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DIFFRACTION OF A PLANE WAVE BY A RIGHT ANGLED WEDGE WHICH SUSTAINS SURFACE WAVES ON ONE FACE*

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1. Introduction. We discuss the diffracted field that arises when an incident plane electromagnetic wave strikes a right angled wedge. One face of the wedge is a perfect conductor and the other face supports and sustains surface waves. The solution on which the discussion is based was obtained in [1] by using an elementary technique developed by the authors. Since this method has been adequately treated in the literature [2]–[7], most of the calculations are omitted.† However, the structure of the field along the surface of the wedge at external grazing incidence ($\theta_0 \pm \pi/2$ in Fig. 1 below) is treated in some detail. This is because there are certain difficulties and subtleties associated with this special case which do not seem to have been noticed in the literature and are not discussed in our previous papers. These results are particularly interesting and significant since special features of the field found in the present simple problem may reasonably be expected to emerge from a careful study of the more general, but less detailed, analyses of wedges of arbitrary angle (see Maluzhinets [8] and Williams [9]). A consequence of this detailed analysis is that the amplitude of the surface wave for small values of the parameter characterizing the surface differs from that given by Williams [9] by a factor of 2. Finally, we plot several curves that illustrate the results obtained. These numerical results are important since they help considerably in the understanding of the interaction between the electromagnetic field and the characteristic surface parameter.

The structure of the paper is as follows. Section 2 contains a precise statement of the problem solved. The results of the analysis, which are exact, are summarized. In Sec. 3 the magnitude of the surface wave is treated for the special cases of small and large values of the characteristic surface parameter. These two cases correspond physically to the limits of loosely and tightly bound surface waves, respectively. A brief discussion of the far field cylindrical wave is given in Sec. 4 and the results are summarized.

We also give a graphical discussion of the field produced by a plane wave incident along the perfectly conducting face ($\theta_0 = 0$ in Fig. 1). Figure 5 shows the radiation

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†The material contained in this paper is a much improved version of an EM-Report by the authors [1]. It contains a considerable simplification of the results derived in [1], omits the analysis and corrects an error made in one of the Sections. This error was pointed out at a later date, but not entirely corrected, by Williams [9].

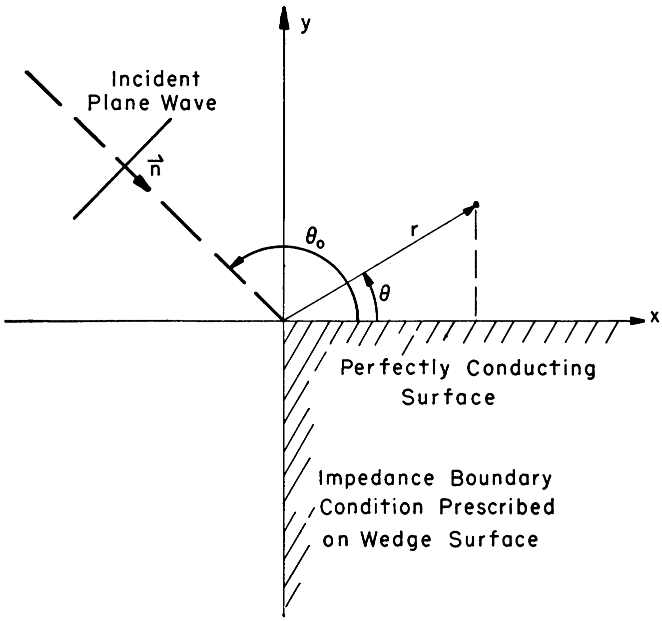

FIG. 1

pattern when the surface reactance X_k of the other face is 0, 1/2, and ∞, respectively. The value 1/2 is a typical value, and the other two values are included for comparison. In Fig. 6 the ratio of the amplitude of the surface wave to that of the incident plane wave is shown as a function of the dimensionless surface reactance when the angle of incidence of the plane wave is zero. In the text we deduce analytically the interesting fact that there is an optimum value of the surface reactance which gives rise to a maximum value of the induced surface wave amplitude. Figure 6 illustrates this result in the case of $\theta_0 = 0$.

