FOURIER SERIES AND EIGENFUNCTION EXPANSIONS ASSOCIATED WITH A NON-SELF-ADJOINT DIFFERENTIAL EQUATION*

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1. Introduction. In a paper by B. Friedman and L. Mishoe [1]** it is shown that if $u_1(x)$ and $u_2(x)$ are fundamental solutions of the non-self-adjoint equation

$$u'' + q(x)u + \lambda[p(x)u - u'] = 0 \quad (1)$$

and $v_1(x)$ and $v_2(x)$ corresponding solutions of the adjoint equation

$$v'' + q(x)v + \lambda[p(x)v + v'] = 0, \quad (2)$$

then for $f(x)$ of bounded variation on $(0, 1)$ and $0 < x < 1$ the series

$$\sum_{-\infty}^{\infty} u_n(x) \int_0^1 f(\xi) \frac{p(\xi)v_n(\xi) + v_n'(\xi)}{C'(\lambda_n)} d\xi \equiv \sum_{-\infty}^{\infty} a_n u_n(x) \quad (3)$$

converges to

$$\frac{1}{2}[f(x+) + f(x-)] - \frac{1}{2}[f(0+) + f(1-)] \exp[-P(0, 1)] \exp[P(0, x)],$$

where $u_n(x)$ and v_n are the eigenfunctions of (1) and (2) associated with the eigenvalues $\lambda = \lambda_n$. $C(\lambda) = w(0, \lambda)$ where $w(x, \lambda)$ is the Wronskian associated with the fundamental solutions u_1 and u_2 . The λ_n are the zeros of $C(\lambda)$ and

$$P(\xi, x) \equiv \int_{\xi}^x p(t) dt.$$

In the expansion (3) the function $f(x)$ was required to be of bounded variation on $(0, 1)$.

In this paper we generalize the result of [1] by allowing $f(x)$ to belong to the Class $L^2(0, 1)$ and then proving that $\sum_{-\infty}^{\infty} a_n u_n(x)$ is convergent or uniformly convergent with the Fourier series

$$\sum_{-\infty}^{\infty} \exp(2n\pi ix) \int_0^1 f(\xi) \exp(-2n\pi i\xi) d\xi \equiv \sum_{-\infty}^{\infty} b_n \exp(2n\pi ix);$$

thus connecting expansions associated with non-self-adjoint differential equations with the study of Fourier series.

We express our result more precisely in the following theorem.†

2. Theorem. If $u_n(x)$ are eigenfunctions associated with the system

$$u'' + qu + \lambda(pu - u') = 0, \quad u(0) = u(1) = 0, \quad (4)$$

and $p''(x)$ exists and $q(x)$ is continuous on $(0, 1)$ then the series $\sum_{-\infty}^{\infty} a_n u_n(x)$ is convergent

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**In [1] $H(x, \lambda)$ and $J(x, \lambda)$ are denoted as $\gamma(x, \lambda)$ and $\theta(x, \lambda)$ respectively.

†Similar equiconvergence theorems for Fourier series and self-adjoint eigenfunction expansions are well known e.g. see Ref. [2], Vol. 2, pp. 771-772 and Ref. [3], pp. 276-278.

or uniformly convergent with the Fourier series $\sum_{-\infty}^{\infty} b_n \exp (2n\pi i x)$ for $0 < x < 1$ and $f(x)$ of Class $L^2(0, 1)$. Furthermore if $\sum_{-\infty}^{\infty} b_n \exp (2n\pi i x) = t(x)$ then

$$\sum_{-\infty}^{\infty} a_n u_n(x) = t(x) - t_0 \exp [P(0, x)],$$

where

$$t_0 = \sum_{-\infty}^{\infty} \int_0^1 [f(\xi) \exp [-2xP(0, 1) + P(0, x)]] \exp (-2n\pi i \xi) d\xi \text{ and } P(\xi, x) \equiv \int_{\xi}^x p(t) dt.$$

