to prove uniqueness of the coupled problem. The reader is therefore referred to other sources, e.g., [5, 6], for more detailed treatments of the uncoupled problems.

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THE MOTION OF A THERMOELASTIC SOLID*

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Introduction. It is the purpose of this paper to set down formally the relevant equations of thermo-elasticity within the approximation of the theory of elasticity of infinitesimal displacements and displacement derivatives. It is not claimed that the following equations of thermo-elasticity are original; Duhamel [1] and Neumann [2] derived similar equations many years ago, but due to the fact that they did not derive their equations from thermodynamic considerations and also that elasticians generally are not aware of the role of thermodynamics in their field, it was felt that a derivation of the general equations and application to a particular problem were in order. The present work is a refinement of Refs. [3] and [4] and the application was inspired by the work of Synge [5] in connection with the motion of a viscous, heat conducting fluid. The author is indebted to the reviewer for his extensive commentary which assisted materially in the revised version of this paper.

Analysis. For the case of small displacements and displacement derivatives, the momentum and energy (First Law) equations for a continuous, homogeneous medium may be written as

$$\text{Momentum} \quad \rho \frac{\partial^2 u_i}{\partial t^2} = \tau_{ki,k}, \quad (1)$$

$$\text{Energy} \quad \rho \frac{\partial U}{\partial t} = K_{mn} T_{,mn} + \tau_{mn} \frac{\partial u_{n,m}}{\partial t}, \quad (2)$$

where ρ is density, u_i is displacement vector, t is time coordinate, τ_{ki} is stress tensor, U is specific internal energy, K_{mn} is thermal conductivity tensor, and T is the temperature. The subscript notation is that of cartesian tensorial form and subscripts following a

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comma indicate partial differentiation with respect to the appropriate independent spatial variables.

The momentum and energy equations constitute four equations connecting the 11 dependent variables:

$$u_i; \tau_{ij}; U; T$$

therefore, seven additional relations must be obtained, in order to reduce the number of dependent variables to four. This reduction can be accomplished thermodynamically.

For an elastic solid, where τ_{ij} ; ϵ_{ij} ; U ; and T are thermodynamic properties of a state, the Gibbs total differential equation may be written as

$$T dS = dU - \frac{1}{\rho} \tau_{ij} d\epsilon_{ij}, \quad (3)$$

where S is the specific entropy, and $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is equal to the "pure" strain. Therefore, from an assumed equation of state of the (normal) form

$$U = U(S, \epsilon_{ij}), \quad (4)$$

T and τ_{ij} can be obtained at once by

$$T = \frac{\partial U}{\partial S}; \quad \tau_{ij} = \rho \frac{\partial U}{\partial \epsilon_{ij}}. \quad (5)$$

Expanding (4) about the reference state $S = S^{(0)}$; $\epsilon_{ij} = 0$, including second order terms, we obtain

$$U = U^{(0)} + sU^{(1)} + A_{ij}\epsilon_{ij} + \frac{1}{2}s^2U^{(2)} + sB_{ij}\epsilon_{ij} + \frac{1}{2\rho}C_{ijmn}\epsilon_{ij}\epsilon_{mn}, \quad (6)$$

$$A_{ij} = \rho \frac{\partial^2 U}{\partial S^2},$$

$$\tau_{ij} = \rho A_{ij} + \rho s B_{ij} + C_{ijmn}\epsilon_{mn}. \quad (8)$$

$U^{(1)}$ is seen in (7) to be the reference temperature $T^{(0)}$.

Substituting (7) and (8) in (1) and (2) yields (to the first order)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \rho B_{ij}s_{,j} + C_{ijmn}u_{m,nj} \quad (9)$$

and

$$\rho T^{(0)} \frac{\partial s}{\partial t} = K_{mn}(sU^{(2)} + B_{ij}u_{i,j})_{,mn}. \quad (10)$$

Thus, the reduction to four variables, s and u_i , has been accomplished.

