_i, \eta)$ given by Eq. (22). The equations for $g_l(\eta)$ are linear, non-homogeneous and of first order. In particular, for $g_1(\eta)$,

$$(g_0' - \xi_i)g_1' - g_0''g_1 = [1 - g_0' - \eta g_0''/2n + g_0''']\xi_i/2 = I_1(\eta). \tag{27}$$

Since $g_0'(\eta)$ and $I_1(\eta)$ behave at large η like $\eta \exp(-\eta^2/4)$ Eq. (27) indicates that $g_1'(\eta)$ also vanishes exponentially with $g_1(\eta)$ approaching a constant value if $\xi_i \neq 1$. The other $g_l'(\eta)$ behaves likewise at large η. Thus the boundary condition at large η that $f_\eta(\xi, \eta) = 1$ is naturally satisfied. If there is no singularity for $0 < \eta < \infty$, the following quadrature integral would serve to determine the complete profile at any downstream station $\xi - \xi_i > 0$ i.e.,

$$g_l(\eta) = (g_0' - \xi_i) \int_0^\eta I_l(\eta)(g_0' - \xi_i)^{-2}\, d\eta \tag{28}$$

where $I_l(\eta)$ is the inhomogeneous part of the first order equation for $g_l(\eta)$. Since $0 \leq g_0' \leq 1$ for $0 < \eta < \infty$, it is clear that the integrals Eq. (28) are free from singularity if $\xi_i > 1$. But when $0 < \xi_i < 1$ the denominator of the integrand may vanish at $\eta = \eta_0$, where $g_0'(\eta_0) = \xi_i$ and $g_l'(\eta)$ is logarithmically singular.

Let us consider the continuation of the F-series solution about the leading edge. If $c_l(\xi_i)$ in Eq. (22) were replaced by $F_l'(\zeta_0)_i'$ evaluated at $\xi = \xi_i$, it is easily shown, if one permits rearranging terms in the series, that the g-series is actually the Taylor expansion of the solution about $\xi = \xi_i$. Then $g_l'(\eta)$ as evaluated from Eq. (28) can be simply recognized as an alternative means of evaluating the Taylor coefficients without finding $\partial F_l/\partial \xi$ etc. at the station ξ_i. Since F-series expansion about the leading edge is not singular at $\zeta(\xi_i, \eta_0)$, the logarithmic singularity may be recognized as primarily due to the rearrangement of the terms in an infinite series. The presence of this logarithmic singularity does not seem serious in the sense that its influence is more or less localized. In fact, the skin friction is not affected at all by this singularity. Numerical calculations at the station $\xi = 0.40$, where the F-series solutions from the leading edge is still expected to be valid, shows that the profile determined from F-series directly cannot be distinguished from the one calculated from g-series as a perturbation over the profile at $\xi_i = 0.35$ for practically the complete range of $0 \leq \eta < \eta_i = 0.34$. This indicates that the g-series solution will give as accurate a solution of the flow field near the plate in the downstream region as the F-series will near the upstream end of the plate despite the presence of the logarithmic singularity. Accordingly, so far as the flow field along the plate surface (including naturally the possibility of determining skin friction at the wall) is concerned, one can repeat the process indefinitely to downstream infinity along the plate because there are no more singlarities along the plate surface. The logarithmic singularity, which recedes from the plate and eventually disappears completely when $\xi > 1$, will not affect the flow field near the plate so determined. Then from the computational point of view, with the six term representation of the F-series solution from the leading edge, one can determine the velocity profile to within a prescribed accuracy up to a station ξ_i. The accuracy can be checked by redetermining the profile at ξ_i using the g-series and the profile at ξ_{i-1}. The flow field along the entire plate surface with $\eta < \eta_i$, where $f_\eta(\xi_i\eta_i) = \xi_i$, can then be determined with the same accuracy by the repeated use of the g-series. From Fig. 2, it is clear that as ξ increases, the local velocity profiles tend to approach the Rayleigh type profile. The difference is significant only in the region sufficiently far away from the plate surface. It is also clear from Fig. 1 that the six term representation of the F-series cannot be valid for moderately large values of ξ, in fact, the curve for $n = 1$ in Fig. 1 is shown dotted for $\xi > 0.5$. Starting from some reasonable value of $\xi < 0.5$ where the F-series will still give the required accuracy, one can proceed with the g-series as has just been described, to investigate numerically whether and how the velocity profile and the skin friction will approach the Rayleigh type solution with increasing ξ. However, the following qualitative discussion may be sufficient to indicate that a monotonic approach to the Rayleigh type solution must take place.

