STATISTICAL PROPERTIES OF LOW-DENSITY TRAFFIC*

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Abstract. This paper considers an infinitely long line of traffic moving on a highway without traffic lights or other inhomogeneities. It is assumed that each car travels at a constant speed which is a random variable. A further assumption is that when one car overtakes another, passing is always possible and occurs without change of speed. it is shown that any initial headway distribution must relax to a negative exponential distribution in the limit of t becoming infinite. The statistics of passing events are examined, and it is shown that the probability of passing (or being passed by) n cars In time t is described by a Poisson distribution.

1. The most important factors which govern the gross traffic flow properties of a highway are those which relate to the interaction between individual drivers, or between one driver and the pattern of traffic external to him. It is well known that the flow-concentration curve on a real highway can qualitatively be described by the curve of Fig. 1. Under optimal driver behavior, which might call for driver interaction only in emergency conditions, the ideal flow-concentration curve would be a straight line also as shown in Fig. 1. The deviation from this behavior can be attributed to psychological and physical factors which are, at best, only qualitatively understood. Furthermore, the incorporation of any realistic model of driver behavior into a model of highway traffic flow usually leads to intractable mathematical problems. It is, therefore, of some interest to examine the very simplest highway model that can be studied mathematically; a highway in which there are no driver interactions. While this approach might initially appear to be completely artificial, it nevertheless does lead to a result which has been observed on highways with light traffic; that is, an exponential distribution of headway distances.

A first attempt to develop a simple theory along these lines was made by Newell[1] in 1955. However, his model was even more complicated than the one which we propose to treat in the present paper in that he attempted to calculate the first approximate corrections which are a function of traffic concentration. The only form of interaction allowed in Newell's model is passing. Although the assumptions which are made on the delay involved in passing are not based on experimental information, the model does achieve agreement with some data presented in the *Highway Capacity Manual.* A more ambitious attempt to develop a theory of traffic with interactions has recently been made by Prigogine et al,[2,3] using some of the approximations suggested by the kinetic theory of gases. The underlying idea of Prigogine's work is that an individual driver interacts with all of the other drivers on the road. Further approximations are then made for the non-linear Boltzmann equation (i.e., the equation for the conservation of probability in veloicty space). The treatment leads to a flow-concentration curve quali-

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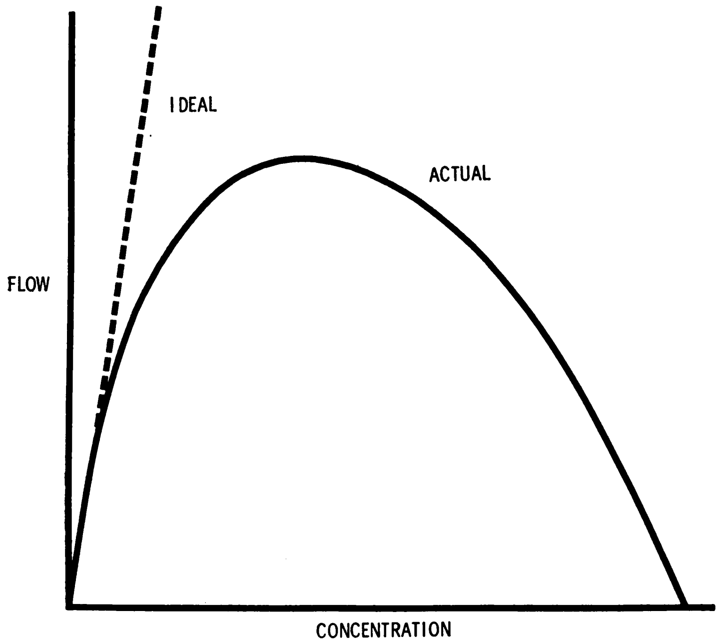

FIG. 1. Actual and Ideal Flow-Concentration Curves

tatively like that of Fig. 1 up to a certain concentration. The restriction on traffic volume is implicit in the formulation as a low density theory. Another theory of traffic is that due to Lighthill and Whitham,[4] but it is a continuum theory, and it does not seem possible to discuss individual interactions within its framework. The work of Chandler et al[5-8] deals with specific interactions between drivers, but only for a fellow-the-leader situation in which cars on a highway remain permanently ordered.