2. Statement of problem and results. Consider a right angled wedge defined by the surfaces $y = 0, x > 0$ and $x = 0, y < 0$, as shown in Fig. 1. In the angular region $0 \leq \theta \leq 3\pi/2$ we assume that we have free space. A plane wave u_I, whose magnetic vector is linearly polarized in the z-direction, is incident upon the wedge. If the angle of incidence is θ_0 and the direction of the normal to the plane wave front is as shown in Fig. 1, then

$$u_I = U_0 \exp\left[-ikr \cos(\theta - \theta_0)\right], \tag{2.1}$$

where U_0 is the magnitude of the incident plane wave and the time dependence $e^{-i\omega t}$ is omitted for convenience. The boundary conditions on the wedge surface are given by

$$\begin{aligned} \frac{\partial u}{\partial y} &= 0, \qquad y = 0, \quad x > 0, \\ \frac{\partial u}{\partial x} - \lambda u &= 0, \qquad x = 0, \quad y < 0, \end{aligned} \tag{2.2}$$

where u is the z-component of the magnetic vector. The quantity λ is given by

$$\lambda = i\omega\epsilon Z = i\omega\epsilon(R - iX), \tag{2.3}$$

where ϵ is the permittivity of free space, ω is the angular frequency and Z, R and X are the impedance, resistance and reactance of the surface, respectively. It is well known that for homogeneous media of large finite conductivity, R and X are positive in sign, small in magnitude and approximately equal. For corrugated or dielectric-coated surfaces, although R and X are positive in sign, R is much smaller than X and hence Z is almost purely reactive. When X is positive it is clear that Re $\lambda > 0$. This condition is employed in our analysis. We also require that the scattered field (total field minus the incident field) at infinity be outgoing and that the energy of the electromagnetic field be finite. We wish to solve Maxwell's equations subject to the prescribed conditions and obtain the amplitude of the reflected surface wave. We restrict θ_0 in our work so that $0 \leq \theta_0 < 3\pi/2$. We leave open the possibility that for $\theta_0 = 3\pi/2$ the problem has no solution for the kind of incident wave described here.

The time dependent form of Maxwell's equation is

$$\text{curl } \mathbf{H} = -i\omega\epsilon\mathbf{E}, \qquad \text{curl } \mathbf{E} = i\omega u\mathbf{H}, \tag{2.4}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities, and ϵ and μ are the permittivity and magnetic permeability of free space. We assume the time dependence to be of the form exp $(-i\omega t)$. Because of the geometry, the field produced is independent of z and hence the field is completely determined by the value of H_z. We have

$$H_x = H_y = E_z = 0 \tag{2.5}$$

and

$$E_x = -\frac{1}{i\omega\epsilon}\frac{\partial H_z}{\partial y}, \qquad E_y = \frac{1}{i\omega\epsilon}\frac{\partial H_z}{\partial x} \tag{2.6}$$

The field component $H_z = u$ satisfies the equation

$$(\nabla^2 + k^2)u = 0 \tag{2.7}$$

where ∇^2 is the rectangular Laplacian and k is the propagation constant of free space. Therefore, the mathematical problem reduces to solving the homogeneous wave equation (2.7) subject to the mixed boundary conditions (2.2) and an incident plane wave of the form given by (2.1). In addition to these requirements, we require that the far field be outgoing and that the electromagnetic field be finite everywhere.

