where δ_{rs} is the Kronecker symbol. If the primed variables satisfy (8), then (6) reduces to

$$\frac{dp'_r}{dt} = -\frac{\partial K'}{\partial q'_r} + Q'_r , \qquad \frac{dq'_r}{dt} = \frac{\partial K'}{\partial p'_r} - P'_r , \qquad (r = 1, \cdots N), \tag{9}$$

which has the same form as (2).

We have shown that (2) is invariant under a canonical transformation

$$p'_r = p'_r(p, q), \qquad q'_r = q'_r(p, q), \qquad (r = 1, \cdots N). \tag{10}$$

Also, if (1) is transformed according to (10), we again obtain (9) where

$$P'_s = \sum_r \frac{\partial q_r}{\partial p'_s} Q_r , \qquad Q'_s = \sum_r \frac{\partial q_r}{\partial q'_s} Q_r .$$

Consequently, (2) rather than (1) is the invariant form under a canonical transformation for systems which contain polygenic forces.

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AN EXTENSION OF POINCARÉ'S CONTINUITY THEOREM*

By WILLIAM R. HASELTINE (*U. S. Naval Ordnance Test Station, China Lake, California*)

Abstract. A theorem is derived which is useful in establishing the existence of periodic solutions of systems of nonlinear ordinary differential equations involving a small parameter in certain cases where first-order methods break down.

1. In studying systems of nonlinear ordinary differential equations by perturbation methods, one of the key questions is the location of periodic solutions; and a principal tool for the search for them is Poincaré's continuity theorem (see for example, Theorem 5.2 of Ref. 1). In some important applications the theorem is inapplicable as it stands, owing to the vanishing of a certain determinant. We present a generalization of one case of the theorem which is useful in some of these degenerate cases. We will also show how the original problem may, if certain conditions are satisfied, be reduced to the case covered by the new theorem

2. *Theorem*: Given the set of $n + 1$ equations

$$\left.\begin{aligned} \frac{d\theta}{dt} &= 1 + \lambda^{r_0}\Theta(\theta, y, \lambda), \\ \frac{dy_i}{dt} &= \lambda^{r_i} Y_i(\theta, y, \lambda), \end{aligned}\right\} \qquad (i = 1, 2, \cdots , n) \tag{1}$$

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where $r_i (i = 0, 1, \cdots, n)$ is a natural number ≥ 1, Θ and all Y_i are periodic of period Ω in θ, and are C' in (θ, y, λ) for all θ and (y, λ) in a neighborhood of zero; and

$$\text{(A)} \qquad \int_0^\Omega Y_i(\theta, 0, 0)\, d\theta = 0,$$

$$\text{(B)} \qquad \det\left(\int_0^\Omega \frac{\partial}{\partial y_i} Y_i(\theta, y, \lambda)\, d\theta \bigg|_{\lambda=y=0}\right) \neq 0.$$

Then there exists a λ neighborhood N of zero such that for $\lambda \in N$ there is a function $T(\lambda)$ which tends to Ω as $\lambda \to 0$ and a vector function $y_0(\lambda)$ (with components y_{0i}) tending to zero as $\lambda \to 0$, such that the solution of system (1) with $\theta(0) = 0$, $y(0) = y_0(\lambda)$ has the property that $\theta(T(\lambda)) = \Omega$, $y(T(\lambda)) = y_0(\lambda)$. Furthermore the set $T(\lambda)$, $y_0(\lambda)$ is unique and C′ in λ.

Proof: Let $\theta = \psi(t, y_0, \lambda)$, $y_i = u_i(t, y_0, \lambda)$ be the solution of (1) for $\theta(0) = 0$, $y_i(0) = y_{0i}$. These functions ψ and u_i have the desired repeating property if the set (2) can be solved for T and y_0

$$\begin{aligned} \psi(T, y_0, \lambda) - \Omega &= 0, \\ h \equiv \int_0^T Y(\psi, u, \lambda)\, dt &= 0. \end{aligned} \tag{2}$$

But at $\lambda = 0$, $\psi = t$, $u = 0$ is a solution of (1) for $y_0 = 0$, and assumption (A) then assures us that (2) has the solution $T = \Omega$, $y_0 = 0$.

Then if

$$J = \begin{vmatrix} \dfrac{\partial\psi}{\partial T} & \dfrac{\partial\psi}{\partial y_0} \\[2ex] \dfrac{\partial h}{\partial T} & \dfrac{\partial h}{\partial y_0} \end{vmatrix},$$

evaluated at $\lambda = y_0 = 0$, $T = \Omega$ does not vanish, the implicit function theorem guarantees the conclusion of our theorem. But at the indicated point, $\partial\psi/\partial T = 1$, $\partial\psi/\partial y_0 = 0$. Also

$$\frac{\partial h_i}{\partial y_{0j}} = \sum_k \int_0^T \left[\frac{\partial}{\partial u_k} Y_i(\psi, u, \lambda)\right] \frac{\partial u_k}{\partial y_{0j}}\, dt.$$

But $u_k = y_{0k} + \lambda^{r_k} \int_0^T Y_k(\psi, u, \lambda)\, dt$ and at $\lambda = 0$, $\partial u_k/\partial y_{0j} = \delta_{jk}$. Hence

$$\frac{\partial h_i}{\partial y_{0j}}\bigg|_{\substack{\lambda=0\\ y_0=0}} = \int_0^\Omega \frac{\partial}{\partial u_j} Y_i(t, u, 0)\, dt \bigg|_{u=0}.$$

Thus the nonvanishing of J reduces to precisely condition (B).