Our present analysis is one which allows no form of interaction between drivers. However, the present model allows for an exact mathematical treatment without approximations. The results to be presented in this paper contain, first of all, the relation between the velocity distribution and the headway distribution. We show that when the velocity distribution is continuous (the individual cars do not travel at one of a finite number of speeds) and passing is freely allowed, the only headway distribution which maintains itself in the long run, is the exponential distribution. This result has also been proved by A. J. Miller by essentially the same methods, in a recently circulated manuscript. A slightly more general statement of the fundamental theorem has been given by L. Breiman, and will appear in a forthcoming paper. The problem we study is somewhat similar to the evolution of the one-particle distribution function for non-interacting molecules. It has been shown that whenever there exists initially a finite dispersion in velocities the spatial distribution for sufficiently long times becomes uniform. As a consequence, in one dimension the probability distribution of first neighbor distances becomes a negative exponential distribution.[9] Related problems have been considered by Bartlett,[10] Carleson,[11] and Miller.[12] The remainder of the paper is devoted to a study of the statistics of passing events. It is possible to write down and solve sets of difference-differential equations for the number of cars which pass and are passed by a car traveling at a given speed. The results lead to a Poisson distribution for both numbers, which is not unexpected due to the negative exponential headway distribution.

2. The model to be discussed consists of a single infinitely long lane of homogeneous traffic in which the individual cars travel at a uniform speed. However, it is assumed that the speeds are themselves independent random variables governed by a probability density $\varphi(v)$. We will assume that $\varphi(v)$ is not given by a sum of delta functions or equivalently that $\int_0^v \varphi(v')\,dv'$ is continuous at every point in $v = (0, \infty)$. The major assumption of this work is that whenever a car with speed v comes into coincidence with one moving at a slower speed, it can pass without further delay. That is to say, there are no interactions at all in the model. The cars themselves will be assumed to be of zero length.

So far we have allowed only for a distribution of speeds; we must now say something about the distribution of distances between cars, or the headway distribution. While it might initially be thought that we are at liberty to choose any arbitrary distribution, this is not the case; after a sufficiently long time all "reasonable" distributions will relax to an exponential. In order to see this let us imagine that at time $t = 0$ cars are set down on the highway at random in such a way that the probability of a gap of length X between two successive cars, where $x < X \leq x + dx$, is given by $\zeta(x, 0)\,dx$. We thus assume that the gaps at $t = 0$ are independent random variables. We assume that $\zeta(x, 0)$ is such that

$$\int_0^\infty x^2\zeta(x, 0)\,dx < \infty. \tag{1}$$

Qualitatively speaking, we require that there are no infinite intercar gaps. Let us now imagine that we are sitting in a marked car and designate the initial position of this car by $x = 0$. We define a function $\rho(x, 0)\,dx$ to be the probability that at time $t = 0$ there is another car at a position x with respect to the marked car. The conditions of the problem assure us that

$$\rho(x, 0) = \rho(-x, 0), \tag{2}$$

or $\rho(x, 0)$ is a function only of $|x|$. The function $\rho(x, t)\,dx$ will be defined analogously for time t.

Let us now consider the expression for $\rho(x, t)$ in terms of $\rho(x, 0)$. After a time t the marked car will be at a position vt, and a car initially at x' will now be at a position $x' + v't$. Hence the relative coordinate of the second car at time t will be

$$x = x' - (v - v')t. \tag{3}$$

An expression for $\rho(x, t)$ is therefore given by

$$\begin{aligned}\rho(x, t) &= \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \rho(x', 0)\varphi(v)\varphi(v')\ \delta[x + (v - v')t - x']\,dv\,dv'\,dx \\ &= \int_0^\infty \int_0^\infty \rho\{x + (v - v')t, 0\}\varphi(v)\varphi(v')\,dv\,dv'.\end{aligned} \tag{4}$$

