For the stationary case (25) is identical, except for a change in notation, with Eq. (79) of Ref. [1]. In terms of the matrix $g(\tau, \mu; \eta)$ the characteristic function can be written

$$F(\eta) = \exp i \int_0^\eta d\xi \int_0^t Tr[g(\sigma, \sigma; \xi)h(\sigma)]\, d\sigma, \tag{26}$$

where $Tr[\ ]$ is the trace of the matrix.

References

1. A. J. F. Siegert, *A systematic approach to a class of problems in the theory of noise and other random phenomena*—Part III, IRE Transactions on Information Theory **IT-4**, 4–14 (March 1958)
2. The symmetry of (1) allows $h^{(m)}(\tau)$ to be symmetric without loss of generality
3. Whittaker and Watson, *Modern analysis*, Cambridge Univ. Press, 1927, Sec. 11.61
4. K. Karhunen, *Über lineare Methoden in der Wahrscheinlichkeitsrechnung*, Ann. Acad. Sci. Fennicae *Ser. A I.* Math. Phys., 37 (1947)
5. M. Loeve, *Probability theory*, D. Van Nostrand Co., Inc., 1955
6. E. Wong, *Vector stochastic processes in problems of communication theory*, Ph.D. dissertation, Princeton University, May 1959
7. J. K. Wolf, *On the detection and estimation problem for multiple nonstationary random processes*, Ph.D. dissertation, Princeton University, October 1959

FURTHER EXTENSIONS OF SCHUSTER'S INTEGRAL*

By E. T. KORNHAUSER** (*H. H. Wills Physics Laboratory, University of Bristol*)

The integral,

$$I = \int_0^\infty [C^2(x) + S^2(x)]\, dx,$$

where $C(x)$ and $S(x)$ are Fresnel integrals defined by

$$C(x) = \int_x^\infty \cos t^2\, dt,$$

$$S(x) = \int_x^\infty \sin t^2\, dt,$$

was conjectured by Schuster[1] to have the value $(\pi/8)^{\frac{1}{2}}$. Proof that I does in fact have this value was given by Hardy[2] and more elegantly by Ingham[3]. More recently Bateman[4] has extended Ingham's treatment to evaluate integrals of the form

$$\int_0^\infty C(x)C(ax)\, dx, \qquad \int_0^\infty C(x)S(ax)\, dx,$$

*Received September 1, 1960.
**On leave from Brown University.
[1]Sir A. Schuster, Proc. Roy. Soc. **107**, 15 (1925).
[2]G. H. Hardy, Proc. London Math. Soc. **24**, xxx (1926)
[3]A. E. Ingham, J. London Math. Soc. **1**, 34 (1926).
[4]H. Bateman, Proc. Natl. Acad. Sci. **32**, 70 (1946).
[5]C. M. Sparrow, Astrophys. J. **49**, 65 (1919).

etc. An additional generalization, which is of interest in the theory of imperfect diffraction gratings[5], involves the inclusion of additive rather than multiplicative constants in the argument of the Fresnel functions.

Consider the integral

$$I_1 = \int_0^\infty C^2(x + a)\, dx = \int_a^\infty C^2(x)\, dx = [xC^2(x) |_a^\infty + 2 \int_a^\infty x \cos x^2 C(x)\, dx$$

$$= -aC^2(a) + [\sin x^2 C(x) |_a^\infty + \int_a^\infty \sin x^2 \cos x^2\, dx$$

$$= -aC^2(a) - \sin a^2 C(a) + \tfrac{1}{2}\, 2^{-1/2} S(2^{1/2} a).$$

In like manner

$$I_2 = \int_0^\infty S^2(x + a)\, dx = -aS^2(a) + \cos a^2 S(a) - \tfrac{1}{2}\, 2^{-1/2} S(2^{1/2} a).$$

Furthermore,

$$I_3 = \int_0^\infty [C^2(x + a) + S^2(x + a)]\, dx = I_1 + I_2$$

$$= -a[C^2(a) + S^2(a)] + \cos a^2 S(a) - \sin a^2 C(a).$$

Changing the sign of a yields

$$I_4 = \int_0^\infty [C^2(x - a) + S^2(x - a)]\, dx$$

$$= a[C^2(-a) + S^2(-a)] + \cos a^2 S(-a) - \sin a^2 C(-a),$$

and since $C(-a) = (\pi/2)^{\frac{1}{2}} - C(a)$ and similarly for $S(-a)$,

$$I_3 + I_4 = (\pi/2)^{1/2}(\cos a^2 - \sin a^2) + 2a(\pi/2)^{1/2}[(\pi/2)^{1/2} - C(a) - S(a)].$$

