in which $\lambda_{on}(N)$ is the associated eigenvalue, also tabulated in Refs. 2 and 3. For a centrally clamped plate only the antisymmetric modes, i.e., those corresponding to n odd are pertinent. By using the tables and graphs of Ref. 2 the results obtained with such difficulty in Ref. 1 may be quickly checked and found to be correct. It may also be seen directly from eq. (4), rewritten as

$$(1 - \alpha^2)U'' - 2\alpha U' + N(1 - \alpha^2)U = 0, \tag{7}$$

that, since S_{on} is analytic, the solution given by Eq. (5) satisfies Eq. (3).

4. Solution for an unclamped ellipsoid. To demonstrate the utility of recognizing that the general solutions to Eq. (1) are expressible as tabulated functions, consider the case of the free ellipsoid. Then Eqs. (5) and (6) are still valid but even values of n are also permissible.

As an example, the lowest non-zero root N_2 satisfying Eq. (6) is obtained from Table 10 of Ref. 2 after a simple linear interpolation as

$$N_2 = 14.36. \tag{8}$$

It may also be useful to point out that all of the results obtained above are valid for a bar whose cross-sectional area variation is given by

$$A(\alpha) = k(1 - \alpha^2), \tag{9}$$

in which k is any constant. The ellipsoid is a special case of Eq. (9).

CANONICAL EQUATIONS FOR SYSTEMS HAVING POLYGENIC FORCES*

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Lanczos [1] uses the term "polygenic" to identify forces which are not derivable from a scalar potential function. Dynamical systems which contain polygenic forces are nonconservative; however, a system is also nonconservative if time appears explicitly in the Hamiltonian. Synge [2] and Ames and Murnaghan [3] have derived Hamilton's equations,

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} + Q_r\,, \qquad \frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}\,, \qquad (r = 1, \cdots N), \tag{1}$$

directly from Lagrange's equations for systems containing the generalized polygenic forces Q_r. The p_r and q_r are the generalized momenta and coordinates, respectively. The forces which are derivable from a scalar potential function are taken into account in the Hamiltonian H. The polygenic forces do not appear in the second set of (1); the reason is that the Lagrangian formulation (from which these equations were derived)

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admits only point transformations. We will show that (1) is a special form of a general set of equations

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} + Q_r\,, \qquad \frac{dq_r}{dt} = \frac{\partial H}{\partial p_r} - P_r\,, \qquad (r = 1, \cdots N), \tag{2}$$

where both P_r and Q_r are generalized polygenic forces, and that (2) is invariant under a canonical transformation.

We start with the transformation

$$\alpha_i = \alpha_i(p, q), \qquad (i = 1, \cdots 2N). \tag{3}$$

Differentiation of the α_i with respect to time, and substitution of (2) into the resulting expressions provide

$$\frac{d\alpha_i}{dt} = [\alpha_i H] + \sum_r \left(\frac{\partial \alpha_i}{\partial p_r} Q_r - \frac{\partial \alpha_i}{\partial q_r} P_r\right), \qquad (i = 1, \cdots 2N), \tag{4}$$

where the $[\alpha_i H]$ are the Poisson brackets representing

$$\sum_r \left(\frac{\partial \alpha_i}{\partial q_r}\frac{\partial H}{\partial p_r} - \frac{\partial \alpha_i}{\partial p_r}\frac{\partial H}{\partial q_r}\right), \qquad (i = 1, \cdots 2N).$$

In terms of the Poisson brackets $[\alpha_i \alpha_j]$ and the new Hamiltonian $K(\alpha) = H(q, p)$, we have

$$[\alpha_i H] = \sum_j [\alpha_i \alpha_j] \frac{\partial K}{\partial \alpha_j}\,,$$

$$\frac{\partial \alpha_i}{\partial p_r} = -\sum_j [\alpha_i \alpha_j] \frac{\partial q_r}{\partial \alpha_j}\,, \qquad \frac{\partial \alpha_i}{\partial q_r} = \sum_j [\alpha_i \alpha_j] \frac{\partial p_r}{\partial \alpha_j}.$$