3. Proof of the theorem. If we let

$$\psi_1 = pv_1 + v'_1, \quad \psi_2 = pv_2 + v'_2, \quad H(x, \lambda)^* = \frac{u_2(x)}{C(\lambda)} \int_0^x f(\xi) \psi_1(\xi) d\xi$$

and

$$J(x, \lambda) = \frac{u_1(x)}{C(\lambda)} \int_x^1 f(\xi) \psi_2(\xi) d\xi,$$

it is shown in [1] that

$$\begin{aligned} \sum_{-\infty}^{\infty} u_1(x, \lambda_n) \int_0^1 f(\xi) \left[\frac{p(\xi)v_2(\xi, \lambda_n) + v'_2(\xi, \lambda_n)}{C'(\lambda_n)} \right] d\xi \\ = \sum_{-\infty}^{\infty} u_2(x, \lambda_n) \int_0^1 f(\xi) \left[\frac{p(\xi)v_1(\xi, \lambda_n) + v'_1(\xi, \lambda_n)}{C'(\lambda_n)} \right] d\xi \\ = \frac{1}{2\pi i} \oint H(x, \lambda) d\lambda + \frac{1}{2\omega i} \oint J(x, \lambda) d\lambda = \sum_{-\infty}^{\infty} a_n u_n(x), \end{aligned} \quad (4a)$$

where the contour is a circle with radius tending to infinity [1].

Asymptotic evaluations for H and J have been determined in [1], where it is shown that

$$\begin{aligned} H(x, \lambda) = \left[\frac{\exp [\lambda(x-1) - P(1, x)] - \exp [P(1, x)] + O(1/\lambda)}{\exp [-P(0, 1)] - \exp [-\lambda + P(0, 1)] + O(1/\lambda)} \right] \\ \cdot \int_0^x f(\xi) \left[\exp [-\lambda\xi + P(0, \xi)] + O\left(\frac{\exp [-\lambda\xi]}{\lambda}\right) + O\left(\frac{1}{\lambda^2}\right) \right] d\xi \quad (5a) \\ \text{for } \operatorname{Re} \lambda > 0, \end{aligned}$$

$$\begin{aligned} H(x, \lambda) = \left[\frac{\exp [P(1, x)] - \exp [\lambda(1-x) + P(1, x)] + O(1/\lambda)}{\exp [\lambda - P(0, 1)] - \exp [P(0, 1)] + O(1/\lambda)} \right] \\ \cdot \int_0^x f(\xi) \left[\exp [\lambda(x-\xi) + P(0, \xi)] + O\left(\frac{\exp [\lambda(x-\xi)]}{\lambda}\right) \right] d\xi \quad (5b) \\ \text{for } \operatorname{Re} \lambda < 0; \end{aligned}$$

$$\begin{aligned} J(x, \lambda) = \left[\frac{\exp [-P(0, x)] - \exp [-\lambda x + P(0, x)] + O(1/\lambda)}{\exp [-P(0, 1)] - \exp [-\lambda + P(0, 1)] + O(1/\lambda)} \right] \\ \cdot \int_x^1 f(\xi) \left[\exp [\lambda(x-\xi) + P(1, \xi)] + O\left(\frac{\exp [\lambda(x-\xi)]}{\lambda}\right) \right] d\xi \quad (5c) \end{aligned}$$

$$J(x, \lambda) = \left[\frac{\exp [\lambda x - P(0, x)] - \exp [P(0, x)] + 0(1/\lambda)}{\exp [\lambda - P(0, 1)] - \exp [P(0, 1)] + 0(1/\lambda)} \right] \cdot \int_x^1 f(\xi) \left[\exp [\lambda(1 - \xi) + P(1, \xi)] + O\left(\frac{\exp [\lambda(1 - \xi)]}{\lambda}\right) \right] d\xi \quad (5d)$$

for $\operatorname{Re} \lambda > 0$,
for $\operatorname{Re} \lambda < 0$.

