ISOTHERMAL SQUEEZE FILMS*

BY

W. E. LANGLOIS

International Business Machines Corporation, San Jose, California

Abstract. A derivation is presented of the equation governing the pressure in a thin, flat film of ideal gas under isothermal conditions, when the surfaces bounding the film are in relative normal and tangential motion. When tangential motion is absent, the pressure equation reduces to a nonlinear heat equation, which admits of very few closed-form solutions. Various approximation methods are discussed, and two problems involving small periodic variation of the gap between parallel plates are solved by a perturbation method for a film in which fluid inertia is negligible.

Introduction. Because of its wide technological application, the theory of fluid film lubrication between surfaces in relative lateral motion has been extensively studied since its formulation by Osborne Reynolds almost eighty years ago. The recent development of air-lubricated bearings has stimulated the extension of the theory to the case of compressible lubricating films.

Less attention has been given to the pressure generated in a fluid film by relative *normal* motion of the surfaces. Most of the published work on this subject has been confined to the study of incompressible films between parallel surfaces and in journal bearings [1]. Gas **squeeze films,** as they are called, have remained, for the most part, a curiosity. The earliest reference appears to be a paper written by Tipei [2] in 1954.

The purpose of this paper is to derive and apply the equation governing the pressure in a thin, flat film of ideal gas, under isothermal conditions, when the surfaces bounding the film are in relative normal motion.

In a recent paper, Elrod [3] derived the equation governing the steady-state pressure in a journal bearing lubricated by an incompressible fluid with constant viscosity. Although such derivations are usually carried out with the help of several *ad hoc* order-of-magnitude assumptions, Elrod used a perturbation approach, with the ratio of film thickness to bearing length as the small parameter. This approach not only allowed him to derive Reynold's lubrication equation in a more convincing manner, but also enabled him to retain the terms resulting from the film curvature. Elrod found that the Reynolds equation, with the first order correction for film curvature included, can be written

$$\frac{\partial}{\partial x}\left\{h^3\left(1 - \frac{h}{D}\right)\frac{\partial p}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{h^3\left(1 + \frac{h}{D}\right)\frac{\partial p}{\partial z}\right\} = 6\mu U \frac{\partial}{\partial x}\left\{h\left(1 - \frac{h}{3\,D}\right)\right\},$$

in which

D = shaft diameter,
h = film thickness,
p = fluid pressure,
U = shaft surface velocity,
x = distance around shaft in direction of rotation,
z = distance parallel to shaft axis,
μ = fluid viscosity.

*Received July 20, 1961; revised manuscript received October 16, 1961.

Elrod's result led a reviewer [4] to conclude that only the film curvature, and not the fluid inertia, has an influence in the approximation of the second order. However, the importance of inertia depends upon the film Reynolds number, which does not appear in Elrod's result. His dimensionless formulation is aimed at determining the effect of film geometry, not the effect of film dynamics. By way of example, his normalized pressure π is defined by

$$\pi = \frac{p\rho}{(\mu/h_0)^2},$$

where ρ is the lubricant density and h_0 is a typical value of the film thickness. Since the pressure in a journal bearing film varies according to h_0^{-2}, this formulation is quite adequate for Elrod's purposes. However, to represent properly the dependence of pressure on the parameters of the fluid, as predicted by lubrication theory, it is correct to use

$$\pi' = \frac{p}{\mu UD/h_0^2},$$

which differs from Elrod's π precisely by the factor $\rho UD/\mu$, a Reynolds number. That inertia effects are negligible is, then, a consequence of the particular limiting procedure postulated by Elrod, rather than a conclusion to be drawn from his results.

In Sec. 1 of the present paper, we use an approach quite similar to Elrod's in order to derive the equation governing the pressure in a thin film when both lateral and relative normal motions of the surfaces are present. We consider only flat films: i.e., although the surfaces bounding the film may be curved, it is possible to choose a plane such that the distance from any point in the film to the plane is small compared with the lateral dimensions of the film. The analysis does, however, apply to journal bearings, subject to curvature corrections of the type derived by Elrod. We assume that the density of the lubricant is proportional to its pressure, a condition which is met when the lubricant is an ideal gas and isothermal conditions obtain. Although fluid inertia is usually negligible in lubricating films, we do not assume this *a priori*. Rather, we derive criteria under which inertia can validly be neglected, and indicate a method of procedure which can be followed when these criteria are not met.