The solution of the above-mentioned problem can be found by using a method developed earlier by the authors [3]. It can be shown that the solution desired here is related to the solution of a simpler problem in which the boundary conditions on the wedge faces are not mixed and the incident excitation is a unit plane wave. In the simpler problem, the field vanishes on the front face ($x = 0$, $y < 0$) of the wedge and the derivative of the field vanishes on the top face ($y = 0$, $x > 0$) of the wedge. Using this result (see [10]), which we denote by $F(x, y)$, the solution of the originally stated mixed boundary value problem is given by

$$\begin{aligned} u(x, y) = {} & U_0(ik \cos \theta_0 + \lambda) \exp (\lambda x) \int_x^\infty \exp (-\lambda\xi)F(\xi, y)\, d\xi \\ & - c_1 \exp (\lambda x) \int_x^\infty \exp (-\lambda\xi)H_{1/3}^{(1)}(kr) \cos \frac{\theta}{3}\, d\xi \\ & + \begin{cases} c_2 \exp [\lambda x - i(k^2 + \lambda^2)^{1/2}y], & y < 0 \\ 0, & y > 0 \end{cases} \end{aligned} \tag{2.8}$$

where

$$F(x, y) = \frac{8}{3} \sum_{n=0}^{\infty} J_{(2n+1)/3}(kr) \exp[-i(2n+1)\pi/6] \cos(2n+1)\theta_0/3 \cos(2n+1)\theta/3. \quad (2.9)$$

Note that $r = (x^2 + y^2)^{1/2}$, $\theta = \arctan x/y$ and $0 \leq \theta \leq 2\pi$. $J_{(2n+1)/3}(kr)$ is a Bessel function of order $(2n + 1)/3$ and argument kr and $H_{1/3}^{(1)}(kr)$ is a Hankel function of the first kind of order $(2n + 1)/3$ and argument kr. The constants c_1 and c_2 are given by

$$c_1 = U_0 k \exp(i\pi/6) \frac{[\cos\theta_0 - \cos\theta_t]E(\theta_t + \pi, \theta_0)}{\cos(\theta_t + \pi)/3} \quad (2.10)$$

and

$$c_2 = \frac{2U_0 \cos\theta_t}{\cos(\theta_t + \pi)/3} \frac{\cos\theta/3}{[\cos\theta_0 + \cos\theta_t]}. \quad (2.11)$$

$E(\theta, \theta_0)$, which is the far field pattern function corresponding to (2.9), is given by

$$E(\theta, \theta_0) = \frac{1}{3^{1/2}} \left[\frac{\sin 2(\theta_0 - \theta)/3}{\sin(\theta_0 - \theta)} + \frac{\sin 2(\theta_0 + \theta)/3}{\sin(\theta_0 + \theta)} \right] \quad (2.12)$$

and the complex angle θ_t is defined by the relation

$$\theta_t = -(\pi/2) + i \sinh^{-1}(\lambda/k). \quad (2.13)$$

When $\theta_0 = \pi/2$ and λ is arbitrary, c_1 and c_2 are given by

$$c_1 = U_0 k \exp(i\pi/6) \frac{\cos 2(\theta_t + \pi)/3}{\cos(\theta_t + \pi)/3} \quad (2.14)$$

and

$$c_2 = \frac{3^{1/2} U_0}{\cos(\theta_t + \pi)/3}. \quad (2.15)$$

The amplitude of the surface wave is given by c_2 .

3. Limiting cases for the amplitudes of the surface waves. The value of λ is related to the impedance Z by means of the relation $\lambda = i\omega\epsilon Z = i\omega\epsilon(R - iX)$, where ϵ is the permittivity, R is the resistance and X is the reactance of the surface. For corrugated or dielectric-coated surfaces, R and X are both positive and R is very small compared to X. We now assume that λ is real and positive and treat the cases when (1) λ approaches zero, and (2) λ approaches infinity.

When λ/k is small we find that

$$\theta_t \to -(\pi/2) + i(\lambda/k). \quad (3.1)$$

Using (2.10) the limiting value of c_2 becomes

$$c_2(\theta_0) \to \frac{4(3)^{1/2}}{3} U_0 \exp(i\pi/2)\left(\frac{\lambda}{k}\right) \frac{\cos\theta_0/3}{\cos\theta_0}, \quad (3.2)$$

where $\theta_0 \neq \pi/2$. We note that the amplitude of the surface wave approaches zero as λ/k approaches zero. When $\theta_0 = \pi/2$, this formula fails. However, if we make use of (2.15) we find that

$$c_2(\pi/2) \to 2U_0 . \quad (3.3)$$

A formula for c_2 that is uniformly valid for all θ_0 and small λ is given by

$$c_2 = \{4U_0(i\lambda/k)[3^{1/2} + i\lambda/(3k)] \cos \theta_0/3\}/\{3[\cos \theta_0 + i\lambda/k]\}. \tag{3.4}$$