Since $f(x)$ is $L^2(0, 1)$ we may apply the Schwarz inequality and obtain

$$\left| \int_0^x f(\xi) O\left(\frac{1}{\lambda^2}\right) d\xi \right| \leq \left(\int_0^x f^2(\xi) d\xi \int_0^x \left[O\left(\frac{1}{\lambda^2}\right) \right]^2 d\xi \right)^{1/2} = O\left(\frac{1}{\lambda^2}\right).$$

Similarly

$$\int_0^x f(\xi) O\left(\frac{\exp - \lambda \xi}{\lambda}\right) d\xi = O\left(\frac{\exp - \lambda x}{\lambda}\right).$$

It is shown in [1] that if C_1 is the semicircle for which $\operatorname{Re} \lambda > 0$ and whose radius tends to infinity then

$$\int_{C_1} O\left(\frac{\exp - \lambda \xi}{\lambda}\right) d\lambda = 0 \quad \text{and} \quad \int_{C_1} O\left(\frac{1}{\lambda^2}\right) d\lambda = 0$$

Also

$$\lim_{|\lambda| \rightarrow \infty} \left[\frac{\exp [\lambda(x - 1) - P(1, x)] - \exp [P(1, x)] + 0(1/\lambda)}{\exp [-P(0, 1)] - \exp [-\lambda + P(0, 1)] + 0(1/\lambda)} \right] = -\exp [P(0, x)] \quad \text{for } \operatorname{Re} \lambda > 0$$

and we conclude that

$$\frac{1}{2\pi i} \int_{C_1} H(x, \lambda) d\lambda = -\frac{\exp [P(0, x)]}{2\pi i} \quad (6a)$$

$$\cdot \int_{C_1} \int_0^x f(\xi) \exp [-\lambda \xi + P(0, \xi)] d\xi d\lambda \quad \text{for } f(x) L^2(0, 1)$$

and similarly as in [1]

$$\frac{1}{2\pi i} \int_{C_2} H(x, \lambda) d\lambda = -\frac{\exp [P(0, x)]}{2\pi i} \int_{C_2} \int_0^x f(\xi) \exp [\lambda(x - \xi) + P(0, \xi)] d\xi d\lambda \quad (6b)$$

$$\frac{1}{2\pi i} \int_{C_1} J(x, \lambda) d\lambda = \frac{\exp [P(x, 1)]}{2\pi i} \int_{C_1} \int_x^1 f(\xi) \exp [\lambda(x - \xi) + P(1, \xi)] d\xi d\lambda \quad (6c)$$

$$\frac{1}{2\pi i} \int_{C_2} J(x, \lambda) d\lambda = \frac{\exp [-P(x, 1)]}{2\pi i} \int_{C_2} \int_x^1 f(\xi) \exp [\lambda(1 - \xi) + P(1, \xi)] d\xi d\lambda \quad (6d)$$

where C_2 is the semi-circle for which $\operatorname{Re} \lambda < 0$ and whose radius tends to infinity.

Consider now the new functions $\mu_1(x, \lambda)$, $\mu_2(x, \lambda)$ and $K(\lambda)$ derived from $u_1(x, \lambda)$, $u_2(x, \lambda)$ and $C(\lambda)$ by dropping all order terms [1]. We have

$$\begin{aligned} \mu_1 &= \frac{\exp [\lambda x - P(0, x)] - \exp [P(0, x)]}{\lambda} \\ \mu_2 &= \frac{\exp [\lambda(x - 1) - P(1, x)] - \exp [P(1, x)]}{\lambda}, \\ K(\lambda) &= \frac{\exp [-P(0, 1)] - \exp [-\lambda + P(0, 1)]}{\lambda}. \end{aligned}$$

It is easy to see that $K(\lambda)$ has zeros at $\lambda = \lambda_n = 2P(0, 1) + 2n\pi i$ and that $\mu_2(x, \lambda_n) = \mu_1(x, \lambda_n) \exp [-P(0, 1)]$ for $\lambda_n = 2P(0, 1) + 2n\pi i$.