The reduction to the variables T and u_i may be accomplished by solving for s in (7),

$$s = \frac{1}{U^{(2)}} (T - T^{(0)} - B_{mn}u_{m,n}) \quad (11)$$

and substitution in (9) and (10)

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{U^{(2)}} B_{ij} T_{,i} + \left(\frac{1}{\rho} C_{ijmn} - \frac{1}{U^{(2)}} B_{ij} B_{mn} \right) u_{m,nj}, \quad (12)$$

$$\frac{\partial T}{\partial t} - B_{ij} \frac{\partial u_{i,j}}{\partial t} = D_{mn} T_{,mn}, \quad (13)$$

where

$$D_{mn} = \frac{U^{(2)}}{\rho T^{(0)}} K_{mn}. \quad (14)$$

For the case of isotropy,

$$B_{ij} = B \delta_{ij}; \quad D_{mn} = D \delta_{mn}, \\ C_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm})$$

and (12) and (13) become

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{B}{U^{(2)}} T_{,i} + \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}} \right) u_{m,mi} + \frac{\mu}{\rho} u_{i,mm}, \quad (15)$$

$$\frac{\partial T}{\partial t} - B \frac{\partial u_{m,m}}{\partial t} = D T_{,mm}. \quad (16)$$

It should be noted that Eq. (16) resembles the customary heat conduction equation except for the term $B \partial u_{m,m} / \partial t$. Duhamel included a similar term in his considerations, but did so because he reasoned that the rate of dilatation would have a linear effect on the rate of temperature change. While it is true that the effect of B is small for many elasticity problems, many elasticians are unaware of the nature of the approximation they make when they use the Fourier equation to obtain a space-time temperature distribution.

Propagation of waves. In the manner of Synge, let us now study solutions of Eqs. (15) and (16) of the form

$$u_i = u_i^* \exp(a_k x_k + b t), \\ T - T^{(0)} = T^* \exp(a_k x_k + b t),$$

where a_i, b are complex.

Substituting in Eqs. (15) and (16) and letting $a_k a_k = A^2$; one obtains

$$\left(b^2 - \frac{\mu}{\rho} A^2 - \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}} \right) a_m a_m \right) a_m u_m^* - \frac{B}{U^{(2)}} a_i T^* = 0, \quad (17)$$

$$-B b a_m u_m^* + (b - D A^2) T^* = 0. \quad (18)$$

If $a_i = a'_i + i a''_i$ where a'_i, a''_i are real, then if we choose the x_3 axis perpendicular to the vectors a'_i and a''_i , $a_3 = 0$. The determinantal equation for the system then is

$$\begin{vmatrix} \left[b^2 - \frac{\mu}{\rho} A^2 - \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}} \right) a_1^2 \right] & -\left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}} \right) a_1 a_2 & -\frac{B}{U^{(2)}} a_1 \\ -\left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}} \right) a_1 a_2 & \left[b^2 - \frac{\mu}{\rho} A^2 - \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}} \right) a_2^2 \right] & -\frac{B}{U^{(2)}} a_2 \\ -B b a_1 & -B b a_2 & (b - D A^2) \end{vmatrix} = 0 \quad (19)$$

or, letting

$$\frac{\mu}{\rho} = V^2; \quad \frac{\lambda + 2\mu}{\rho} - \frac{B^2}{U^{(2)}} = V_r^2$$

and

$$\frac{\lambda + 2\mu}{\rho} = V_s^2,$$

the determinantal equation may be written

$$(b^2 - V^2 A^2)[(b^2 - V_s^2 A^2)b - (b^2 - V_r^2 A^2)DA^2] = 0. \quad (20)$$

V , V_s and V_r are clearly velocities of propagation of transverse, isentropic-longitudinal, and isothermal-longitudinal disturbances, respectively. The transverse mode is uncoupled from the longitudinal mode of motion.

A few limiting cases are of interest. For $B \rightarrow 0$; $V_s \rightarrow V_r$ and the mechanical mode uncouples from the temperature mode. The propagation of a temperature disturbance is then given by

$$b - DA^2 = 0. \quad (21)$$

For $D \rightarrow 0$; a longitudinal disturbance is given by

$$b^2 - V_s^2 A^2 = 0. \quad (22)$$

For $D \rightarrow \infty$; a longitudinal disturbance is given by

$$b^2 - V_r^2 A^2 = 0. \quad (23)$$

The general equation for a longitudinal disturbance is seen, from Eq. (20), to be biquadratic in A . The two pairs of roots, therefore, represent advancing and receding waves of two families, each family having a different velocity of propagation. It is, therefore, observed that the herein studied continuum model displays a "second sound" phenomenon and that disturbances of both families exhibit frequency dependent damping and velocities of propagation.

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