The result (3.3) must be interpreted with care since the amplitude of the surface wave should clearly be zero in the limit of small λ. The explanation of the finite limit (and the coefficient of 2) in (3.3) can be explained by examining (2.8). When $\theta_0 = \pi/2$ and $\lambda = 0$, the surface wave has the same form as the incident plane wave and hence is indistinguishable from it. Thus (3.3) yields a term $2U_0 \exp[-iky]$ in this limit. Actually, a field of the form $U_0 \exp[-iky]$ is to be expected for the problem when $\lambda = 0$, i.e., the perfectly conducting wedge. However, if we examine the integrals appearing in (2.8) when $\theta_0 = \pi/2$ and λ is small, we find another incident plane wave contribution from the first integral. Its magnitude is U_0 and its sign is negative. The sum therefore yields the expected incident plane wave contribution and hence there is nothing left to contribute to the magnitude of the surface wave. The surface wave is therefore zero, as expected. Therefore, it is interesting to note that Williams [9], who solves this problem using a different method, obtains a different expansion for the amplitude of the surface wave. Reducing his equation (22) to our notation we find agreement with our (3.2) except for a factor of 2, i.e., our result is twice his result.

We next consider the case of large λ/k and obtain the limiting values of c_1 and c_2. When λ/k is large we find that

$$\theta_t \to -\pi/2 + i \log 2\lambda/k. \tag{3.5}$$

Substituting (3.4) into (2.11) we obtain

$$c_2 \to 4U_0 \exp(i\pi/6)(2\lambda/k)^{-1/3} \cos \theta_0/3, \tag{3.6}$$

where θ_0 is unrestricted. We note that the amplitude c_2 of the surface wave approaches zero as λ approaches infinity. Since the amplitude approaches zero for both small and large λ, there is some optimum value of λ for which the magnitude of the surface wave is a maximum. This optimum value may depend on the angle of incidence θ_0. We illustrate this point in Figure 6 when the angle of incidence is taken to be zero and the dimensionless surface reactance $X_k = \lambda/k$ ranges from zero to infinity. A maximum occurs when X_k is approximately 2.5.

4. Determination of the radiated far field. In the preceding sections we gave the exact solution of the diffraction problem. A simple expression for the far field, however, was not given. In this section we shall indicate briefly how the far field can be obtained by using a simple method developed by the authors [2]–[7] that avoids the asymptotic evaluation of the complicated integrals appearing in (2.8). We note first that the original electromagnetic field component u in the far field can be written as the sum of two terms. The first term can be found by using geometrical optics. The second term is due to the diffracted field and has the form

$$(u)_{\text{diff}} = m(\theta)r^{-1/2} \exp(ikr) \tag{4.1}$$

The coefficient $m(\theta)$ can be constructed algebraically. In using the method it is again convenient to make use of the complex far field amplitude corresponding to a plane wave diffracted by a right angled wedge when the field *vanishes* on one surface ($\theta = 3\pi/2$) and the *derivative of the field vanishes* on the other ($\theta = 0$). Reiche [10] gives the solution of a plane wave diffracted by a right angled wedge where the field vanishes on both