If we now fix an x and let t tend to infinity the argument of $\rho\{x + (v - v')t, 0\}$ must tend to plus or minus infinity except when $v = v'$. But this gives no contribution to the integral. Under various assumptions it is possible to prove that

$$\lim_{x\to\infty} \rho(x, 0) = \rho, \tag{5}$$

where ρ is a constant.[13] For this to be true it is necessary that $\zeta(x, 0)$ have a continuous component, i.e., be a probability density in the ordinary sense rather than a sum of delta functions. A further condition for Eq. (5) to hold is that Eq. (1) be valid, although the conclusion can also follow from weaker assumptions. We can now write the estimate

$$\lim_{t\to\infty} \rho(x, t) = \lim_{t\to\infty} \int_0^\infty \int_0^\infty \rho\{x + (v - v')t, 0\}\varphi(v)\varphi(v')\, dv\, dv'$$

$$= \rho \int_0^\infty \int_0^\infty \varphi(v)\varphi(v')\, dv\, dv' = \rho. \tag{6}$$

Finally, we can note that if $\rho(x, t)$ were identically a constant then the headway distribution would necessarily be exponential. As a result we can write for $\lim_{t\to\infty} \zeta(x, t)$:

$$\lim_{t\to\infty} \zeta(x, t) = \rho\, e^{-\rho x}, \tag{7}$$

for a fixed value of x. Hence, we can assert that the only headway distribution which maintains itself in the long run is the exponential, provided that we consider only distributions for which the second moment is finite and for which the cars do not all travel at one of a finite set of speeds. If $\varphi(v) = \delta(v - V)$ then, from Eq. (4)

$$\rho(x, t) = \rho(x, 0), \tag{8}$$

identically. Notice that if $\varphi(v)$ is chosen to be a combination of two delta functions, e.g., if

$$\varphi(v) = \lambda\, \delta(v - V) + (1 - \lambda)\, \delta(v - V^*), \tag{9}$$

then

$$\rho(x, t) = [\lambda^2 + (1 - \lambda)^2]\rho(x, 0) + \lambda(1 - \lambda)\rho[x + (V - V^*)t, 0] + \lambda(1 - \lambda)\rho[x + (V^* - V)t, 0], \tag{10}$$

in which the last two terms approach constants as $t \to \infty$, but the first term does not do so unless $\rho(x, 0)$ is chosen to be a constant.

3. We may now proceed to a closer consideration of some of the features of our model. In what follows we will assume that the headway distribution is given by an exponential of the form of Eq. (7). Let us first ask for the probability density of intercar arrival times at a fixed point on the highway. This density would be required in any discussion of the delay to cars either trying to cross a lane of traffic or to merge with it.[14,15]

We first compute the probability that a car will pass a fixed point P in the time interval $(t, t + dt)$. This probability can be expressed as

$$\beta(t)\, dt = \rho \int_0^\infty \left[\Phi\left(\frac{x}{t}\right) - \Phi\left(\frac{x}{t + dt}\right)\right] dx, \tag{11}$$

where ρ is the constant traffic density and $\Phi(x/t)$ is the probability that the speed, v, of a given car satisfies $v \leq (x/t)$. The function $\Phi(v)$ is related to $\varphi(v)$ by

$$\Phi(v) = \int_0^\infty \varphi(v')\, dv'. \tag{12}$$

Equation (11) is derived by noting that if a car passes P in $(t, t + dt)$ it must have started at $t = 0$ in the space interval $(x, x + dx)$. This occurs with probability $\rho\, dx$.

The terms $\Phi(x/t) - \Phi[x/(t + dt)]$ then represent the probability that the car speed is between x/t and $x/(t + dt)$. When Eq. (11) is expanded and only the terms proportional to dt retained, it is found that

$$\beta(t) = \rho\langle v\rangle, \tag{13}$$

where $\langle v\rangle$ is the mean speed,

$$\langle v\rangle = \int_0^\infty v\varphi(v)\, dv. \tag{14}$$

The probability density for the arrival time of the first car to arrive at P, if observations are begun at $t = 0$, will be denoted by $\psi(t)$. Then $\psi(t)$ satisfies

$$\rho\langle v\rangle = \psi(t) + \rho\langle v\rangle \int_0^t \psi(\tau)\, d\tau. \tag{15}$$