In Sec. 2, we introduce the restriction that lateral motion be absent, and thereafter consider only pure squeeze films.

A few exact solutions to the squeeze-film equation are presented in Sec. 3. In Sec. 4, we consider limiting forms of the squeeze film equation at high and low frequency of the squeeze motion. We find, in particular, that at low frequencies, the pressure in an isothermal squeeze film is governed, to first approximation, by the incompressible squeeze-film equation.

In Sec. 5, we calculate the pressure field generated by small, periodic variation of the gap between infinitely long parallel plates, and in Sec. 6, we treat the equivalent axisymmetric problem.

1. The mathematical foundations of isothermal gas-film theory. Assume that a thin continuous film of ideal gas is contained between the surfaces

$$x_3 = \mathcal{H}(x_1, x_2, t), \tag{1.1}$$

$$x_3 = \mathcal{H}'(x_1, x_2, t), \tag{1.2}$$

where x_1, x_2, x_3 are right-handed Cartesian coordinates and t is time. The film thickness h, defined by

$$h(x_1, x_2, t) = \mathcal{H}'(x_1, x_2, t) - \mathcal{H}(x_1, x_2, t), \tag{1.3}$$

is positive for all values of x_1, x_2, t; the surfaces move relative to the ambient gas with velocity components V_α, V'_α.

The surfaces bounding the film may be either rigid or flexible, but are assumed continuous. At each point of each surface, three components of velocity provide one degree of freedom too many, and must therefore be related through a kinematic constraint. With the convention that Latin indices extend over the values 1, 2, and Greek indices over 1, 2, 3, the kinetic constraints are

$$V_3 = \partial\mathcal{H}/\partial t + V_i\, \partial\mathcal{H}/\partial x_i\,, \tag{1.4}$$

$$V'_3 = \partial\mathcal{H}'/\partial t + V'_i\, \partial\mathcal{H}'/\partial x_i\,, \tag{1.5}$$

repeated indices denoting summation.

The motion of the gas in the film is governed by the equations of viscous hydrodynamics. The viscosity coefficients μ and λ are assumed constant, since, within the range of interest of gas lubrication theory, their variation is slight. As a consequence, we can write the Navier-Stokes equation

$$\rho \frac{Dv_\kappa}{Dt} = \frac{\partial}{\partial x_\kappa}\left[-p + (\lambda + \mu)\frac{\partial v_\alpha}{\partial x_\alpha}\right] + \mu \frac{\partial^2 v_\kappa}{\partial x_\alpha\, \partial x_\alpha}\,, \tag{1.6}$$

in which ρ and p denote, respectively, the density and pressure of the gas, and D/Dt denotes material differentiation:

$$D/Dt = \partial/\partial t + v_\alpha\, \partial/\partial x_\alpha\,. \tag{1.7}$$

The gas also obeys the continuity equation

$$\Delta + \partial v_\alpha/\partial x_\alpha = 0, \tag{1.8}$$

where the dilation Δ is defined by

$$\Delta = \frac{1}{\rho}\frac{D\rho}{Dt}. \tag{1.9}$$

Thus, Eq. (1.6) can also be written

$$\rho \frac{Dv_\kappa}{Dt} = -\frac{\partial}{\partial x_\kappa}[p + (\lambda + \mu)\,\Delta] + \mu \frac{\partial^2 v_\kappa}{\partial x_\alpha\, \partial x_\alpha}. \tag{1.10}$$