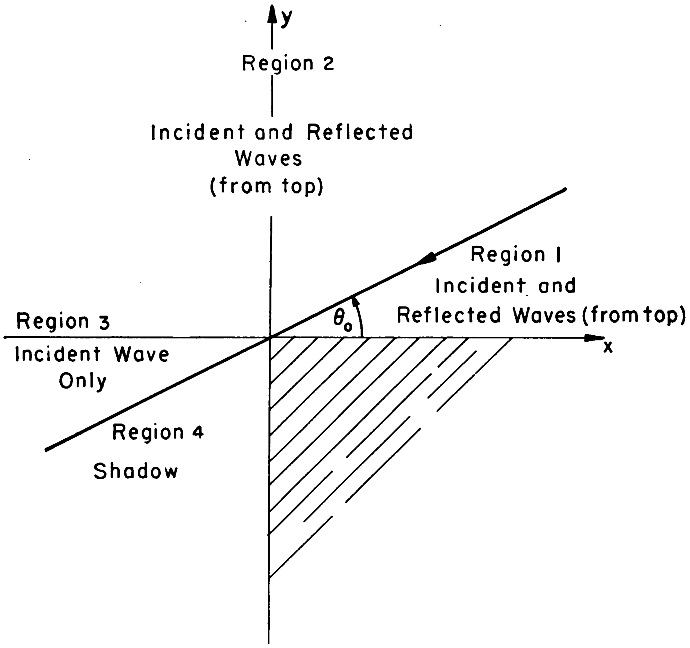

FIG. 2. Case A: $0 < \theta_0 < \pi/2$

surfaces. By using some of his intermediate results, the solution of the desired problem can easily be found. Proceeding in the manner indicated in the references, the diffracted angular dependence of the far field can be shown to be

$$m(\theta) = \sqrt{\frac{2}{\pi k}}\, U_0 \frac{(ik \cos \theta_0 + \lambda) \exp (i\pi/4)}{(ik \cos \theta - \lambda) \cos (\theta_t + \pi)/3} \cdot [E(\theta, \theta_0) \cos (\theta_t + \pi)/3 - E(\theta_t + \pi, \theta_0) \cos \theta/3]. \qquad (4.2)$$

The function $E(\theta, \theta_0)$ is given in equation (2.11).

It is of interest to show that the above expression for the diffracted far field reduces to the well known result for the diffraction of an incident plane wave by a perfectly conducting wedge. This special case occurs in the limit as λ/k approaches zero. In order to show this, we first make use of the identity

$$K(\theta, \theta_0) \cos \theta + E(\theta, \theta_0) \cos \theta_0 = \frac{2}{3^{1/2}} \cos 2\theta_0/3 \cos \theta/3, \qquad (4.3)$$

where

$$K(\theta, \theta_0) = \frac{1}{3^{1/2}} \left[\frac{\sin (\theta_0 - \theta)/3}{\sin (\theta_0 - \theta)} + \frac{\sin (\theta_0 + \theta)/3}{\sin (\theta_0 + \theta)} \right], \qquad (4.4)$$

is the well known pattern function due to Reiche [10] for $\lambda = 0$. It is, therefore, the pattern function corresponding to the solution N say of the problem for the same incident wave as F, but with Neumann condition on both faces.* Substituting (4.3) into (4.2)

*The existence of such an identity is forced a priori, since N_x and F both obey the same boundary condition. Since $N_x + F \cos \theta_0$ has no incident wave term, it must be a multiple of $H_{1/3}(kr) \cos \theta/3$. Therefore, there must be an identify for all x and y, of the form $N_x + F \cos \theta_0 \to B(\theta_0) H_{1/3}(kr) \cos \theta/3 \equiv 0$ where $B(\theta_0)$ can also be determined. The far field form of this identify yields (4.3).