This is derived by the usual renewal argument: If a car arrives in $(t, t + dt)$, it is either the first with probability $\psi(t)\, dt$, or the first arrived at time τ and another followed after a time $t - \tau$. The unique differentiable solution of Eq. (16) is

$$\psi(t) = \rho\langle v\rangle \exp\,(-\rho\langle v\rangle t). \tag{16}$$

That is to say, if the distribution of distances is exponential, then the distribution of arrival times is also exponential under quite general assumptions on the form of the velocity distribution function. Thus, it can be inferred that deviations from an exponential (time) headway distribution are due to interactions due to passing, or to boundary effects due to traffic entrance or exit onto the highway. The Poisson distribution is often used to describe traffic, and has often been found experimentally for uncrowded highways.[16] Many results concerning the ability of Poisson traffic to delay intersecting traffic have been discussed in the literature.[17,18]

We now turn our attention to the statistical description of passing under the assumption of no delay on passing. Let us denote the probability that a car traveling at speed v will pass n cars in time T by $P_n(v, T)$. Then, it is possible to derive a set of difference-differential equations for these functions by increasing v to $v + dv$. It will be assumed that by increasing v by an infinitesimal amount the number of cars that are passed can be increased by no more than one. We shall use $\theta(v, T)\, dv$ to denote the probability that be increasing v by dv, a single additional car will be passed in time T. This quantity can be calculated from the kinematics of the situation. The car doing the passing will be assumed to be at $x = 0$ at $t = 0$. After a time t this car will be at $x = vT$. Therefore, the car will pass only those cars which are located in the x interval $(0, vT)$ both at time $t = 0$ and at $t = T$. If v is increased to $v + dv$ then a single additional car will be passed in time T if it is initially located in $(0, vT)$ and has a velocity such that the passing car overtakes it at time T. Thus we require that the car which is passed at time T have an initial distance x' and a velocity v' satisfying

$$x' + v'T = vT. \tag{17}$$

The situation is illustrated in Fig. 2. The probability that a car located at x' will just be overtaken at T, given that the passing car has speed $v + dv$ is therefore

$$\theta(v, T)\, dv = \rho \int_0^{vT} \varphi\left(v - \frac{x'}{T}\right) dx'\, dv, \tag{18}$$

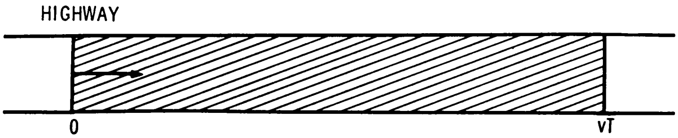

FIG. 2. All cars in the shaded region at $t = 0$ which remain in the shaded region at $t = T$, are passed by the car at $x = 0$

or

$$\theta(v, T) = \rho T[1 - \Phi(v)], \tag{19}$$

where $\Phi(v)$ is defined by Eq. (12). Having established the value of $\theta(v, T)$ we can write for $P_n(v, T)$,

$$\begin{aligned} P_0(v + dv, T) &= [1 - \theta(v, T)\, dv]P_0(v, T) \\ P_n(v + dv, T) &= \theta(v, T)P_{n-1}(v, T)\, dv + [1 - \theta(v, T)\, dv]P_n(v, T), \end{aligned} \tag{20}$$

corresponding to the two possibilities that by increasing v infinitesimally an additional car is, or is not, passed in time T. Passing to the limit $dv = 0$ we find that $P_n(v, T)$ satisfies

$$\begin{aligned} \frac{\partial P_n}{\partial v} &= \theta(v, T)[P_{n-1} - P_n] \\ \frac{\partial P_0}{\partial v} &= -\theta(v, T)P_0\,. \end{aligned} \tag{21}$$

It is easily verified that these equations are satisfied by a Poisson distribution

$$P_n(v, T) = \frac{[M(v, T)]^n}{n!} e^{-M(v,T)}, \tag{22}$$

where

$$M(v, T) = \int_0^v \theta(v', T)\, dv' = \rho T \int_0^v (v - v')\varphi(v')\, dv'. \tag{23}$$