Since the film thickness is small compared with the bearing breadth B, the hydrodynamic equations can be simplified by using

$$\epsilon = h_0/B \tag{1.11}$$

as a perturbation parameter, where h_0 is a typical value of the film thickness. Using the technique employed by Elrod [3] to study curved incompressible films, we introduce a dimensionless coordinate system that "stretches" the coordinate normal to the film:

$$X_i = x_i/B \qquad (i = 1, 2), \tag{1.12}$$

$$z = x_3/h_0 = x_3/\epsilon B. \tag{1.13}$$

The $\kappa = 1, 2$ components of Eq. (1.10) then become

$$\mu \frac{\partial^2 v_i}{\partial z^2} = \epsilon^2 \left\{ -\mu \frac{\partial^2 v_i}{\partial X_j \, \partial X_j} + B \frac{\partial}{\partial X_i} [p + (\lambda + \mu) \Delta] + B^2 \rho \frac{D v_i}{Dt} \right\}, \tag{1.14}$$

and the $\kappa = 3$ component becomes

$$\mu \frac{\partial^2 v_3}{\partial z^2} = \epsilon B \frac{\partial}{\partial z} [p + (\lambda + \mu) \Delta] - \epsilon^2 \left[\mu \frac{\partial^2 v_3}{\partial X_j \, \partial X_j} - B^2 \rho \frac{D v_3}{Dt} \right]. \tag{1.15}$$

We now introduce dimensionless variables in such a way that the magnitude of each term in the equations of motion is represented by its coefficient, bearing in mind that we intend to apply the equations to a lubricating film.

For the time scale, we choose the reciprocal of a typical frequency ω of the squeeze component of surface motion. Thus, we assume

$$V_3 = h_0 \omega W, \qquad V_3' = h_0 \omega W', \tag{1.16}$$

where W and W' are dimensionless velocities of order unity, and introduce a dimensionless time T, defined by

$$T = \omega t. \tag{1.17}$$

To be consistent with Eq. (1.16), we let

$$v_3 = h_0 \omega w. \tag{1.18}$$

The scale of the lateral velocity components is not necessarily related to the scale of the squeeze component. While it is true that the squeeze motion forces gas outward or sucks it inward, at a characteristic velocity ωB, there is also a contribution to the lateral velocity arising from the lateral motion of the bearing surfaces. It is always possible to choose an instantaneous orientation of the coordinate system such that the components of surface motion in the x_1 and x_2 directions are of the same order. We shall assume that the time variation of these components is sufficiently slow that the same orientation can always be used. Thus, we introduce a reference velocity V such that

$$V_i = V U_i , \qquad V_i' = V U_i' , \tag{1.19}$$

where the dimensionless velocities U_i , U_i' are of order unity. To account for both contributions to the lateral velocity of the fluid, we let

$$v_i = (\omega B + V) u_i , \tag{1.20}$$

and expect the u_i to be of order unity.

In terms of our dimensionless quantities, the constraints (1.4) and (1.5) become, respectively,

$$\omega W = \omega \frac{\partial \bar{\mathfrak{H}}}{\partial T} + \frac{V}{B} U_i \frac{\partial \bar{\mathfrak{H}}}{\partial X_i} , \tag{1.21}$$

$$\omega W' = \omega \frac{\partial \bar{\mathfrak{H}}'}{\partial T} + \frac{V}{B} U_i' \frac{\partial \bar{\mathfrak{H}}'}{\partial X_i} , \tag{1.22}$$

in which

$$\bar{\mathfrak{H}} = \mathfrak{H}/h_0 , \qquad \bar{\mathfrak{H}}' = \mathfrak{H}'/h_0 . \tag{1.23}$$

The factor $V/\omega B$ implicit in Eqs. (1.21) and (1.22) provides a measure of the relative magnitude of the two reciprocal times characteristic of the bearing kinematics: V/B represents a shear rate characteristic of the lateral motion; ω, as defined above, is a typical frequency of the squeeze motion.