Let us first observe that $I_1 = 0$ at $\eta = 0$, and that $I_1(\eta)$ becomes positive, increasing roughly linearly with increasing η for small η near the wall, so long as the local skin friction remains larger than that of the Rayleigh type solution. With $g_0''(\eta)$ positive and decreasing with increasing η it is not difficult to show that $\partial v/\partial\xi \simeq \alpha_1^2\, g_1'(\eta)$ at small η is always negative and directly proportional to $I_1(\eta)/\xi_i^2$. Now $I_1(\eta)$ is identically zero if $g_0'(\eta)$ is the Rayleigh type solution $f_\eta^{(0)}$ (see Eq. (8)). Therefore, the velocity at a given small value of η and the skin friction at the wall will approach those of the Rayleigh type solution monotonically and will be identical at least when ξ is infinitely large. Once the profile in the immediate vicinity of the plate has approached the Rayleigh type profile within sufficient accuracy, similar observation will apply to the next slices of the stream in the vicinity of the plate. Thus, the approach to the Rayleigh type solution will be "slower" for larger values of η. Therefore, from the engineering point of view, it appears reasonable to expect that the solution starting from the leading edge, as expounded thus far, will eventually approach the Rayleigh type solution as a limit when ξ is sufficiently large, at least for some distance not too far away from the plate.

Consequently, we shall consider the solution of the viscous flow field in the vicinity of the plate to be given by the F-series starting from the leading edge $\xi = 0$ up to some intermediate values of ξ_i where the finite term approximation is still considered satisfactory and is then continued by the g-series successively for the flow region downstream of the station ξ_i up to $\xi \to \infty$. The station ξ_i must correspond to a value of $\tau/\tau_R > 1$. For practically all values of n, if not too close to zero, a good value of ξ_i would be 0.4. The accuracy of the F-series at a selected value of ξ_i can be checked by the g-series constructed from a profile at a slightly upstream station. This formal solution is appreciably in error for detailed description of the flow near the leading edge where $\xi \ll 1$ but it satisfies both the requirement of continuity of the incoming uniform stream at $\xi = 0$

and the requirement of approaching the Rayleigh type solution at a very large distance downstream $\xi \to \infty$. Therefore, this formal solution, besides being self-consistent, meets all the physical requirements upstream and downstream at any instant t.

This method of approach as described in the present paper has been applied to a flat plate moving with arbitrary velocity $U_0(t)$ with $U_0(t) = 0$ when $t = 0$ (the only requirement is that $U_0(t)$ should be differentiable), including oscillating motion of finite magnitude provided that there is no flow reversal in the field [4]. With this solution it is possible to calculate the flow field in an unsteady laminar boundary layer which can also be measured under experimental conditions, thus providing a direct check of the method presented in the present paper.

6. Discussion and summary of results. The flow in the laminar boundary layer over a semi-infinite flat plate moving in air at rest with a velocity $U_0 = At^n$ has been analyzed in considerable detail using two different methods of approach. The first method is based upon the idea of perturbing the Rayleigh type solution which can be visualized either as a solution valid at a very large distance downstream of the leading edge at any instant t or as a solution valid for all downstream positions during the initial period $t \gtrsim 0$. This method is shown in Sec. 3 to fail completely in producing any perturbations which are not identically zero under the only restriction that the velocity component u parallel to the plate surface must be continuous while the velocity gradient $\partial u/\partial x$ along the surface is permitted to become infinite. Alternatively, if there is a region in which the flow field varies along the surface, the series solution obtained from the boundary layer equation in this region could not be expected to converge to the Rayleigh type solution for any finite value of X. This fact certainly raises serious questions as to the validity of the process of perturbing the Rayleigh type solutions for non-flat plate problems. (This does not necessarily imply that the Rayleigh type solution falls to be a valid approximation of the flow field at any instant t at a very large distance downstream of the leading edge.)