3. Suppose the original equations to be studied have the form (1) with all $r_i = 1$, as in common problems, (and θ, Y be C^2) and that condition (A) is satisfied, but (B) fails.

Let B be the matrix

$$\left(\int_0^\Omega \frac{\partial}{\partial y_j} Y_i(\theta, y, \lambda)\, dt \bigg|_{\lambda=y=0}\right).$$

There is a constant matrix C, such that the substitution $y = Cz$ transforms our Equations into (3)

$$\frac{d\theta}{dt} = 1 + \lambda\Theta^*(\theta, z, \chi)$$
$$\frac{dz}{dt} = \lambda Z(\theta, z, \lambda), \tag{3}$$

with

$$(A') \int_0^\Omega Z(\theta, 0, 0)\, dt = 0,$$

and

$$B^* \equiv \int_0^\Omega \frac{\partial}{\partial z} Z(\theta, z, \lambda)\, dt \Big|_{\lambda=z=0} = C^{-1}BC,$$

is in Jordan normal form. There will be at least one zero row in B^*. Let there be m such, and renumber the z_i so that these m rows become the last. Set $z_i = w_i$ for $1 \leqq i \leqq n - m$ and

$$z_i = w_i + \lambda \int_0^\theta z_i(\theta, w, \lambda)\, d\theta,$$

for $n - m < i \leqq n$.

Then

$$\frac{d\theta}{dt} = 1 + \lambda\Theta^*(\theta, w, \lambda) + 0(\lambda^2)$$
$$= 1 + \lambda\hat{\theta}(\theta, w, \lambda),$$

For $i \leqq i \leqq n - m$

$$\frac{dw_i}{dt} = \lambda Z(\theta, w, \lambda) + 0(\lambda^2)$$
$$= \lambda W_i(\theta, w, \lambda).$$

For $n - m < i \leqq n$

$$\left\{\frac{dw_i}{dt} + \lambda Z_i(\theta, w, \lambda)[1 + \lambda\Theta^*(\theta, w, \lambda) + 0(\lambda^2)]\right\}$$
$$= \lambda\left\{\begin{array}{l} Z_i(\theta, w, \lambda) + \lambda \sum_{n-m<i<n} \frac{\partial}{\partial zj} Z_i \int_0^\theta Z_i(\theta, w, \lambda)\, dt \\ + 0(\lambda^2) \end{array}\right\}$$
$$\frac{dw_i}{dt} = \lambda^2 W_i(\theta, w, \lambda).$$

And these equations are of the form (1), with

$$r_j = 1 \qquad (j = 0 \quad \text{to} \quad n - m)$$
$$= 2 \qquad (n - m < j \leqq n).$$

[Clearly for $1 \leqq j \leqq n - m$, $\int_0^\Omega W_j(\theta, 0, 0)\, d\theta = 0$.]
Then *if* this is also true for $n - m < j \leqq n$, condition (A) is satisfied. We may proceed to test condition (B). If new condition (A) is satisfied but the new (B) is not, this process can be repeated (given enough additional differentiability), in the hope that at some finite step either the unfavorable case of not- (A) will occur or the favorable one (A) and (B) will.

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ON AN EIGENVALUE PROBLEM*

By H. BÜLENT ATABEK (*University of Minnesota*)

The purpose of this note is to consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \frac{2}{b^2 - a^2}[b\phi'(b) - a\phi'(a)] = -\lambda\phi, \qquad a \leq x \leq b; \tag{1}$$

$$\phi(a) = \phi(b) = 0. \tag{2a,b}$$

Here $\phi'(b)$ and $\phi'(a)$ denote the value of $d\phi/dx$ at the points $x = a$ and $x = b$.

This eigenvalue problem arises in considering unsteady flow of an incompressible viscous fluid in the entrance region of an annular tube [1]. A special case of the equation (1) ($a = 0$ case), has been given in [2].

1. Compatibility condition for non-homogeneous equation. *Theorem*: The non-homogeneous linear differential equation

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \frac{2}{b^2 - a^2}[b\phi'(b) - a\phi'(a)] = R(x) \tag{3}$$

subject to the conditions ($2a$, b) has a solution if, and only if

$$\int_a^b xR(x)\, dx = 0. \tag{4}$$

Proof: Let $\phi(x)$ be a solution. Multiplying both sides of (3) by x and then integrating with respect to x from a to b we get (4). This gives us the necessity part of the proof. Sufficiency of the condition (4) however will follow from the construction of a *Green's function in the generalized sense* in Sect. 5.

As a consequence of this theorem, replacing function $R(x)$ by $-\lambda_i\varphi_i$, where λ_i is a non-zero eigenvalue and φ_i is the corresponding normalized eigenfunction, we see that

$$\int_a^b x\varphi_i(x)\, dx = 0. \tag{5}$$

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