MAXIMUM ERROR CAUSED BY USING COMPLETELY CLIPPED FUNCTIONS IN THE COMPUTATION OF FOURIER TRANSFORMS AND CORRELATION FUNCTIONS*

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1. Introduction. A function is said to be completely clipped if its value is replaced by $+1$ whenever it is positive and by -1 whenever it is negative. As realized by Faran and Hills [1] and by Riisnaes [2] the circuitry of electronic devices is often simplified if the inputs are completely clipped.

Suppose, for example, that an electronic device containing adding and multiplying units is used to compute a Fourier transform or correlation coefficient by means of an approximating sum of a series of products. If every factor of each product is completely clipped then multiplication is equivalent to a test for like polarity, and addition is equivalent to a count of unit inputs.

If digital equipment is used to add or multiply terms the number of significant figures retained in each term determines the time required for each operation and also the amount of storage required to hold the data. Complete clipping of all inputs leads to rapid computation and enables storage of a large number of different terms. Data may often be sampled and recorded more quickly if only the sign of each sample is recorded.

Since complete clipping does not alter the number of changes of sign of a function, it might be expected that complete clipping preserves to some extent the frequency distribution and correlation properties of functions. Some experience with digital equipment designed to operate upon clipped functions has led to the observation that for many functions of a fairly random amplitude variation complete clipping does not have much effect upon the form of their Fourier transforms or correlation functions. It is of interest to know the effect of complete clipping when the functions involved are of general form not necessarily of a random nature.

A simple illustration of the effect of complete clipping upon a Fourier transform results from consideration of the function

$$f(t) = \begin{cases} \cos t & \text{for } |t| < 10\pi \\ 0 & \text{for } |t| > 10\pi \end{cases} \quad (1)$$

whose Fourier transform $F(\omega)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt = \frac{2\omega}{\omega^2 - 1} \sin 10\pi\omega. \quad (2)$$

Let $f'(t)$ and $c'(\omega t)$ be the functions obtained by complete clipping of $f(t)$ and $\cos \omega t$. Then

$$f'(t) = \begin{cases} +1 & \text{if } f(t) > 0 \\ -1 & \text{if } f(t) < 0 \end{cases} \quad (3)$$

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with a similar definition for $c'(\omega t)$ in terms of $\cos \omega t$. If both functions in the integrand of $F(\omega)$ are replaced by their clipped values over the range $|t| < 10\pi$ then $F(\omega)$ is replaced by

$$F'(\omega) = \int_{-10\pi}^{10\pi} f'(t)c'(\omega t) dt \quad (4)$$

$$= 4 \sum_{r=1}^{n-1} (-1)^{r+1} z_r + 2(-1)^{n+1} z_n \quad (5)$$

where the z_r are the zeros of $f(t)$ and $\cos \omega t$ between 0 and 10 chosen so that

$$0 < z_1 \leq z_2 \leq z_3 \leq \dots \leq z_{n-1} < z_n = 10. \quad (6)$$

The functions $F(\omega)$ and $\frac{1}{2} F'(\omega)$ are drawn for comparison in Fig. 1. The normalising factor of $\frac{1}{2}$ is chosen in accordance with the procedure to be explained in Sec. 2. It is apparent that $\frac{1}{2} F'(\omega)$ forms a good approximation to $F(\omega)$ except for values of ω close to $\frac{1}{3}$.

FIG. 1. Comparison of $F(\omega)$ and $F'(\omega)$ when $f(t)$ is according to Eq. (1). The curve is through values of $F(\omega)$. The crosses are values of $F'(\omega)$.

Suppose that the integral (2) were computed numerically by means of summation of 120 values of the integrand chosen at intervals $\pi/6$ apart. Suppose also that values of $f(t)$ and $\cos \omega t$ were given to three significant decimal figures, and hence to ten significant binary figures. Operations with the values of $f(t)$ and $\cos \omega t$ required for a single integration would then involve 2,400 binary bits of data. Numerical computation of the integral (4) with the same sampling interval would involve only 480 binary bits of data for the 120 values of $f'(t)$ and $c'(\omega t)$. Similarly, if the 2,400 binary bits of storage required for the 120 values of $f(t)$ and $\cos \omega t$ chosen at intervals of $\pi/6$ apart were used to store clipped values $f'(t)$ and $c'(\omega t)$ then 600 values could be chosen at intervals of $\pi/30$ apart.

