[Clearly for $1 \leq j \leq n - m$, $\int_0^a W_j(\theta, 0, 0) d\theta = 0$.]

Then if this is also true for $n - m < j \leq n$, condition (A) is satisfied. We may proceed to test condition (B). If new condition (A) is satisfied but the new (B) is not, this process can be repeated (given enough additional differentiability), in the hope that at some finite step either the unfavorable case of not- (A) will occur or the favorable one (A) and (B) will.

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ON AN EIGENVALUE PROBLEM*

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The purpose of this note is to consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \frac{1}{x} \frac{d\phi}{dx} - \frac{2}{b^2 - a^2} [b\phi'(b) - a\phi'(a)] = -\lambda\phi, \quad a \leq x \leq b; \quad (1)$$

$$\phi(a) = \phi(b) = 0. \quad (2a,b)$$

Here $\phi'(b)$ and $\phi'(a)$ denote the value of $d\phi/dx$ at the points $x = a$ and $x = b$.

This eigenvalue problem arises in considering unsteady flow of an incompressible viscous fluid in the entrance region of an annular tube [1]. A special case of the equation (1) ($a = 0$ case), has been given in [2].

1. Compatibility condition for non-homogeneous equation. *Theorem:* The non-homogeneous linear differential equation

$$\frac{d^2\phi}{dx^2} + \frac{1}{x} \frac{d\phi}{dx} - \frac{2}{b^2 - a^2} [b\phi'(b) - a\phi'(a)] = R(x) \quad (3)$$

subject to the conditions (2a, b) has a solution if, and only if

$$\int_a^b xR(x) dx = 0. \quad (4)$$

Proof: Let $\phi(x)$ be a solution. Multiplying both sides of (3) by x and then integrating with respect to x from a to b we get (4). This gives us the necessity part of the proof. Sufficiency of the condition (4) however will follow from the construction of a *Green's function in the generalized sense* in Sect. 5.

As a consequence of this theorem, replacing function $R(x)$ by $-\lambda_i\varphi_i$, where λ_i is a non-zero eigenvalue and φ_i is the corresponding normalized eigenfunction, we see that

$$\int_a^b x\varphi_i(x) dx = 0. \quad (5)$$

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2. Eigenvalues and eigenfunctions. Having established (5) we can prove easily that all non-zero eigenvalues of $[(1), (2a, b)]$ are positive (see [3], p. 112.) Therefore we have to consider only the following two cases.

I. $\lambda > 0$: The general solution of (1) is

$$\phi(x) = A \left\{ J_0(\sqrt{\lambda} x) - \frac{2}{\lambda(b^2 - a^2)} [\sqrt{\lambda} b J_1(\sqrt{\lambda} b) - \sqrt{\lambda} a J_1(\sqrt{\lambda} a)] \right\} \\ + B \left\{ Y_0(\sqrt{\lambda} x) - \frac{2}{\lambda(b^2 - a^2)} [\sqrt{\lambda} b Y_1(\sqrt{\lambda} b) - \sqrt{\lambda} a Y_1(\sqrt{\lambda} a)] \right\}.$$

Using boundary conditions $(2a, b)$ and seeking for non-trivial solutions we obtain the following characteristic equation:

$$[J_0(\eta) - J_0(c\eta)][Y_2(\eta) - c^2 Y_2(c\eta)] - [Y_0(\eta) - Y_0(c\eta)][J_2(\eta) - c^2 J_2(c\eta)] = 0, \quad (6)$$

where

$$\eta = \sqrt{\lambda} b \quad \text{and} \quad c = a/b.$$

Let η_n denote the positive roots of (6). Then the normalized eigenfunctions for $\lambda > 0$ are

$$\varphi_n(x) = \frac{\sqrt{2} \eta_n}{b[\Phi_{n1}\Phi_{n2}]^{1/2}} \left\{ \left[J_0\left(\eta_n \frac{x}{b}\right) - J_0(c\eta_n) \right] [Y_0(\eta_n) - Y_0(c\eta_n)] \right. \\ \left. - \left[Y_0\left(\eta_n \frac{x}{b}\right) - Y_0(c\eta_n) \right] [J_0(\eta_n) - J_0(c\eta_n)] \right\}, \quad (7)$$

where

$$\Phi_{n1} = \eta_n [J_1(\eta_n) Y_0(c\eta_n) - J_0(c\eta_n) Y_1(\eta_n)] + c\eta_n [J_1(\eta_n) Y_1(c\eta_n) - J_1(c\eta_n) Y_0(\eta_n)], \\ \Phi_{n2} = \eta_n [J_1(\eta_n) Y_0(c\eta_n) - J_0(c\eta_n) Y_1(\eta_n)] - c\eta_n [J_1(\eta_n) Y_1(c\eta_n) - J_1(c\eta_n) Y_0(\eta_n)] - \frac{4}{\pi}.$$

The limit of the characteristic equation (6) as $c \rightarrow 0$ (keeping b fixed) is

$$J_2(\eta) = 0. \quad (8)$$

Thus when $c \rightarrow 0$, the roots of (6) converge to the roots of $J_2(\eta) = 0$.