The second method is based upon the idea that within the framework of the boundary layer approximation, the solution of the u velocity component from the boundary layer equations immediately downstream of the leading edge must pass into the uniform free stream velocity U_0 as $x \to 0$. Since the boundary layer equation is improper for describing the flow field near the leading edge $x = 0$, the derivatives of u with respect to x are permitted to be discontinuous so as to obtain the best approximation to the actual flow field throughout the downstream range of x in the rectangular cartesian coordinates adopted in the present analysis. (The use of rectangular cartesian coordinates is not necessary.) The solution near the leading edge as determined from this condition is in fact the so-called "quasi-steady state solution" i.e., the steady state solution corresponding to the instantaneous free stream velocity. It is shown in Secs. 4 and 5 that the quasi-steady state solutions can be perturbed to produce a consistent solution at any station downstream of the leading edge including the region where $\xi > 1$. Without the support of experimental data, it is difficult to conclude that the solution obtained by this second method is the proper description of the actual flow field in the region sufficiently far downstream of the leading edge. From another point of view, the success of the second method in producing a self-consistent solution and the complete failure of the first method to produce any solution at all demonstrated an interesting behavior of the boundary layer equation which is parabolic either in y and t or in y and x. It admits perturbation solutions in the positive direction of X axis (the plate) but not in the opposite direction starting from a given initial station. This behavior is closely analogous to that of the classical diffusion equation.

Despite considerable recent developments in the mathematical theory of diffusion, the question of the pertinent boundary conditions applicable to a linear partial differential equation of parabolic type is still open [5]. Therefore, in the case of the quasi-linear, parabolic boundary layer equations, the chance of resolving mathematically whether it is permissible to enforce the boundary conditions at both ends of a finite or semi-infinite interval in x-axis, $x = 0$ to $x = \infty$, is rather remote. It seems, however, that this peculiar behavior of the present boundary value problem might be different from that of the classical diffusion equation.

If it were possible to obtain a complete solution of the Navier-Stokes equation for the flow of a viscous fluid over a semi-infinite plate moving in its own plane, the flow velocity at a given instant t along a given value of ζ as observed from the moving coordinates, defined previously, would look qualitatively like the one shown in Fig. 3.

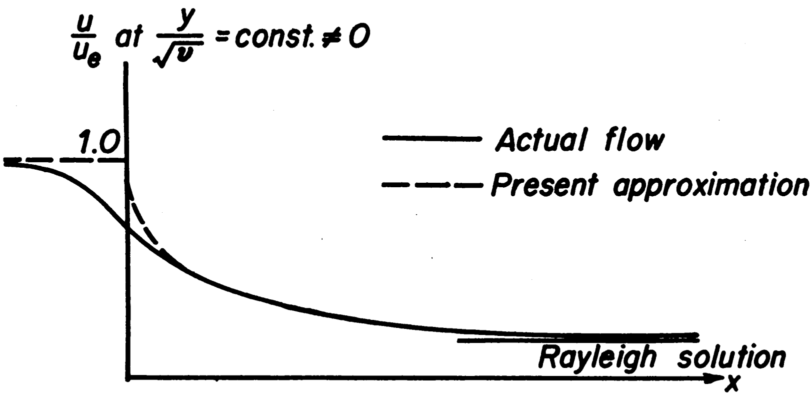

FIG. 3.