Lawson and Uhlenbeck [3] have obtained a formula for the autocorrelation function of completely clipped Gaussian noise, while McFadden [4] has derived a formula

for the fourth product moment. The effect of complete clipping upon certain signal detection systems has been investigated by Riisnaes [2].

The subject of the present note is an investigation of the extent to which complete clipping of functions may change the values of their Fourier cosine transforms and correlation functions. In consideration of the Fourier transform it is supposed that the cosine term is also completely clipped.

The results describe the greatest possible changes that could be caused by complete clipping and are not dependent upon assumptions concerning the statistical behavior of the functions involved.

2. The Fourier transform. Let $F(\omega)$ denote the Fourier cosine transform of a function $f(t)$. Then

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t dt. \quad (7)$$

Suppose that $f(t)$ is real and that the integral defining $F(\omega)$ exists and may be approximated with arbitrary small error by a finite sum of the form

$$F(\omega) = \Delta \sum_n f_n c_n(\omega), \quad (8)$$

where $c_n(\omega) = \cos n\omega \Delta$ and $f_n = f(n\Delta)$. The summation in n is from n_1 to n_2 , where n_1 , n_2 , and Δ are chosen so that the terms in the summations are negligible except for $n_1 < n < n_2$. Denote $(n_2 - n_1)\Delta$ by T .

Let quantities f'_n be defined in terms of f_n by

$$f'_n = \begin{cases} +1 & \text{if } f_n > 0 \\ -1 & \text{if } f_n < 0 \\ \pm 1 & \text{if } f_n = 0 \end{cases} \quad (9)$$

with a similar definition of quantities $c'_n(\omega)$ in terms of $c_n(\omega)$. Put

$$F'(\omega) = \Delta \sum_n f'_n c'_n(\omega). \quad (10)$$

The functions $F(\omega)$ and $F'(\omega)$ of (8) and (10) have the period $2\pi/\Delta$. If $f(t)$ is multiplied by a constant factor then $F(\omega)$ is changed by the same factor whereas $F'(\omega)$ remains unchanged. Therefore in order to compare $F(\omega)$ with $F'(\omega)$ the latter should be normalised by multiplication by a factor proportional to the square root of

$$\int_0^{2\pi/\Delta} [F(\omega)]^2 d\omega = \Delta^2 \sum_n \sum_m f_n f_m \int_0^{2\pi/\Delta} c_n(\omega) c_m(\omega) d\omega \quad (11)$$

$$= \Delta^2 \sum_n f_n^2 \int_0^{2\pi/\Delta} [c_n(\omega)]^2 d\omega = \Delta\pi \sum_n f_n^2. \quad (12)$$

Throughout the present paper a normalisation constant k has been chosen so that

$$\Delta^2 \sum_n f_n^2 \int_0^{2\pi/\Delta} [c_n(\omega)]^2 d\omega = \Delta^2 \sum_n k^2 f_n^2 \int_0^{2\pi/\Delta} [c'_n(\omega)]^2 d\omega \quad (13)$$

$$= 2\Delta\pi k^2 \sum_n f_n'^2 = 2\Delta\pi k^2 N, \quad (14)$$

where N is the number of terms in the summation. Thus

$$k^2 = (1/2N) \sum_n f_n^2. \quad (15)$$

It may be noted that the functions $c'_n(\omega)$ are not orthogonal. Hence, although the left-hand side of (13) is equal to the area under the function $F(\omega)^2$, the right hand side of (13) is not equal to the area under $F'(\omega)^2$.

For a fixed ω , in order to find the function $f(t)$ for which the normalised function (10) provides the least accurate approximation to (8) let the f_n 's be chosen to maximize the quantity

$$\begin{aligned} Q^2(\omega) &= \frac{[F(\omega) - kF'(\omega)]^2}{\Delta \int_{-\infty}^{\infty} f(t)^2 dt} \\ &= \frac{[\sum_n f_n c_n - A(\sum_n f_n^2)^{1/2}]^2}{\sum_n f_n^2}, \end{aligned} \quad (16)$$

where

$$A = (1/2N)^{1/2} \sum_n f'_n c'_n. \quad (17)$$

It may be noted that an upper bound for $|Q(\omega)|$ results from the relation

$$\begin{aligned} |Q(\omega)| &= \left| \frac{\sum_n f_n c_n}{(\sum_n f_n^2)^{1/2}} - \frac{\sum_n f'_n c'_n}{(2N)^{1/2}} \right| \\ &\leq (\sum_n c_n^2)^{1/2} + (N/2)^{1/2} \end{aligned} \quad (18)$$

but such an upper bound is not necessarily the least possible.

