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LONGITUDINAL VIBRATION OF A PROLATE ELLIPSOID*

BY FRANK L. DIMAGGIO (*Columbia University*)

1. Introduction. In this paper, the mode shapes and frequency parameters for longitudinal vibration of a bar whose cross-sectional area varies parabolically along its axis are obtained as single tabulated prolate spheroidal angle functions and their eigenvalues. The results tediously obtained previously for the case of a centrally clamped ellipsoid, using a power series expansion, are quickly verified.

2. Previous solution. In a previous paper,[1] Kouvelites obtained the differential equation satisfied by the mode shapes of an ellipsoidal rod of major axis $2a$ vibrating longitudinally as

$$U'' - \frac{2\alpha}{1 - \alpha^2} U' + NU = 0, \tag{1}$$

in which primes denote differentiation with respect to $\alpha = x/a$, x being the coordinate along the longitudinal (major) axis measured from the center of the rod, $U(\alpha)$ is the mode shape, and

$$N = \rho\omega^2 a^2/E, \tag{2}$$

where ρ is the volume density, E is Young's modulus and ω is the circular frequency.

For a centrally clamped rod the solution was obtained as an infinite series in odd powers of α with the coefficients satisfying a recurrence equation containing N. By a tedious process of trial and error, N (and thus U) were determined numerically by requiring that the longitudinal stress be zero at the ends, i.e., by requiring that

$$U'(\pm 1) = 0. \tag{3}$$

Kouvelites noticed, apparently by observation of plots, that the series solution for U was divergent at $\alpha = \pm \perp$ except for that value of N which satisfied Eq. (3).

3. Solution as a prolate spheroidal angle function. If eq. (1) is rewritten as

$$[(1 - \alpha^2)U']' + N(1 - \alpha^2)U = 0, \tag{4}$$

the general solution that is finite at $\alpha = \pm \perp$ is seen to be[2]

$$U = S_{on}(N, \alpha), \tag{5}$$

where S_{on} are the prolate spheroidal angle functions of the first kind of order zero and degree n which have recently been tabulated[2,3] and N satisfies the frequency equation

$$\lambda_{on}(N_n) = N_n^2 , \tag{6}$$

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[1]J. S. Kouvelites, *Free longitudinal vibration of a prolate ellipsoid, clamped centrally*, Q. App. Math. **9**, 105–108 (1951)

[2]See, for example, C. Flammer, *Spheroidal wave functions*, Stanford University Press, 1957. The notation for these functions used in this paper is that of this book.

[3]J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little and F. Corboto, *Spheroidal wave functions*, John Wiley and Sons, Inc., 1956

in which $\lambda_{on}(N)$ is the associated eigenvalue, also tabulated in Refs. 2 and 3. For a centrally clamped plate only the antisymmetric modes, i.e., those corresponding to n odd are pertinent. By using the tables and graphs of Ref. 2 the results obtained with such difficulty in Ref. 1 may be quickly checked and found to be correct. It may also be seen directly from eq. (4), rewritten as

$$(1 - \alpha^2)U'' - 2\alpha U' + N(1 - \alpha^2)U = 0, \qquad (7)$$

that, since S_{on} is analytic, the solution given by Eq. (5) satisfies Eq. (3).

4. Solution for an unclamped ellipsoid. To demonstrate the utility of recognizing that the general solutions to Eq. (1) are expressible as tabulated functions, consider the case of the free ellipsoid. Then Eqs. (5) and (6) are still valid but even values of n are also permissible.

As an example, the lowest non-zero root N_2 satisfying Eq. (6) is obtained from Table 10 of Ref. 2 after a simple linear interpolation as

$$N_2 = 14.36. \qquad (8)$$

It may also be useful to point out that all of the results obtained above are valid for a bar whose cross-sectional area variation is given by

$$A(\alpha) = k(1 - \alpha^2), \qquad (9)$$

in which k is any constant. The ellipsoid is a special case of Eq. (9).

CANONICAL EQUATIONS FOR SYSTEMS HAVING POLYGENIC FORCES*

By RICHARD R. AUELMANN (*Aeronutronic, Newport Beach, Cal.*)

Lanczos [1] uses the term "polygenic" to identify forces which are not derivable from a scalar potential function. Dynamical systems which contain polygenic forces are nonconservative; however, a system is also nonconservative if time appears explicitly in the Hamiltonian. Synge [2] and Ames and Murnaghan [3] have derived Hamilton's equations,

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} + Q_r\,, \qquad \frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}\,, \qquad (r = 1, \cdots N), \qquad (1)$$

directly from Lagrange's equations for systems containing the generalized polygenic forces Q_r . The p_r and q_r are the generalized momenta and coordinates, respectively. The forces which are derivable from a scalar potential function are taken into account in the Hamiltonian H. The polygenic forces do not appear in the second set of (1); the reason is that the Lagrangian formulation (from which these equations were derived)

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