We now turn our attention to the definition of a dimensionless pressure. The theory of lubrication for incompressible films, which provides at least a limiting description of gas films, shows that, in the absence of squeeze motion, the pressure in the interior of a film varies according to $(\mu V/B)\epsilon^{-2}$. For pure squeeze films (lateral motion absent), the variation is as $\mu\omega\epsilon^{-2}$. Therefore, when lateral and squeeze motions are both present, we may find useful the normalization

$$p = \mu(\omega + V/B)\epsilon^{-2}\pi, \tag{1.24}$$

where π is of order unity except near the bearing periphery. Normalizing the density with respect to its ambient value ρ_a,

$$\rho = \rho_a P. \tag{1.25}$$

In isothermal films, however, the density is proportional to the pressure, so that

$$\rho = (\rho_a/p_a)p, \tag{1.26}$$

where p_a is the ambient pressure. Equation (1.24) therefore entails

$$\rho = \mu(\rho_a/p_a)(\omega + V/B)\epsilon^{-2}\pi, \tag{1.27}$$

and Eq. (1.25) yields the usual pressure normalization of gas lubrication theory:

$$P = p/p_a . \tag{1.28}$$

However, it is π, not necessarily P, which is of order unity in the interior of the film.

The dilational stress, $(\lambda + \mu)\,\Delta$, can be expressed

$$(\lambda + \mu)\,\Delta = \mu\left(\omega\theta_S + \frac{V}{B}\,\theta_L\right), \tag{1.29}$$

where

$$\theta_S = (1 + \lambda/\mu)(\partial\pi/\partial T + u_i\,\partial\pi/\partial X_i + W\,\partial\pi/\partial z)/\pi,$$

$$\theta_L = (1 + \lambda/\mu)\,\frac{u_i}{\pi}\,\frac{\partial\pi}{\partial X_i} \tag{1.30}$$

are dimensionless quantities. Near the bearing periphery, where steep gradients of pressure obtain, θ_S and θ_L may be quite large. In the interior of the film, however, they will normally be of order unity. An obvious exception occurs when the bearing undergoes lateral vibration at a frequency large compared with ω. In a step bearing,* θ_S and θ_L may be large in the neighborhood of the step. Both θ_S and θ_L are likely not to be of order unity in the vicinity of a source, unless we consider only the completely viscous film in which fluid inertia is negligible.

We now introduce modified Reynolds numbers corresponding, respectively, to the squeeze motion and to the lateral motion:

$$R_S = \omega\rho_a h_0^2/\mu, \qquad R_L = V\rho_a h_0^2/B\mu. \tag{1.31}$$

*Although we have been considering only films with continuous bounding surfaces, piecewise continous surfaces can be treated in an obvious manner.

In terms of the dimensionless quantities introduced above, the equations of motion (1.14) for the lateral velocity components become

$$\frac{\partial \pi}{\partial X_i} = \partial^2 u_i/\partial z^2 - R_S P(\partial u_i/\partial T + W\, \partial u_i/\partial z) - (R_S + R_L)Pu_j\, \partial u_i/\partial X_j$$
$$- \epsilon^2\left[\frac{\partial \theta_S/\partial X_i}{(1 + V/\omega B)} + \frac{\partial \theta_L/\partial X_i}{(1 + \omega B/V)} - \frac{\partial^2 u_i}{\partial X_j\, \partial X_j}\right]. \qquad (1.32)$$

Equation (1.15), for the normal velocity component, becomes

$$\frac{\partial \pi}{\partial z} = \frac{\epsilon^2}{(1 + V/\omega B)}\,[\partial^2 w/\partial z^2 - \partial\theta_S/\partial z - R_S P(\partial w/\partial T + w\, \partial w/\partial z)$$
$$- (R_S + R_L)Pu_i\, \partial w/\partial X_i] - \frac{\epsilon^2}{(1 + \omega B/V)}\frac{\partial \theta_L}{\partial z} + \frac{\epsilon^4}{(1 + V/\omega B)}\frac{\partial^2 w}{\partial X_i\, \partial X_i}. \qquad (1.33)$$