The function $F(\omega)$ of (8) is a continuous function of each f_n . The function $F'(\omega)$ is a continuous function of f_n except for a discontinuity of magnitude $2 \Delta c'_n(\omega)$ at $f_n = 0$. For a given ω if the f_n are chosen to maximize $Q^2(\omega)$ then $Q^2(\omega)$ is either stationary with respect to $f_1, f_2, \dots, f_n$ or is discontinuous at the point $(f_1, f_2, \dots, f_n)$. Accordingly the stationary values and the discontinuities of $Q^2(\omega)$ will be examined.

The stationary values of $Q^2(\omega)$ result when for each n

$$0 = \frac{\partial Q^2}{\partial f_n} = \frac{(\sum f_n^2)2[c_n - A f_n (\sum f_n^2)^{-1/2}] - []^2 2f_n}{(\sum f_n^2)^2}, \quad (19)$$

the derivative of Q^2 existing except when $f_n = 0$. Equation (19) is satisfied when

$$\sum_n f_n c_n = A (\sum_n f_n^2)^{1/2} \quad (20)$$

which by (15) and (17) implies that

$$\sum_n f_n c_n = k \sum_n f'_n c'_n$$

and hence corresponds to the minimum value of $Q = 0$. Equation (19) is also satisfied when for each n

$$(\sum f_n^2)[c_n - Af_n(\sum f_n^2)^{-1/2}] - [\sum f_n c_n - A(\sum f_n^2)^{1/2}]f_n = 0$$

which implies that

$$\frac{f_n}{c_n} = \frac{\sum f_n^2}{\sum f_n c_n}$$

and hence

$$f_n = \lambda c_n, \quad (21)$$

where the constant λ is independent of n . Then $f'_n = \pm c'_n$ where the $+$ sign applies if λ is positive and the $-$ sign applies if λ negative.

When the f_n 's are chosen according to (21) then $A = \pm(N/2)^{1/2}$ and the resulting stationary value of $Q(\omega)$ is

$$\begin{aligned} Q_s(\omega) &= (\sum_n c_n^2)^{1/2} - (N/2)^{1/2} \\ &= (\tfrac{1}{2}N + \tfrac{1}{2}\sum_n \cos 2n\omega \Delta)^{1/2} - (N/2)^{1/2}. \end{aligned} \quad (22)$$

Let the origin of t be chosen so that only positive values of n occur in the summation (8). Then (see Ref. 5, p. 86) for large N

$$Q_s(\omega) = \left[\tfrac{1}{2}N + \frac{\sin 2\omega T}{4 \sin \omega \Delta} \right]^{1/2} - (N/2)^{1/2}.$$

If $Q_s(\omega)/N^{1/2}$ is plotted as a function of ωT its numerical value is seen to vary between $1 - (\tfrac{1}{2})^{1/2} = 0.293$, at $\omega T = 0$, and zero at $\omega T = \infty$. Thus

$$|Q_s(\omega)| \leq 0.293N^{1/2}. \quad (23)$$

As the f_n vary the greatest value of $Q(\omega)$ occurs either at the stationary value, at a point of discontinuity, or at infinity. However, since $Q(\omega)$ is independent of the scale of the f_n 's then putting some f_n 's equal to infinity is equivalent to putting other f_n 's equal to zero and hence leads to a discontinuity in $Q(\omega)$.