Now define

$$\begin{aligned} \nu_1 &= \mu_1 \exp(-\lambda x), & \nu_2 &= \mu_2 \exp(-\lambda x), & \varphi_1 &= p\nu_1 + \nu'_1, & \varphi_2 &= p\nu_2 + \nu'_2, \\ h(x, \lambda) &= \frac{\mu_2(x)}{K(\lambda)} \int_0^x f(\xi) \varphi_1(\xi) d\xi, & j(x, \lambda) &= \frac{\mu_1(x)}{K(\lambda)} \int_x^1 f(\xi) \varphi_2(\xi) d\xi. \end{aligned}$$

From (4a) it is also clear that,

$$\begin{aligned} \frac{1}{2\pi i} \oint h(x, \lambda) d\lambda + \frac{1}{2\pi i} \int j(x, \lambda) d\lambda &= \sum_{-\infty}^{\infty} \mu_1(x, \lambda_n) \int_0^1 f(\xi) \left[\frac{p(\xi) \nu_2(\xi, \lambda_n) + \nu'_2(\xi, \lambda_n)}{K'(\lambda_n)} \right] d\xi \\ &= \sum_{-\infty}^{\infty} \mu_2(x, \lambda_n) \int_0^1 f(\xi) \left[\frac{p(\xi) \nu_1(\xi, \lambda_n) + \nu'_1(\xi, \lambda_n)}{K'(\lambda_n)} \right] d\xi, \end{aligned} \quad (7)$$

where now $\lambda_n = 2P(0, 1) + 2n\pi i$.

From the above definitions we obtain

$$\begin{aligned} h(x, \lambda) &= \frac{\exp[\lambda(x-1) - P(1, x)] - \exp[P(1, x)]}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1)]} \\ &\quad \cdot \int_0^x f(\xi) \left[\exp[-\lambda\xi + P(0, \xi)] - \frac{2p \exp[-\lambda\xi - P(0, \xi)]}{\lambda} \right] d\xi, \\ h(x, \lambda) &= A(x, \lambda) \int_0^x f(\xi) \left[\exp[-\lambda\xi + P(0, \xi)] + O\left(\frac{e^{-\lambda\xi}}{\lambda}\right) \right] d\xi, \quad \text{Re } \lambda > 0, \end{aligned} \quad (8)$$

where

$$A(x, \lambda) = \frac{\exp[\lambda(x-1) - P(1, x)] - \exp[P(1, x)]}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1)]}$$

Equation (8) may be written as

$$h(x, \lambda) = B(x, \lambda) \int_0^x f(\xi) \left[\exp[\lambda(x-\xi) + P(0, \xi)] - \frac{2p \exp[\lambda(x-\xi) + P(0, \xi)]}{\lambda} \right] d\xi, \quad (9)$$

where

$$B(x, \lambda) = \frac{\exp[-P(1, x)] - \exp[(1-x) + P(1, x)]}{\exp[\lambda - P(0, 1)] - \exp[P(0, 1)]},$$

and from (9) we have

$$h(x, \lambda) = B(x, \lambda) \int_0^x f(\xi) \left[\exp[\lambda(x-\xi) + P(0, \xi)] + O\left(\frac{\exp[\lambda(x-\xi)]}{\lambda}\right) \right] d\xi, \quad \text{Re } \lambda < 0$$

Similarly

$$\begin{aligned} j(x, \lambda) &= \frac{\exp[\lambda x - P(0, x)] - \exp[P(0, x)]}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1)]} \\ &\quad \cdot \int_x^1 f(\xi) \left[\exp[-\lambda\xi + P(1, \xi)] - \frac{2p \exp[-\lambda\xi + P(1, \xi)]}{\lambda} \right] d\xi \\ &= C(x, \lambda) \int_x^1 f(\xi) \left[\exp[\lambda(x-\xi) + P(1, \xi)] - \frac{2p \exp[\lambda(x-\xi) + P(1, \xi)]}{\lambda} \right] d\xi \end{aligned}$$

where

$$C(x, \lambda) = \frac{\exp [-P(0, x)] - \exp [-\lambda x + P(0, x)]}{\exp [-P(0, 1)] - \exp [-\lambda + P(0, 1)]}.$$