Equation (1.33) implies that, with neglect only of terms of the second degree or higher in ϵ, the pressure is constant across the film. This conclusion, of course, does not necessarily apply near the periphery of the bearing or in other regions where one or both of the dimensionless dilational stresses θ_S , θ_L become large (of order ϵ^{-2}). Moreover, either PR_S or PR_L could be of order ϵ^{-2}: In this unlikely case also, the pressure may vary significantly across the film. In most cases of interest, however, it is correct to infer from Eq. (1.33) that $\partial\pi/\partial z$ vanishes throughout the interior of the film, and we shall proceed on the assumption that this is the case. Consistent with this assumption is the reduction of Eq. (1.32) to

$$\partial\pi/\partial X_i = \partial^2 u_i/\partial z^2 - R_S P(\partial u_i/\partial T + w\, \partial u_i/\partial z) - (R_S + R_L)Pu_j\, \partial u_i/\partial X_j\ . \qquad (1.34)$$

In most bearing applications, the Reynolds numbers are negligibly small, but for the present we need not assume this is so. In view of Eqs. (1.19), (1.20) and (1.23), the dimensionless velocity components satisfy the boundary conditions.

$$[u_i]_{z=\mathfrak{K}} = \frac{U_i}{(1 + \omega B/V)}\,,$$
$$[u_i]_{z=\mathfrak{K}'} = \frac{U_i'}{(1 + \omega B/V)}. \qquad (1.35)$$

Since π, and consequently P, are assumed constant across the film, Eq. (1.34) and the boundary conditions (1.35) taken together are equivaletn to the integro-differential equation

$$u_i = \frac{1}{2}\frac{\partial \pi}{\partial X_i}(z - \mathfrak{K})(z - \mathfrak{K}') + \frac{U_i\mathfrak{K}' - U_i'\mathfrak{K} + (U_i' - U_i)z}{(1 + \omega B/V)H}$$
$$+ \frac{1}{2}P\int_{\mathfrak{K}}^{\mathfrak{K}'} K(z, z')\left[(R_S + R_L)u_j\frac{\partial u_i}{\partial X_j} + R_S\left(\frac{\partial u_i}{\partial T} + w\frac{\partial u_i}{\partial z'}\right)\right]dz', \qquad (1.36)$$

in which

$$H = h/h_0 = \mathfrak{K}' - \mathfrak{K},$$
$$K(z, z') = |\, z - z'\,| + (z - \mathfrak{K})(z' - \mathfrak{K}')/H + (z - \mathfrak{K}')(z' - \mathfrak{K})/H. \qquad (1.37)$$

Equation (1.36) represents a preliminary result to which we shall return presently.

For the moment, however, let us consider again the continuity equation (1.8). In terms of the original (dimensional) variables, this equation can be writtem

$$\partial\rho/\partial t + \partial(\rho v_i)/\partial x_i + \partial(\rho v_3)/\partial x_3 = 0. \tag{1.38}$$

Since we are neglecting the transverse pressure variation, and consequently the transverse density variation, integration of Eq. (1.38) across the film yields

$$h\frac{\partial\rho}{\partial t} + \int_{\mathcal{H}}^{\mathcal{H}'} \frac{\partial(\rho v_i)}{\partial x_i}\,dx_3 + \rho(V_3' - V_3) = 0. \tag{1.39}$$

However,

$$\int_{\mathcal{H}}^{\mathcal{H}'} \frac{\partial(\rho v_i)}{\partial x_i}\,dx_3 = \frac{\partial}{\partial x_i}\int_{\mathcal{H}}^{\mathcal{H}'} \rho v_i\,dx_3 - \rho V_i'\frac{\partial\mathcal{H}'}{\partial x_i} + \rho V_i\frac{\partial\mathcal{H}}{\partial x_i} \tag{1.40}$$

so that

$$h\frac{\partial\rho}{\partial t} + \rho(V_3' - V_3) - \rho V_i'\frac{\partial\mathcal{H}'}{\partial x_i} + \rho V_i\frac{\partial\mathcal{H}}{\partial x_i} = -\frac{\partial}{\partial x_i}\int_{\mathcal{H}}^{\mathcal{H}'} \rho v_i\,dx_3\,. \tag{1.41}$$