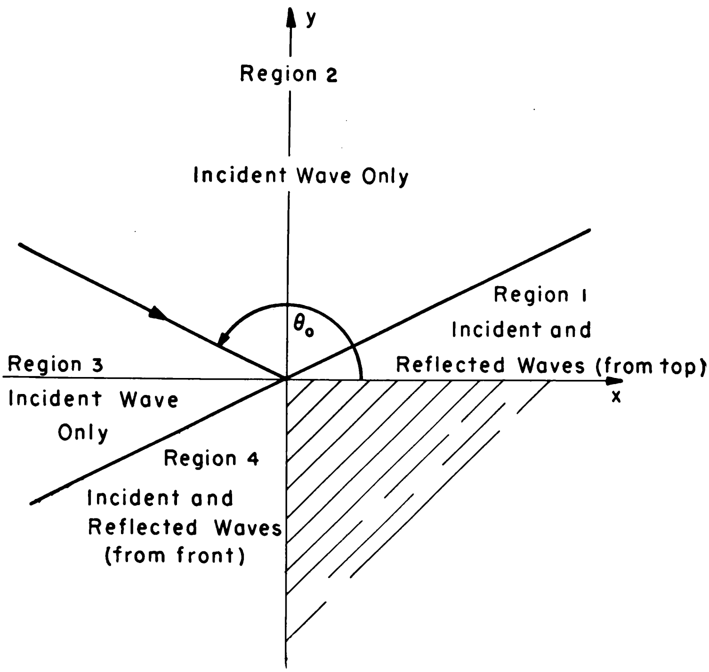

FIG. 3. Case B: $\pi/2 < \theta_0 < \pi$

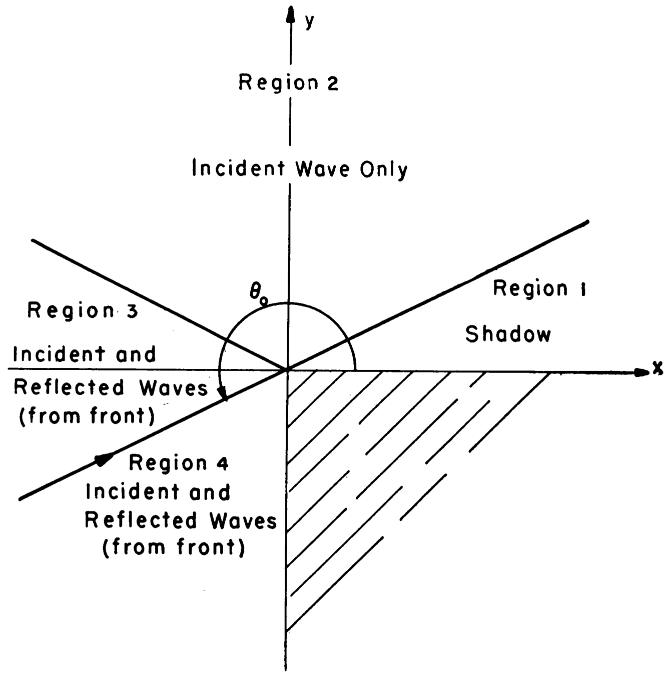

FIG. 4. Case C: $\pi < \theta_0 < 3\pi/2$

we have

$$m(\theta) = -(2/\pi k)^{1/2} U_0 \exp(i\pi/4) \frac{(ik \cos \theta_0 + \lambda) \cos \theta}{(ik \cos \theta - \lambda) \cos \theta_0} K(\theta, \theta_0) \cdot \left[1 - \frac{\cos \theta / 3 \cos (\theta_t + \pi) K(\theta_t + \pi, \theta_0)}{\cos \theta \cos (\theta_t + \pi)/3 \, K(\theta, \theta_0)} \right] \tag{4.5}$$

If we now permit λ/k to be small, it is easily shown that the second expression appearing in the brackets is of the order λ/k and hence vanishes. The far field amplitude then reduces to the simple expression

$$m(\theta) = -(2/\pi k)^{1/2} U_0 \exp(i\pi/4) K(\theta, \theta_0). \tag{4.6}$$

which was to be shown.

In order to examine the special case of large λ/k, it is convenient to write (4.2) in the form

$$m(\theta) = (2/\pi k)^{1/2} U_0 \exp(i\pi/4) \frac{(ik \cos \theta_0 + \lambda)}{(ik \cos \theta - \lambda)} E(\theta, \theta_0) \cdot \left[1 - \frac{\cos \theta/3 \, E(\theta_t + \pi, \theta_0)}{\cos (\theta_t + \pi)/3 \, E(\theta, \theta_0)} \right] \tag{4.7}$$

In the limit of large λ/k, it is easily shown that the second expression appearing in the brackets is of the order $(\lambda/k)^{-2/3}$ and hence vanishes. The far field amplitude then becomes

$$m(\theta) \to -(2/\pi k)^{1/2} U_0 \exp(i\pi/4) E(\theta, \theta_0). \tag{4.8}$$