A set of functions analogous to the $\{P_n(v, T)\}$ are the $\{\pi_n(v, T)\}$ where $\pi_j(v, T)$ is defined as the probability that j cars pass a car traveling at speed v, in time T. To derive the equations satisfied by the $\pi_n(v, T)$ we consider the events which might occur if v is decreased to $v - dv$. If this change is made then either the number of passing cars remains the same or it increases by one. These considerations lead to the equations

$$\begin{aligned} \pi_0(v - dv, T) &= \pi_0(v, T)[1 - \eta(v, T)\, dv] \\ \pi_n(v - dv, T) &= \pi_n(v, T)[1 - \eta(v, T)\, dv] + \pi_{n-1}(v, T)\eta(v, T)\, dv, \end{aligned} \tag{24}$$

where $\eta(v, T)\, dv$ is the probability that a single car will just pass the given car if the latter reduces speed to $v - dv$. Another way of defining $\eta(v, T)\, dv$ is by saying that it is the probability that a single car with speed greater than v will be in $[(v + dv)T, vT]$ at time T, where $x = 0$ is defined as the starting point at $t = 0$; of the passing car. We will calculate $\eta(v, T)$ explicitly, below. When the limit $dv = 0$ is taken in Eq. (24), the following equations result

$$\frac{\partial \pi_0}{\partial v} = \eta(v, T)\pi_0(v, T)$$
$$\frac{\partial \pi_n}{\partial v} = \eta(v, T)[\pi_{n-1}(v, T) - \pi_n(v, T)]. \tag{25}$$

Defining a new independent variable $N(v, T)$ by

$$N(v, T) = \int_v^\infty \eta(v', T)\, dv', \tag{26}$$

we find that the $\pi_n(v, T)$ satisfy the equations for the Poisson distribution, or

$$\pi_n(v, T) = \frac{N^n(v, T)}{n!} e^{-N(v,T)}. \tag{27}$$

We now turn to the evaluation of $\eta(v, T)$. The probability that a car is initially a distance x behind the given car is $\rho\, dx$. The probability that the car at $-x$ ends at vT at time T is just $\varphi(v + x/T)\, dv$, hence the result

$$\eta(v, T) = \rho \int_0^\infty \varphi\left(v + \frac{x}{T}\right) dx = \rho T \Phi(v), \tag{28}$$

which implies that $N(v, T) = \rho T \int_v^\infty (v' - v)\varphi(v')\, dv'$. The results of Eq. (21) and (22) can also be derived from Eq. (16) provided we choose a coordinate system fixed in the car moving at speed v. Then we are dealing with a fixed point P as was the case in the derivation of Eq. (16), however, the velocity distribution must be modified to describe the speeds relative to the test car.

If we define $N^*(v, T)$ to be the sum of cars passing and being passed by a given one traveling at speed v in time T, and

$$P_n^*(v, T) = P_r\{N^*(v, T) = n\} \tag{29}$$

then, since the events of passing and being passed are independent and have a Poisson distribution, we can write

$$P_n^*(v, T) = \sum_{m=0}^{n} P_m(v, T)\pi_{n-m}(v, T)$$
$$= \frac{[M(v, T) + N(v, T)]^n}{n!} e^{-[M(v,T)+N(v,T)]}. \tag{30}$$

When there are N types of cars on the road, e.g., cars and trucks for $N = 2$, all of which travel independently, the velocity density function of type i being denoted by $\varphi_i(v)$ and the density being ρ_i cars per unit length then some modifications of the results are necessary. The number of cars passing a given fixed point is again a Poisson variable, however, in Eq. (16) we must set

$$\rho\langle v\rangle = \sum \rho_i\langle v_i\rangle. \tag{31}$$

If the assumption is made that cars of each type are placed independently on the highway then, for example, the probability that a car traveling at speed v will pass n_1 cars of type 1, n cars of type 2, etc., is

$$P_{n_1, n_2, \cdots n_N}(v, T) = \prod_{\kappa=1}^{N} \frac{[M_\kappa(v, T)]^{n_\kappa}}{n_\kappa!} e^{-M_\kappa(v,T)}, \tag{32}$$

with a similar result holding for the $\pi_{n_1, n_2, \cdots, n_N}(v, T)$ and the $P_{n_1, n_2, \cdots, n_N}(v, T)$. The probability that a car traveling at speed v will pass a total of r cars in time T is

$$P_r(v, T) = \frac{1}{r!} \left[\sum_{\kappa=1}^{N} M_\kappa(v, T) \right]^r \exp \left[-\sum_{\kappa=1}^{N} M_\kappa(v, T) \right], \tag{33}$$

a result which is obvious because of our assumption that the different types of cars are independent.