In view of the kinematic conditions (1.4) and (1.5), Eq. (1.41) becomes

$$\frac{\partial(\rho h)}{\partial t} = -\frac{\partial}{\partial x_i}\int_{\mathcal{H}}^{\mathcal{H}'} \rho v_i\,dx_3\,. \tag{1.42}$$

If we again neglect the density variation across the film, Eq. (1.42) becomes, in terms of the dimensionless variables,

$$\frac{1}{(1 + V/\omega B)}\frac{\partial(\pi H)}{\partial T} + \frac{\partial}{\partial X_i}\left(\pi\int_{\bar{\mathcal{H}}}^{\bar{\mathcal{H}}'} u_i\,dz\right) = 0. \tag{1.43}$$

Integrating Eq. (1.36) across the film yields

$$\int_{\bar{\mathcal{H}}}^{\bar{\mathcal{H}}'} u_i\,dz = \frac{H(U_i + U_i')}{2(1 + \omega B/V)} - \frac{H^3}{12}\frac{\partial\pi}{\partial X_i}$$
$$+ \tfrac{1}{2}P\int_{\bar{\mathcal{H}}}^{\bar{\mathcal{H}}'} (z - \bar{\mathcal{H}})(z - \bar{\mathcal{H}}')\left[(R_S + R_L)u_j\frac{\partial u_i}{\partial X_j} + R_S\left(\frac{\partial u_i}{\partial T} + w\frac{\partial u_i}{\partial z}\right)\right]dz. \tag{1.44}$$

Substituting Eq. (1.44) into Eq. (1.43) yields

$$\frac{12}{(1 + V/\omega B)}\frac{\partial(\pi H)}{\partial T} + \frac{6}{(1 + \omega B/V)}\frac{\partial}{\partial X_i}[\pi H(U_i + U_i')]$$
$$- \frac{\partial}{\partial X_i}\left(H^3\pi\frac{\partial\pi}{\partial X_i}\right) = R_L I_L + R_S I_S\,, \tag{1.45}$$

where

$$I_L = 6\frac{\partial}{\partial X_i}\left[\pi P\int_{\bar{\mathcal{H}}}^{\bar{\mathcal{H}}'} (z - \bar{\mathcal{H}})(z - \bar{\mathcal{H}}')u_j\frac{\partial u_i}{\partial X_j}\,dz\right] \tag{1.46}$$

and

$$I_S = I_L + 6\frac{\partial}{\partial X_i}\left[\pi P\int_{\bar{\mathcal{H}}}^{\bar{\mathcal{H}}'} (z - \bar{\mathcal{H}})(z - \bar{\mathcal{H}}')\left(\frac{\partial u_i}{\partial T} + w\frac{\partial u_i}{\partial z}\right)dz\right]. \tag{1.47}$$

We note that, if H, U_i and U_i' are specified functions of X_i and T, the dimensionless pressure π is the only dependent variable appearing on the left side of Eq. (1.45). Consequently, if conditions are such that the right side can be neglected, we obtain a single partial differential equation for the pressure in an isothermal gas film: a generalized Reynolds equation.

The right side of Eq. (1.45) is comprised of terms arising from the inertia terms in the Navier-Stokes equation. The inertia factors, defined by Eqs. (1.46) and (1.47), involve the dependent variables π, U_i and w, and hence cannot be specified *a priori*. However, the various dimensionless quantities have been defined in such a way that, under normal circumstances, I_S and I_L are of unit order. Thus, the significance of the right side of Eq. (1.45) is measured solely by the modified Reynolds numbers R_S and R_L. If they are very small compared with unity, Eq. (1.45) reduces to

$$\frac{\partial}{\partial X_i}\left(H^3\pi\,\frac{\partial \pi}{\partial X_i}\right) = \frac{12}{(1 + V/\omega B)}\,\frac{\partial(\pi H)}{\partial T} + \frac{6}{(1 + \omega B/V)}\,\frac{\partial}{\partial X_i}\,[\pi H(U_i + U_i')], \tag{1.48}$$

or, in terms of the original variables,

$$\frac{\partial}{\partial x_i}\left(h^3 p\,\frac{\partial p}{\partial x_i}\right) = 6\mu\left\{2\,\frac{\partial(ph)}{\partial t} + \frac{\partial}{\partial x_i}\,[ph(V_i + V_i')]\right\}. \tag{1.49}$$