Consider a discontinuity at which

$$f_r = 0 \text{ (for } M \text{ values of } r) \quad (24)$$

but at which the remaining f_n 's, henceforth denoted by f_s , are not zero. Such a discontinuity will be referred to as being of order M . For the fixed zero values of f_r the stationary values of Q^2 result when (19) is satisfied for each $n = s$ and hence when $f_n = \lambda c_n$ for each $n = s$. The resulting stationary value of $Q(\omega)$ is

$$(\sum_s c_s^2)^{1/2} - \frac{(N-M)}{(2N)^{1/2}} - \frac{\sum_r (\pm 1)}{(2N)^{1/2}}. \quad (25)$$

Of the stationary values assumed by (25) the least value results when every ± 1 term is $+1$ and thus the least value is

$$(\sum_s c_s^2)^{1/2} - (N/2)^{1/2}. \quad (26)$$

Thus

$$Q'(\omega) \geq -(N/2)^{1/2} = -0.707N^{1/2}. \quad (27)$$

The greatest of the M stationary values (25) results when every ± 1 term is -1 and thus the greatest value is

$$Q = (\sum_i c_i^2)^{1/2} + \frac{2M - N}{(2N)^{1/2}}. \quad (28)$$

For a fixed M if (28) is evaluated at all discontinuities of order M its greatest value results when the M values of r appearing in (24) are chosen to include the $N - M$ greatest values of c_n^2 in (28). Let such a choice of the f_r be made for each value of M and let $(N - M)^{-1/2}(\sum_i c_i^2)^{1/2}$ be denoted by σ_{N-M} . Then σ_{N-M} is a non-decreasing function of M . Thus

$$Q = (N - M)^{1/2} \sigma_{N-M} + \frac{2M - N}{(2N)^{1/2}} \quad (29)$$

$$= \left(\frac{N}{2}\right)^{1/2} (1 + \sigma_{N-M}^2/4) - 2\left(\frac{N}{2}\right)^{1/2} \left[(1 - M/N)^{1/2} - \frac{\sigma_{N-M}}{2\sqrt{2}} \right]^2$$

$$\leq \left(\frac{N}{2}\right)^{1/2} (1 + \sigma_{N-M}^2/4) \quad (30)$$

$$\leq \frac{5}{4} \left(\frac{N}{2}\right)^{1/2} = 0.884N^{1/2}.$$

The Eqs. (23), (27), and (30) show that

$$|Q(\omega)| \leq \frac{5}{4} \left(\frac{N}{2}\right)^{1/2}$$

and hence that

$$|F(\omega) - kF'(\omega)| \leq \frac{5}{4} \left(\frac{N}{2}\right)^{1/2} \Delta (\sum_n f_n^2)^{1/2}. \quad (31)$$

It may be noted that if the interval T is a whole multiple of $2\pi/\omega$, and if Δ is allowed to tend to zero, then

$$\sigma_{N-M}^2 = \frac{2}{(1 - M/N)\pi} \int_0^{(1-M/N)\pi/2} \cos^2 t \, dt \quad (32)$$

$$= \frac{1}{2} \left[1 + \frac{\sin(1 - M/N)\pi}{(1 - M/N)\pi} \right].$$

Accordingly (29) may be written in the form

$$Q = \left(\frac{N}{2}\right)^{1/2} \left\{ (1 - M/N)^{1/2} \left[1 + \frac{\sin(1 - M/N)\pi}{(1 - M/N)\pi} \right]^{1/2} - 2(1 - M/N) + 1 \right\}$$

which has a maximum value of

$$1.2470(N/2)^{1/2} \quad (33)$$

when $M/N = 0.88$. The expression (33) is the least upper bound for $Q(\omega)$ since a corresponding function $f(t)$ may be constructed to satisfy (24) for $f_n = f_r$ and to satisfy (21) for $f_n = f_s$.

It is clear that if the above analysis is repeated for the Fourier sine transform the results (31) and (33) are again obtained.

By allowing Δ to tend to zero the results obtained in the present section may be expressed by Theorem 1 below.

Theorem 1. Suppose that $f(t)$ is zero except within a finite interval of length T . Denote by $F(\omega)$ and $F'(\omega)$ the integrals

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt,$$

$$F'(\omega) = \int_T f'(t) c'(\omega t) \, dt,$$

where

$$f'(t) = \begin{cases} 1 & \text{if } f(t) > 0, \\ -1 & \text{if } f(t) < 0 \\ \pm 1 & \text{if } f(t) = 0 \end{cases} \quad c'(\omega t) = \begin{cases} 1 & \text{if } \cos(\omega t) > 0 \\ -1 & \text{if } \cos(\omega t) < 0 \\ \pm 1 & \text{if } \cos(\omega t) = 0 \end{cases}$$