Now consider the integral,

$$I_5 = \int_0^\infty [C(x + a)C(x - a) + S(x + a)S(x - a)]\, dx.$$

Successive integrations by parts yield

$$I_5 = \tfrac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) + a \int_0^\infty [C(x + a) \cos (x - a)^2$$

$$- C(x - a) \cos (x + a)^2 + S(x + a) \sin (x - a)^2 - S(x - a) \sin (x + a)^2]\, dx,$$

which may be written

$$I_5 = \tfrac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) + a \int_0^\infty \frac{\partial}{\partial a} [C(x + a)C(x - a) + S(x + a)S(x - a)]\, dx$$

Thus it is clear that I_5 obeys the differential equation,

$$I_5 - a \frac{dI_5}{da} = \tfrac{1}{2}(\pi/2)^{1/2}(\cos a^2 - \sin a^2) \equiv F(a)$$

which may be solved by quadrature,

$$I_5 = a \int_a^\infty u^{-2} F(u)\, du = \tfrac{1}{2}(\pi/2)^{1/2}\{\cos a^2 - \sin a^2 - 2a[C(a) + S(a)]\}.$$

Finally, the most significant integral from the point of view of the physical problem

$$I_6 = \int_0^\infty \{[C(x+a) - C(x-a)]^2 + [S(x+a) - S(x-a)]^2\}\, dx$$
$$= I_3 + I_4 - 2I_5 = \pi a,$$

a remarkably simple result.

NOTE ON THE POINCARÉ BOUNDARY-VALUE PROBLEM*

By E. E. JONES (*University of Nottingham, England*)

1. This note is concerned with the solution of a modified form of the Poincaré boundary-value problem [1]. It is required to solve the Poisson differential equation $\nabla^2 \phi_i = f(x, y)$ for $\phi_i(x, y)$ defined in S_i, the region enclosed by the circle C of equation $|z| = a$, $(z = x + iy)$, such that on C

$$k \frac{\partial \phi_i}{\partial n} + l \frac{\partial \phi_i}{\partial s} + m\phi_i = g(x, y), \tag{1}$$

where k, l, m are constants, $f(x, y)$ is prescribed in S_i, $g(x, y)$ is prescribed on C, and $\partial/\partial n$, $\partial/\partial s$ denote differentiations along the inward normal and positive tangential directions respectively to C.

It is assumed that $\phi_i = \phi_{i0} + \Phi_i$, where $\phi_{i0}(x, y)$ is a particular integral of the Poisson equation reflecting all the singularities of the complete solution ϕ_i, and $\Phi_i(x, y)$ is harmonic in S_i, being together with its first partial derivatives single-valued and continuous in S_i. It is thus possible to write $\Phi_i = re\, W_i(z)$, where $W_i(z)$ is a regular function of z in S_i. If $z = ae^{i\theta} = \zeta$, $\bar{z} = a^2/\zeta$ on C, and by definition

$$h(\zeta) = -g - k \frac{\partial \phi_{i0}}{\partial r} + \frac{l}{a} \frac{\partial \phi_{i0}}{\partial \theta} + m\phi_{i0}, \tag{2}$$

then

$$h_i(z) = \frac{1}{2\pi i} \int_C \frac{(\zeta + z) h(\zeta)}{\zeta - z}\, d\zeta, \tag{3}$$

is the Schwarz integral representation of a function $h_i(z)$ regular in S_i with a real part equal to $h(\zeta)$ on C—for this to be so it is necessary for $h(\zeta)$ to satisfy the Lipschitz condition on C [2]. The boundary condition (1) then takes the form

$$re \left\{ \frac{\zeta}{\alpha} \frac{dW_i(\zeta)}{d\zeta} - mW_i(\zeta) - h_i(\zeta) \right\} = 0, \tag{4}$$

*Received September 30, 1959.