The set (4) can now be written as

$$\frac{d\alpha_i}{dt} = \sum_j [\alpha_i \alpha_j]\left(\frac{\partial K}{\partial \alpha_j} - \sum_r \frac{\partial p_r}{\partial \alpha_j} P_r - \sum_r \frac{\partial q_r}{\partial \alpha_j} Q_r\right), \qquad (i = 1, \cdots 2N). \tag{5}$$

With

$$\alpha_1 = p_1'\,, \cdots \alpha_N = p_N'\,, \alpha_{N+1} = q_1'\,, \cdots \alpha_{2N} = q_N'$$

and the formation of $K'(p', q') = K(\alpha)$, the set (5) becomes

$$\begin{aligned} \frac{dp_r'}{dt} &= \sum_s [p_r' p_s']\left(\frac{\partial K'}{\partial p_s'} - P_s'\right) + \sum_s [p_r' q_s']\left(\frac{\partial K'}{\partial q_s'} - Q_s'\right), \\ \frac{dq_r'}{dt} &= \sum_s [q_r' p_s']\left(\frac{\partial K'}{\partial p_s'} - P_s'\right) + \sum_s [q_r' q_s']\left(\frac{\partial K'}{\partial q_s'} - Q_s'\right), \qquad (r = 1, \cdots N), \end{aligned} \tag{6}$$

where P_s' and Q_s' are defined by

$$P_s' = \sum_r \left(\frac{\partial p_r}{\partial p_s'} P_r + \frac{\partial q_r}{\partial p_s'} Q_r\right), \qquad Q_s' = \sum_r \left(\frac{\partial p_r}{\partial q_s'} P_r + \frac{\partial q_r}{\partial q_s'} Q_r\right). \tag{7}$$

The necessary and sufficient conditions for p_r' and q_r' to be canonical variables are

$$\begin{aligned} [p_r' p_s'] &= [q_r' q_s'] = 0, \\ [q_r' p_s'] &= -[p_r' q_s'] = \delta_{rs}\,, \end{aligned} \tag{8}$$

where δ_{rs} is the Kronecker symbol. If the primed variables satisfy (8), then (6) reduces to

$$\frac{dp'_r}{dt} = -\frac{\partial K'}{\partial q'_r} + Q'_r, \qquad \frac{dq'_r}{dt} = \frac{\partial K'}{\partial p'_r} - P'_r, \qquad (r = 1, \cdots N), \tag{9}$$

which has the same form as (2).

We have shown that (2) is invariant under a canonical transformation

$$p'_r = p'_r(p, q), \qquad q'_r = q'_r(p, q), \qquad (r = 1, \cdots N). \tag{10}$$

Also, if (1) is transformed according to (10), we again obtain (9) where

$$P'_s = \sum_r \frac{\partial q_r}{\partial p'_s} Q_r, \qquad Q'_s = \sum_r \frac{\partial q_r}{\partial q'_s} Q_r.$$

Consequently, (2) rather than (1) is the invariant form under a canonical transformation for systems which contain polygenic forces.

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AN EXTENSION OF POINCARÉ'S CONTINUITY THEOREM*

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Abstract. A theorem is derived which is useful in establishing the existence of periodic solutions of systems of nonlinear ordinary differential equations involving a small parameter in certain cases where first-order methods break down.

1. In studying systems of nonlinear ordinary differential equations by perturbation methods, one of the key questions is the location of periodic solutions; and a principal tool for the search for them is Poincaré's continuity theorem (see for example, Theorem 5.2 of Ref. 1). In some important applications the theorem is inapplicable as it stands, owing to the vanishing of a certain determinant. We present a generalization of one case of the theorem which is useful in some of these degenerate cases. We will also show how the original problem may, if certain conditions are satisfied, be reduced to the case covered by the new theorem

2. *Theorem*: Given the set of $n + 1$ equations

$$\left.\begin{aligned} \frac{d\theta}{dt} &= 1 + \lambda^{r_0}\Theta(\theta, y, \lambda), \\ \frac{dy_i}{dt} &= \lambda^{r_i} Y_i(\theta, y, \lambda), \end{aligned}\right\} \qquad (i = 1, 2, \cdots, n) \tag{1}$$

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