Let

$$k = (1/2T)^{1/2} \left[\int_{-\infty}^{\infty} f(t)^2 \, dt \right]^{1/2}, \quad (34)$$

then

$$|F(\omega) - kF'(\omega)| \leq \frac{5}{4} (T/2)^{1/2} \left[\int_{-\infty}^{\infty} f(t)^2 \, dt \right]^{1/2}. \quad (35)$$

Furthermore, if T is a whole multiple of $2\pi/\omega$ then

$$|F(\omega) - kF'(\omega)| \leq 1.2470(T/2)^{1/2} \left[\int_{-\infty}^{\infty} f(t)^2 \, dt \right]^{1/2} \quad (36)$$

and the right-hand side of (36) is the least upper bound for $|F(\omega) - kF'(\omega)|$.

An example of a function $f(t)$ for which $|F(\omega) - kF'(\omega)|$ is close to the upper bound (28) is provided by the function

$$f(t) = \begin{cases} \cos t & \text{for } |t - n\pi| < \pi/16 \text{ and } |t| < 4\pi \\ -\epsilon \cos t & \text{for } |t - n\pi| > \pi/16 \text{ and } |t| < 4\pi \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

where $\epsilon \ll 1$. Then $T = 8\pi$ and

$$\int_{-\infty}^{\infty} f(t)^2 \, dt = (\pi/2) + 4 \sin(\pi/8) = 3.1016.$$

The right-hand side of (36) takes the value of 7.785. Since $F(1) = 3.1016$, $k = 0.2484$, and $F'(1) = -6\pi$, then $F(1) - kF'(1) = 7.784$ which is very close to the upper bound of 7.785.

3. The correlation function. The correlation function of two functions $f(t)$ and $g(t)$ is defined to be

$$\int_{-\infty}^{\infty} f(t_1) g(t_1 + t) \, dt. \quad (38)$$

With arbitrary small error the integral (37) may be approximated by a sum of the form

$$\Delta \sum_n f_n g_{n+m}, \quad (39)$$

where $m \Delta = t$. A normalised correlation function ϕ_m may be derived from (39) by writing

$$\phi_m = \frac{\sum_n f_n g_{n+m}}{(\sum_n f_n^2)^{1/2} (\sum_n g_n^2)^{1/2}}. \quad (40)$$

The normalised correlation function of the completely clipped functions f'_n and g'_n is then

$$\phi'_m = \frac{\sum_n f'_n g'_{n+m}}{N} \quad (41)$$

The functions (40) and (41) assume their maximum values of unity if $f_n = g_{n+m}$ for all n . For fixed values of the g_n and of m the expression (41) provides the least accurate approximation to (40) when the f_n are chosen to maximize the quantity

$$P_m^2 = \left[\frac{\sum_n f_n g_{n+m}}{(\sum_n f_n^2)^{1/2} (\sum_n g_n^2)^{1/2}} - \frac{\sum_n f'_n g'_{n+m}}{N} \right]^2. \quad (42)$$

Stationary values of P_m^2 occur when

$$0 = \frac{\partial P_m^2}{\partial f_n} \quad (43)$$

which has the solution

$$f_n = \frac{(\sum_n f_n^2)}{\sum_n f_n g_{n+m}} g_{n+m} = \lambda_m g_{n+m} \quad (44)$$

and leads to the minimum value of $P_m = 0$. The maximum value of P_m^2 must thus occur at a point of discontinuity of the form (24).

At a discontinuity of order M a stationary value of P_m^2 results when $f_n = \lambda_m g_{n+m}$ for each $n = s$. The resulting value of P_m is

$$P_m = \frac{(\sum_s g_{s+m}^2)^{1/2}}{(\sum_n g_n^2)^{1/2}} - \frac{N - M + \sum_r (\pm 1)}{N}. \quad (45)$$

The least value of (45) is

$$\frac{(\sum_s g_{s+m}^2)^{1/2}}{(\sum_n g_n^2)^{1/2}} - 1 \quad (46)$$

which is equal to -1 if each $f_n = 0$ except when $g_{n+m} = 0$.