It is of some interest to examine in greater detail, the consequence of the expressions that we have just derived. The following discussion will refer to only a single species of car; the appropriate modifications to N species follow simply from the results of the last paragraph.

It is evident that the expected number of cars passed in time T by one which is traveling at speed v, will be an increasing function of v. Likewise, the expected number of cars which pass this car is a decreasing function of v. We can now ask, what speed v will insure that the sum of the number of cars passing, and the number of cars being passed, be a minimum. From Eq. (31) we find that this expected number, for a time T, is just $M(v, T) + N(v, T) = E(v, T)$. The complete expression for $E(v, T)$ is, from Eqs. (23) and (26),

$$\begin{aligned} E(v, T) &= \rho T \left\{ \int_0^v [1 - \Phi(v')] \, dv' + \int_v^\infty \Phi(v') \, dv' \right\} \\ &= \rho T \left\{ \langle v \rangle + \int_0^v [1 - 2\Phi(v')] \, dv \right\}. \end{aligned} \tag{34}$$

Setting $\partial E / \partial v$ equal to zero, yields the condition for a minimum sum of interactions with other cars

$$\Phi(v) = \tfrac{1}{2}, \tag{35}$$

i.e., the optimal speed to minimize interactions is just the median speed. The expected numbers of cars which are passed, and which pass, a car of the ensemble are denoted by $\langle \geqq (T) \rangle$ and $\langle N(T) \rangle$ respectively. These functions are given by

$$\begin{aligned} \langle M(T) \rangle &= \int_0^\infty \varphi(v) M(v, T) \, dv = \rho T \int_0^\infty \Phi(v)[1 - \Phi(v)] \, dv \\ \langle N(T) \rangle &= \int_0^\infty \varphi(v) N(v, T) \, dv = \langle M(T) \rangle, \end{aligned} \tag{36}$$

where both relations are derived by an integration by parts. The equality $\langle M(T) \rangle = \langle N(T) \rangle$ is, of course, obvious on the consideration that we are dealing with an equilibrium situation.

Let us now look at the results for a specific choice of $\varphi(v)$. We consider a rectangular distribution

$$\begin{aligned} \varphi(v) &= 0 & 0 \leq v < V_1 \\ &= (V_2 - V_1)^{-1} & V_1 \leq v \leq V_2 \\ &= 0 & v > V_2 \end{aligned} \tag{37}$$

Then the functions $\geqq (v,\ T)$ and $N(v,\ T)$, which are, respectively, the expected number of cars which are passed by, and pass, a car traveling at speed v in time T, are given by

$$\begin{aligned} M(v, T) &= 0, \qquad 0 \leq v < V_1 , \\ &= \tfrac{1}{2}\rho T \frac{(v - V_1)^2}{V_2 - V_1}, \qquad V_1 \leq v \leq V_2 , \\ &= \rho T\left(v - \frac{V_1 + V_2}{2}\right), \qquad v > V_2 , \\ N(v, T) &= \rho T\left(\frac{V_1 + V_2}{2} - v\right), \qquad 0 \leq v < V_1 \\ &= \frac{\rho T}{2} \frac{(V_2 - v)^2}{V_2 - V_1}, \qquad V_1 \leq v \leq V_2 \\ &= 0, \qquad v > V_2 . \end{aligned} \tag{38}$$

The average number of cars passing a given member of the ensemble in time T is given by $\langle \geqq (T) \rangle$ which may be calculated from the formula of Eq. (36), and is found in this particular case to be

$$\langle M(T) \rangle = \langle N(T) \rangle = \frac{\rho T}{6} (V_2 - V_1). \tag{39}$$

The results which we have presented refer only to a very crude model which is likely to be informative only for low density traffic. A calculation of the next corrections to the present theory would involve some form of interaction between drivers. However, any such extension of the theory would seem to be an order of magnitude more difficult than the present calculations.

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