Consequently

$$j(x, \lambda) = C(x, \lambda) \int_x^1 f(\xi) \left[\exp [\lambda(x - \xi) + P(1, \xi)] + O\left(\frac{\exp [\lambda(x - \xi)]}{\lambda}\right) \right] d\xi, \operatorname{Re} \lambda > 0$$

We may write $j(x, \lambda)$ as

$$j(x, \lambda) = \frac{\exp [\lambda x - P(0, x)] - \exp [P(0, x)]}{\exp [\lambda - P(0, 1)] - \exp [P(0, 1)]} \cdot \int_x^1 f(\xi) \left[\exp [\lambda(1 - \xi) + P(1, \xi)] - \frac{2p \exp [\lambda(1 - \xi) + P(1, \xi)]}{\lambda} \right] d\xi$$

and obtain

$$j(x, \lambda) = D(x, \lambda) \int_x^1 f(\xi) \left[\exp [\lambda(1 - \xi) + P(1, \xi)] + O\left(\frac{e^{\lambda(1-\xi)}}{\lambda}\right) \right] d\xi, \operatorname{Re} \lambda < 0$$

where

$$D(x, \lambda) = \frac{\exp [\lambda x - P(0, x)] - \exp [P(0, x)]}{\exp [\lambda - P(0, 1)] - \exp [P(0, 1)]}.$$

As previously shown

$$\begin{aligned} \int_0^x f(\xi) O\left(\frac{e^{-\lambda\xi}}{\lambda}\right) d\xi &= O\left(\frac{e^{-\lambda\xi}}{\lambda}\right), \\ \int_0^x f(\xi) O\left(\frac{e^{\lambda(x-\xi)}}{\lambda}\right) d\xi &= O\left(\frac{e^{\lambda(x-\xi)}}{\lambda}\right), \\ \int_x^1 f(\xi) O\left(\frac{e^{\lambda(1-\xi)}}{\lambda}\right) d\xi &= O\left(\frac{e^{\lambda(1-\xi)}}{\lambda}\right), \end{aligned}$$

and these integrals tend to zero when integrated over their contours.

Also,

$$\begin{aligned} \lim_{|\lambda| \rightarrow \infty} A(x, \lambda) &= -\exp [P(0, x)] \quad \text{for } \operatorname{Re} \lambda > 0, \\ \lim_{|\lambda| \rightarrow \infty} B(x, \lambda) &= -\exp [-P(0, x)] \quad \text{for } \operatorname{Re} \lambda < 0, \\ \lim_{|\lambda| \rightarrow \infty} C(x, \lambda) &= \exp [P(x, 1)] \quad \text{for } \operatorname{Re} \lambda > 0, \\ \lim_{|\lambda| \rightarrow \infty} D(x, \lambda) &= \exp [-P(x, 1)] \quad \text{for } \operatorname{Re} \lambda < 0. \end{aligned}$$

From these results we conclude that

$$\begin{aligned} \frac{1}{2\pi i} \oint H(x, \lambda) d\lambda + \frac{1}{2\pi i} \oint J(x, \lambda) d\lambda &= \frac{1}{2\pi i} \oint h(x, \lambda) d\lambda \\ &+ \frac{1}{2\pi i} \oint j(x, \lambda) d\lambda \quad \text{for } |\lambda| \rightarrow \infty \end{aligned} \quad (10)$$

and consequently

$$\sum_{-\infty}^{\infty} a_n u_n(x) = \sum_{-\infty}^{\infty} \mu_1(x, \lambda_n) \int_0^1 f(\xi) \frac{[p\nu_2(\xi, \lambda_n) + \nu_2'(\xi, \lambda_n)]}{K'(\lambda_n)} d\xi,$$

where

$$K'(\lambda_n) = \left[\frac{d}{d\lambda} K(\lambda) \right]_{\lambda_n} = \frac{\exp[-P(0, 1)]}{\lambda_n}.$$