On the other hand, as $c \rightarrow 1$, the roots of (6) go to infinity. However if we replace η with $\eta = (a + h)\sqrt{\lambda} \equiv a\sqrt{\lambda} + 2\gamma$ and let $a \rightarrow \infty$ we find

$$\tan \gamma = \gamma \quad (9)$$

Thus, if γ_n 's denote the roots of (9), we see that η_n 's go to infinity as $\gamma_n/(1 - c)$. In Fig. 1, for the first three roots of (6), numerical values of $(1 - c)\eta_n$ are given as a function of the parameter c .

II. $\lambda = 0$: For $\lambda = 0$ the equation (1) has the following solution

$$\varphi_0(x) = \frac{4}{b^2} \frac{\left(1 - \frac{x^2}{b^2}\right) \ln c - (1 - c^2) \ln \frac{x}{b}}{(1 - c^4) \ln c + (1 - c^2)^2} \quad (10)$$

FIG. 1.

which satisfies the boundary conditions $(2a, b)$. Therefore $\lambda = 0$ is an eigenvalue and $\varphi_0(x)$ is an eigenfunction. Here we have adjusted coefficients of (10) so that $\int_a^b x \varphi_0(x) dx = 1$.

3. Multiplicity of the eigenvalues. Following Ince ([4], p. 241) and using (5) we can show that λ_n 's ($=\eta_n^2/b^2$) are the simple roots of equation (6).

4. Orthogonality of the eigenfunctions. Following the standard method given in text books (see [4], p. 237) and again making use of (5) it can be shown that, except $\varphi_0(x)$ all the eigenfunctions are mutually orthogonal with a weight x .

5. A Green's function in the generalized sense. To find the Green's function for (3) we multiply both sides of (3) by x and then integrate with respect to x from a to x . In view of $(2a, b)$ this gives us

$$\phi(x) = \int_a^b K_1(x, t) t R(t) dt \quad (11)$$

$$K_1(x, t) = \frac{[b\phi'(b) - a\phi'(a)][(b^2 - x^2) \ln(a/b) - (b^2 - a^2) \ln(x/b)]}{2(b^2 - a^2) \ln(a/b)}$$

where

$$K_1(x, t) = \begin{cases} \frac{1}{\ln(a/b)} \ln \frac{b}{x} \ln \frac{t}{a} & t \leq x; \\ \frac{1}{\ln(a/b)} \ln \frac{b}{t} \ln \frac{x}{a} & t \geq x. \end{cases} \quad (12)$$

To eliminate the derivatives of $\phi(x)$ from the right hand side of (11) we require the first moment of the solution to be zero [the solution of (3) is determined only up to an arbitrary additive function $C\varphi_0(x)$] and obtain the following expression for the solution of (3).

$$\phi(x) = \int_a^b [K_1(x, t) + B(x, t)]tR(t) dt \quad (13)$$

where

$$B(x, t) = \frac{[(b^2 - t^2) \ln(a/b) - (b^2 - a^2) \ln(t/b)][(b^2 - x^2) \ln(a/b) - (b^2 - a^2) \ln(x/b)]}{(b^2 - a^2)[(b^2 + a^2) \ln(a/b) + (b^2 - a^2)]}.$$

Substitution of (13) into (3) reveals that (13) is a solution only if the first moment of the function $R(t)$ is equal to zero. This completes the sufficiency proof of the theorem given in Section 1.

6. An Expansion theorem. Let the function $w = w(x)$ have a continuous first and a piecewise continuous second derivative, and satisfy the boundary conditions $w(a) = w(b) = 0$. Then the function

$$\bar{w}(x) \equiv w(x) - \varphi_0(x) \int_a^b xw(x) dx \quad (14)$$

will satisfy the conditions

$$\bar{w}(a) = \bar{w}(b) = 0, \quad \int_a^b x\bar{w}(x) dx = 0,$$

and therefore, being in the range of the integral operator (13), can be expanded into an absolutely and uniformly convergent eigenfunction series (Courant and Hilbert [5], p. 360.)

$$\bar{w}(x) = \sum_{n=1}^{\infty} \varphi_n(x) \int_a^b t\bar{w}(t)\varphi_n(t) dt.$$

Moreover, since the kernel $K(x, t)$ is non-negative definite, this expansion is also valid for continuous functions with piecewise continuous first and second derivatives (see [5], p. 361 and [6], p. 245). Thus we have the following result: *Every continuous function $w(x)$ which satisfies the boundary conditions $w(a) = w(b) = 0$ and has piecewise continuous first and second derivatives may be expanded in an absolutely and uniformly convergent series*

$$w(x) = \varphi_0(x) \int_a^b tw(t) dt + \sum_{n=1}^{\infty} \varphi_n(x) \int_a^b t\bar{w}(t)\varphi_n(t) dt. \quad (15)$$

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