Because of the small viscosity of the fluid, the disturbance over the uniform flow field ahead of the leading edge, $x = 0$, does not extend very far to any appreciable magnitude. Physically, it can be conceived that the extent of the disturbed region is largest in the plane of the plate where the flow is completely stopped, and decreases in planes further away from the plate. The velocity in each plane (corresponding to a given value of ζ_0 inside the viscous layer) will decrease from the free stream value with increasing x and approach some asymptotic value which will very likely be determined by the Rayleigh type solution. The curve is therefore expected to be concave upwards and the approach to be gradual. Let us consider the Prandtl boundary layer approximation as a scheme for obtaining approximate solutions of the first order quantity (the u component of velocity in the rectangular Cartesian coordinate) over as large a region as practical by introducing the proper type of singularity at the proper location. The air ahead of the leading edge of the plate $x = 0$ may be assumed to be undisturbed so that u is overestimated significantly in the immediate vicinity of the leading edge $x = 0$. In order to approximate the downstream solution, some singularity at $x = 0$ must be admitted so as to restrict the error in the immediate neighborhood of the singularity. The concave shape of u as a function of x permits an approximation shown by the dotted curve in Fig. 3. The form of the boundary layer equation determines (Sec. 4) the singularity as of the type $x^{-\frac{1}{2}}$ in order to obtain proper downstream approximation. Now consider the approach from downstream by assuming that the x-indepdendent Rayleigh type solution is valid up to $x > 1$, (or any other station). In this case u is underestimated in the vicinity of $x \gtrsim 1$ by the Rayleigh type solution and the concave shape of u as a function of x is not compatible with any approximate curve, with singular behavior of the x-derivatives, that would result in significant errors only in the neighborhood of the singularity of $x = 1$. Any such approximate curve would diverge eventually. In other words the error of the underestimation of the initial velocity at any point by the Rayleigh type solution can not be made up properly by introducing some singular behavior. The best overall approximation would be to tolerate the error, indicated by the discontinuity between the actual velocity u and the velocity given by the Rayleigh type solution, and let it persist all the way upstream as if there were no leading edge. In such a case, the perturbation over the Rayleigh type solution becomes identically zero. Trying to obtain better approximation upstream by perturbing the Rayleigh type solution is therefore fruitless. The peculiar situation in solving the boundary layer equation as demonstrated in Sec. 3 can therefore be expected with whatever type of singular behvaior that one may introduce in perturbing the Rayleigh type solution. Physically, if one modifies the geometry of the plate near the leading edge, for example, if a certain finite length of the plate from the leading edge is bent, the flow is disturbed to a great extent locally. But at very large distance downstream, the deviation of the flow from the Rayleigh type solution is expected to be very small. However, it is this very small difference which preserves the characteristics of the particular flow. This physical situation in the scheme of perturbing the Rayleigh type solution indicates that the perturbation must diverge in the upstream direction. Consequently, a slight error in determining the first perturbation results in a totally different physical flow field in the upstream region. Therefore such perturbations cannot be carried out under any approximate formulation such as the Prandtl boundary layer approximation. On the other hand, one can tolerate a significant error at a certain upstream station in determining the flow field sufficiently far downstream of that station because the error initially introduced must eventually converge to a negligible magnitude. A valid downstream approximation can then be obtained. This is precisely what the Prandtl boundary layer approximation is supposed to accomplish. Though the mathematical behavior of the boundary layer equation is considerably more complicated than the classical diffusion equation, the physical situations of the boundary layer flow and the classical thermal diffusion problem are analogous in this respect. It is probably pertinent at this point to remark that, according to the above graphical and physical argument, the method of perturbing some approximate downstream solution of the boundary layer equation to obtain upstream solution may be equally invalid in steady state problems and in unsteady state problems.

The series solution of u component velocity as given in Sec. 4 reduces to the quasi-steady solution when $\xi = x/\int_0^t U_0(t)dt$ is very small. This does not necessarily mean that if ξ is small, so that the higher order terms in ξ are negligible, the actual flow field is really quasi-steady. The two limiting processes (i) $x \to 0$ with t constant and (ii) $t \to \infty$ with x constant both leading to $\xi \to 0$ and to the formal solution of quasi-steady flow should be carefully distinguished. In the case of the first limit, $x \to 0$, the quasi-steady state solution as a limiting form of the result in Sec. 4 does not represent quantitatively the actual flow field. Under the present interpretation of the boundary layer approximation, the quasi-steady state solution at the leading edge is simply an artificial consequence of the approximate initial condition $U(0,Y,t) = U_0(t)$ in obtaining a valid approximation sufficiently far downstream of the leading edge. The error committed by the quasi-steady state solution U/U_0 at the leading edge station is very likely of the order of unity. It is not known to what downstream range, this error will remain significant but it is believed that this spatial range in which the present series solution is invalid is more or less related to the criterion, $R_{ex} = U_0x/\nu \gg 1$. For the downstream region with $R_{ex} \gg 1$, but $\xi < 1$ the series solution obtained in Sec. 4 may be expected to serve as a good approximation in so far as the first order quantity U/U_0 is concerned. It is rather difficult to visualize the physical argument that in this region, (not the leading edge) that the convection effect must predominate and the quasi-steady state solution is a valid approximation.

Let us consider the second case of the limit with $x =$ constant but $t \to \infty$. The formal limit of quasi-steady state solution is a valid approximation for $R_{ex} \gg 1$ and the larger t is, the better the quasi-steady state approximation as $| \alpha_0\xi_0 |^2 = xt^{-n-1}$ becomes smaller for larger t. It should be noticed, however that the fact that quasi-steady state solution becomes a better approximation at larger t is primarily due to the particular behavior of the assumed motion of the plate in that the acceleration of the plate becomes less significant as compared with the instantaneous velocity, $U_0'/U_0 \sim t^{-1}$. The motion of the plate is approaching the steady state condition itself as t becomes large. If the plate is moving with a velocity different from the ones considered in the present paper, for example a uniform flow of air over an oscillating plate without having any flow reversal in the flow field, the motion of the air will not, in any sense, approach the steady state condition no matter how large t may be. In fact the analysis of such general motion of the plate [4] shows that while $\xi = x/\int_0^t U_0(t)dt$ does approach zero as t becomes large, the formal solution of the velocity field does not converge to the quasi-steady solution but to some other distinctly different velocity field depending upon the nature of the unsteady motion of the plate. The fundamental difference between the two limiting processes $x \to 0$ with t constant and $t \to \infty$ with x constant, both leading to $\xi \to 0$ should therefore be distinguished carefully.