Using $\lambda_n = 2P(0, 1) + 2n\pi i$ we obtain

$$p\nu_2(\xi, \lambda_n) + \nu_2'(\xi, \lambda_n) = \exp[-\lambda_n \xi + P(1, \xi)] - \frac{2p \exp[-\lambda_n \xi + P(1, \xi)]}{\lambda_n}$$

and therefore

$$\sum_{-\infty}^{\infty} a_n u_n(x) = \sum_{-\infty}^{\infty} \left[\mu_1(x, \lambda_n) \int_0^1 f(\xi) \frac{\exp[-\lambda_n \xi + P(1, \xi)]}{K'(\lambda_n)} d\xi - T_n \right],$$

where

$$T_n = \mu_1(x, \lambda_n) \int_0^1 2pf(\xi) \frac{\exp[-\lambda_n \xi + P(1, \xi)]}{\lambda_n K'(\lambda_n)} d\xi;$$

also

$$\frac{\mu_1(x, \lambda_n)}{K'(\lambda_n)} \int_0^1 f(\xi) \exp[-\lambda_n \xi + P(1, \xi)] d\xi = \beta_n \exp g(x) (\exp 2n\pi i x) - \beta_n \exp [P(0, x)],$$

where

$$g(x) = 2x \int_0^1 p dt - \int_0^x p dt$$

and β_n is the complex Fourier coefficient

$$\int_0^1 f(\xi) \exp[-g(\xi)] \exp(-2n\pi i \xi) d\xi.$$

We conclude then that

$$\sum_{-\infty}^{\infty} a_n u_n(x) = \sum_{-\infty}^{\infty} [\beta_n \exp g(x) [\exp 2n\pi i x] - \beta_n \exp [P(0, x)] - T_n]. \quad (11)$$

Now

$$\begin{aligned} \sum_{-\infty}^{\infty} T_n &= \frac{1}{2\pi i} \oint d\lambda \left[\int_0^x \frac{2pf(\xi) \exp[-\lambda \xi + P(0, \xi)]}{\lambda K(\lambda)} \mu_2(x, \lambda) d\xi \right. \\ &\quad \left. + \int_x^1 \frac{2pf(\xi) \exp[-\lambda \xi + P(1, \xi)]}{\lambda K(\lambda)} \mu_1(x, \lambda) d\xi \right]. \quad (12) \end{aligned}$$

From the results of (8) and (9) both of the integrals in (12) tend to zero when the radius of the contour tends to infinity and we conclude that

$$\sum_{-\infty}^{\infty} T_n = 0.$$

Now, if $\sum_{-\infty}^{\infty} b_n \exp (2n\pi ix)$ converges for $0 < x < 1$, then $\sum_{-\infty}^{\infty} b_n$ converges also and (11) may be written as

$$\sum_{-\infty}^{\infty} a_n u_n(x) = \left[\exp g(x) \sum_{-\infty}^{\infty} \beta_n \exp (2n\pi ix) \right] - \left[\exp [P(0, x)] \sum_{-\infty}^{\infty} \beta_n \right], \quad (13)$$

since $\sum_{-\infty}^{\infty} T_n = 0$.

Now, if

$$\sum_{-\infty}^{\infty} \exp (2n\pi ix) \int_0^1 f(\xi) \exp (-2n\pi i\xi) d\xi = t(x), \quad 0 < x < 1$$

then

$$\exp g(x) \sum_{-\infty}^{\infty} \beta_n \exp (2n\pi ix) = \exp g(x)[t(x) \exp [-g(x)]] = t(x).$$

Now $\sum_{-\infty}^{\infty} \beta_n$ converges to some value, say t_0 , and therefore

$$\sum_{-\infty}^{\infty} a_n u_n(x) = t(x) - t_0 \exp [P(0, x)]. \quad (14)$$

If our Fourier series converges uniformly to $t(x)$ for $0 < x < 1$, clearly then the convergence of (14) is uniform and our theorem is proved.

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