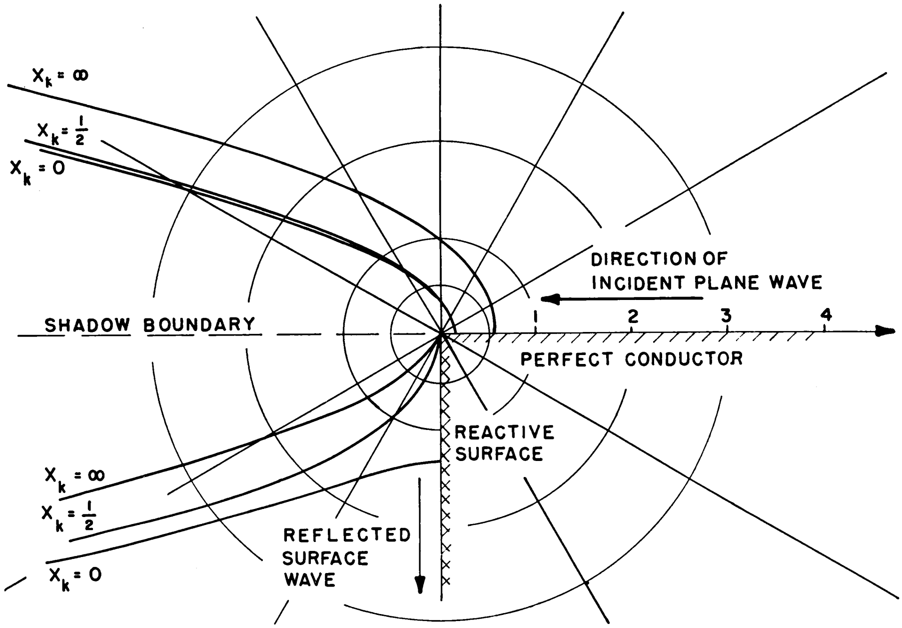

FIG. 5. Radiation pattern for incident plane wave.

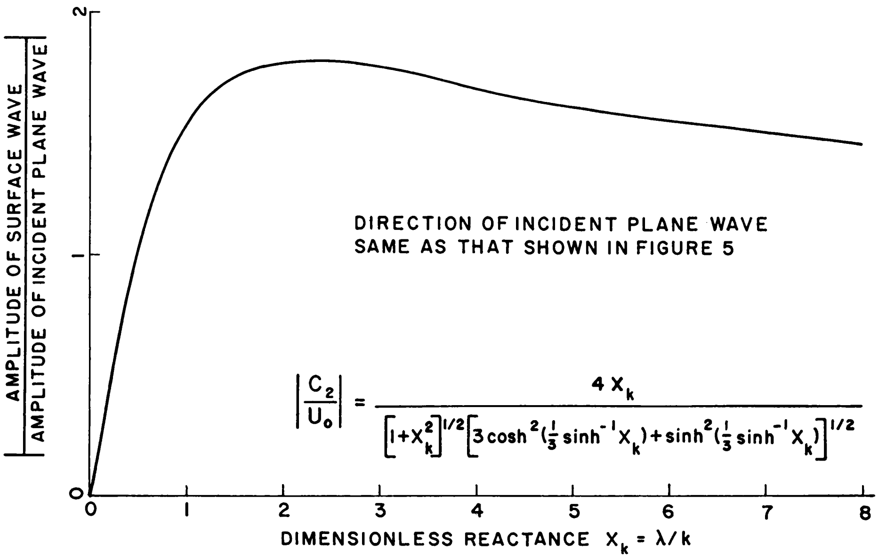

FIG. 6. Normalized surface wave amplitude versus the dimensionless reactance.

Hence, for the case of infinite λ/k, the far field amplitude reduces as expected to the appropriate result for an incident plane wave diffracted by a right angled wedge in which a Neumann boundary condition is prescribed on the upper face and a Dirichlet boundary condition is prescribed on the front face.