Now considering the limiting form of the series solution in which t becomes very small and in the limit $t \to 0$. In this case the series solution as an expansion in terms of $\xi = x/\int_0^t U_0(t)dt$ could be valid only when $x \to 0$ with ξ remaining small, i.e., very close to the leading edge well within the limit of $R_{ex} \sim 0(1)$ where the series solution cannot represent the actual flow field. In the limit of $t = 0$, $u(0) = 0$, the so-called "quasi-steady state solution" simply leads to $u = 0$ everywhere throughout the flow field, which is, after all, the pertinent initial condition. Formally, for small t, the method as described in Sec. 5 will still permit the determination of the flow field away from the leading edge. Only that, in this case, the value of $\int_0^t U_0(t)dt$ is extremely small so that the region where $R_{ex} \gg 1$ corresponds to $\xi \gg 1$. According to the results in Secs. 4 and 5 the present solution is essentially the same as the Rayleigh type solution for the flow region downstream of $\xi \gtrsim 1$ up to infinity which, during the initial interval of motion, covers the entire length of the plate except, of course, the "leading edge $R_{ex} = 0(1)$".

To summarize the results and discussions:

1. It is found (Sec. 3) that by permitting the x-derivatives to become locally infinite according to some fractional power or logarithmic law, the x-independent Rayleigh type solution cannot be joined continuously with the x-dependent solution in the

upstream region at any station downstream of the leading edge. Despite the fact that the Rayleigh type solution is an exact solution of both the Navier-Stokes equation and the boundary layer equation, the method of obtaining solutions valid in the upstream region at any instant t by perturbing the Rayleigh type solution with or without conventional types of singularity is invalid. While the mathematical evidences are not conclusive, it can be conceived physically that the method of perturbing the Rayleigh type solution is, in general, not a valid process for determining the upstream solution based upon the boundary layer equation.

2. By assuming a uniform stream just upstream of the leading edge and imposing the condition that the u-component velocity in the downstream region must pass into the uniform value at the leading edge, a self-consistent approximate solution downstream of the leading edge can be obtained based on the boundary layer equation (Sec. 4). This solution can be continued as far downstream as one wishes (Sec. 5). The formal difficulty of the appearance of an "essential singularity" at $\xi = 1$ and $\eta = \infty$, encountered in the method of perturbing the Rayleigh type solution, disappears. It reflects itself in the present solution as a logarithmic singularity at any intermediate station ξ_i where the F-series is no longer valid and where the local velocity is equal to ξ_i. Its presence has little effect on the solution near the plate.

3. Besides satisfying the usual boundary conditions at the plate surface and at large distances away from the plate, the solution determined in Secs. 4 and 5 approaches the Rayleigh type solution within any degree of engineering accuracy at sufficiently far distance downstream of the leading edge at any instant. Therefore, this solution possesses the proper downstream behavior, as appeared physically necessary. This solution satisfies also the timewise initial condition that at $t = 0$, there is no flow; and at very small time during the initial period, the unsteady flow field over the entire surface of the plate (except for the vicinity of the leading edge) is approximately the same as that of the Rayleigh type solution. Thus the present solution possesses the proper initial behavior timewise. The initial behavior of the present solution spacewise at very small x or $x \simeq 0$, is not proper in describing the actual flow field (i.e., in the immediate vicinity of the leading edge). This is tolerated as self-consistent within the framework of the boundary layer approximation.

4. For a uniformly accelerated plate, the skin friction on the surface is practically the value given by the Rayleigh type solution from the point $\xi = \frac{1}{2}$ to downstream infinity with 2% error. The station $\xi = \frac{1}{2}$ is midway between the leading edge at any instant t and the leading edge position at $t = 0$. Similar results are obtained for other accelerated motions (Fig. 1).

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