The greatest value of (45) is

$$P'_m = \frac{(\sum_s g_{s+m}^2)^{1/2}}{(\sum_n g_n^2)^{1/2}} + \frac{2M - N}{N}. \quad (47)$$

For each value of M the values of f_i may be chosen to include the $N - M$ greatest values of g_{i+m}^2 in (47). With such choice of the f_i let σ_{N-M} denote $(N - M)^{-1/2}(\sum g_{i+m}^2)^{1/2}$. Then σ_{N-M} is a non-decreasing function of M , and (47) may be written in the form

$$\begin{aligned} P'_m &= 1 + \sigma_{N-M}^2/8\sigma_N^2 - 2[(1 - M/N)^{1/2} - \sigma_{N-M}/4\sigma_N]^2 \\ &\leq 1 + \frac{1}{8} \frac{\sigma_{N-M}^2}{\sigma_N^2} \end{aligned} \quad (48)$$

$$\leq 1 + \frac{1}{8} \frac{g_{\max}^2}{\sigma_N^2}, \quad (49)$$

where $g_{\max}$ denotes the maximum value of $g(t)$ over the interval of length T .

The relation (49) provides an upper bound for P_m . A lower bound for the greatest value of P_m may be obtained as follows. Denote $g_{\max}^2/\sigma_N^2$ by k . Since $\sigma_{N-M}^2/16\sigma_N^2$ is a non-decreasing function of M and varies from $1/16$ at $M = 0$ to $k/16$ at $M = N - 1$ then the expression

$$(1 - M/N)^{1/2} - \sigma_{N-M}/4\sigma_N$$

becomes zero for a value of M/N between $1 - k/16$ and 1 . For such a value of M/N then

$$\begin{aligned} P'_m &= 1 + \sigma_{N-M}^2/8\sigma_N^2 \\ &\geq \begin{cases} 1 + \frac{1}{8} \frac{\sigma_{Nk/16}^2}{\sigma_N^2} & \text{if } k < 16 \\ 1 + \frac{1}{8} & \text{if } k \geq 16. \end{cases} \end{aligned} \quad (50)$$

By allowing Δ to tend to zero the results of the present section may be expressed in the form of Theorem 2 below.

Theorem 2. Suppose that $f(t)$ and $g(t)$ are zero except within a finite interval of length T . Let $\phi(t)$ denote the normalised correlation function of $f(t)$ and $g(t)$ over the interval T . Let $\phi'(t)$ denote the normalised correlation function of $f'(t)$ and $g'(t)$ where

$$f'(t) = \begin{cases} 1 & \text{if } f(t) > 0 \\ -1 & \text{if } f(t) < 0 \\ \pm 1 & \text{if } f(t) = 0 \end{cases}$$

with a similar definition for $g'(t)$.

Let $g_{\max}$ and σ denote respectively the maximum and the root mean square of $g(t)$ over the interval T . Then

$$|\phi(t) - \phi'(t)| \leq 1 + \frac{1}{8} \frac{g_{\max}^2}{\sigma^2}. \quad (51)$$

Also, if k denotes $g_{\max}^2/\sigma^2$ then there exists a function $f(t)$ such that

$$|\phi(t) - \phi'(t)| \geq \begin{cases} 1 + \frac{1}{8} \frac{\sigma_{(k/16)}^2}{\sigma^2}, \\ 9/8 & \text{if } k \geq 16 \end{cases}$$

where $\sigma_{(k/16)}$ is the root mean square of the largest $k/16$ values of $g(t)$.

In order to apply the relations (51) and (52) to a specific example suppose that the distribution of values of $g(t)$ is as for $\cos t$. It follows from (32) that $g_{\max}^2/\sigma^2 = 2(=k)$ and that $\sigma_{(k/16)}^2/\sigma^2 = 1.975$. Hence

$$|\phi(t) - \phi'(t)| \leq 1.250$$

but there exists a function $f(t)$ for which

$$|\phi(t) - \phi'(t)| \geq 1.247.$$

For a further example consider the instance in which the values of $g(t)$ are uniformly distributed between the extreme values. Then

$$\sigma_{N-M}^2/g_{\max}^2 = [1 + M/N + (M/N)^2]/3$$

so that $g_{\max}^2/\sigma^2 = 3(=k)$ and $\sigma_{(k/16)}^2/\sigma^2 = 2.473$. Hence

$$|\phi(t) - \phi'(t)| \leq 1.375$$

and there exists an $f(t)$ for which

$$|\phi(t) - \phi'(t)| \geq 1.309.$$

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