We shall now summarize the results for the far field. See Figures 2, 3, and 4 for definitions of the regions referred to in Cases A, B, and C, respectively. The expression for $m(\theta)$ is given by (4.2) when λ/k is arbitrary, by (4.6) when λ/k is zero, and by (4.8) when λ/k is infinite. Figure 5 illustrates the radiation pattern $| (2/\pi k)^{1/2} m(\theta)/U_0 |^2$.

Case A: $0 < \theta_0 < \pi/2$

1. Regions 1 and 2: $(0 < \theta < \pi - \theta_0)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + U_0 \exp[-ikr\cos(\theta + \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

2. Region 3: $(\pi - \theta_0 < \theta < \pi + \theta_0)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

3. Region 4: $(\pi + \theta_0 < \theta < 3\pi/2)$

$$u = \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

Case B: $\pi/2 < \theta_0 < \pi$

1. Region 1: $(0 < \theta < \pi - \theta_0)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + U_0 \exp[-ikr\cos(\theta + \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

2. Regions 2 and 3: $(\pi - \theta_0 < \theta < 2\pi - \theta_0)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

3. Region 4: $(2\pi - \theta_0 < \theta < 3\pi/2)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + U_0 \frac{ik\cos\theta_0 + \lambda}{ik\cos\theta_0 - \lambda} \exp[+ikr\cos(\theta + \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

Case C: $\pi < \theta_0 < 3\pi/2$

1. Region 1: $(0 < \theta < \theta_0 - \pi)$

$$u = \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

2. Region 2: $(\theta_0 - \pi < \theta < 2\pi - \theta_0)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr)$$

3. Regions 3 and 4: $(2\pi - \theta_0 < \theta < 3\pi/2)$

$$u = U_0 \exp[-ikr\cos(\theta - \theta_0)] + U_0 \frac{ik\cos\theta_0 + \lambda}{ik\cos\theta_0 - \lambda} \exp[ikr\cos(\theta + \theta_0)] + \frac{m(\theta)}{r^{1/2}} \exp(ikr).$$

References

1. F. C. Karal and S. N. Karp, *Diffraction of a plane wave by a right angled wedge which sustains surface waves on one face*, New York Univ., Inst. Math. Sci., Div. Electromagnetic Res., Res. Rep. EM-123, 1959
2. F. C. Karal and S. N. Karp, *Diffraction of a skew plane electromagnetic wave by an absorbing right angled wedge*, *Comm. Pure Appl. Math.* **11**, 495–533 (1958)
3. S. N. Karp and F. C. Karal, *Vertex excited surface waves on both faces of a right angled wedge*, Comm. Pure Appl. Math., Vol. 12, 1959, pp. 435–455
4. S. N. Karp and F. C. Karal, *A new method for the determination of far fields with application to the problem of radiation of a line source at the tip of an absorbing wedge*, I. R. E. Trans. on Antennas and Propagation, Toronto Symposium on Electromagnetic Theory, AP-7, 1959, pp. 91–102
5. S. N. Karp and F. C. Karal, *Vertex excited surface waves on one face of a right angled wedge*, Quart. Appl. Math., **18**, 235–243 (1960)
6. F. C. Karal, S. N. Karp, T. S. Chu, and R. G. Kouyomjian, *Scattering of a surface wave by a discontinuity in the surface reactance on a right angled wedge*, Comm. Pure Appl. Math. **14**, 1–14 (1961)
7. S. N. Karp, *Two-dimensional Green's function for a right angled wedge under an impedance boundary condition*, Comm. Pure Appl. Math. **13**, 203 (1960)
8. G. D. Maluzhinets, *The excitation, reflection and emission of surface waves from a wedge with given face impedances*, Soviet Physics, Dokl. 3, 752–755 (1958)
9. W. E. Williams, *Propagation of electromagnetic surface waves along wedge surfaces*, Quart. Jour. of Mech. Appl Math., **13**, 278–284 (1960)
10. F. Reiche, *The diffraction of light by a plane right angled wedge of infinite conductivity*, Ann. Physik, **37**, 131–156 (1912)