Thus, when fluid inertia and dilational stresses are negligible, the pressure in a thin, isothermal gas film is governed by a nonlinear partial differential equation in three independent variables. In some problems of practical importance, p is the only dependent variable: h, V_i, and V_i' are specified functions of x_i and t. In other cases, some or all of the quantities h, V_i, V_i' are themselves dependent variables, so that the Reynolds equation (1.49) is coupled to other equations describing the dynamics of the bearing system.

Even if fluid inertia cannot be completely neglected, it may be possible to account for its effect, at least approximately, if the modified Reynolds numbers R_S and R_L are not too large. Estimates of the inertia factors I_S and I_L, based on some reasonable assumption for the velocity profile, can and have been used [5]. Equation (1.47) thus becomes a partial differential equation with the dimensionless pressure as the only dependent variable, instead of an integro-differential equation in four dependent variables.

2. The squeeze-film equation. In deriving the Reynolds equation (1.49), we found it convenient to introduce the dimensionless pressure π, defined by Eq. (1.24), since this quantity is normally of order unity in the interior of the gas film. In this section, however, we are concerned with the implications of Eq. (1.49) rather than with its derivation. Since the load-bearing ability of a gas film is measured by the gage pressure in its interior, it is useful to normalize the pressure with respect to the ambient pressure. The Reynolds equation then becomes

$$\frac{\partial}{\partial X_i}\left(H^3 P\,\frac{\partial P}{\partial X_i}\right) = \Lambda\,\frac{\partial}{\partial X_i}\,[PH(U_i + U_i')] + \sigma\,\frac{\partial(PH)}{\partial T}, \tag{2.1}$$

where the dimensionless parameters Λ (the **bearing number**) and σ (the **squeeze number**) are defined by

$$\Lambda = 6\mu B V/p_a h_0^2, \tag{2.2}$$

$$\sigma = 12\mu B^2\omega/p_a h_0^2. \tag{2.3}$$

If U_i, U_i', and H are specified functions of X_i and T, Eq. (2.1) is a quasilinear parabolic differential equation in one dependent and three independent variables.

Spatial symmetries can sometimes be used to eliminate one of the independent variables. The simplest example of this procedure occurs when the gas film is infinitely long in the X_2 direction and H is independent of X_2. Omitting the subscripts from U_1, U_1' and X_1, we reduce Eq. (2.1) to

$$\frac{\partial}{\partial X}\left(H^3 P \frac{\partial P}{\partial X}\right) = \Lambda \frac{\partial}{\partial X}\left[PH(U + U')\right] + \sigma \frac{\partial(PH)}{\partial T}. \tag{2.4}$$

It is sometimes possible to take advantage of a spatial symmetry by introducing a curvilinear coordinate system Y^i. Equation (2.1) then becomes

$$\begin{aligned} g^{ij} \frac{\partial}{\partial Y^i}\left(H^3 P \frac{\partial P}{\partial y^j}\right) &- g^{ij}\Gamma^k_{ij} H^3 P \frac{\partial P}{\partial y^k} \\ &= \Lambda\left\{\frac{\partial}{\partial Y^i}\left[PH(U^i + U^{i\prime})\right] + PH\Gamma^j_{ji}(U^i + U^{i\prime})\right\} + \sigma \frac{\partial(PH)}{\partial T}, \end{aligned} \tag{2.5}$$

where U^i and $U^{i\prime}$ denote the contravariant components of the lateral surface motion, $g^{\alpha\beta}$ denote the components of the contravariant metric, and $\Gamma^\gamma_{\alpha\beta}$ are the Cristoffel symbols of the second kind.

The most widely used curvilinear coordinate system is the polar system

$$\begin{aligned} y^1 &= R = (X_1^2 + X_2^2)^{1/2}, \\ y^2 &= \theta = \arctan (X_2/X_1), \end{aligned} \tag{2.6}$$

for which Eq. (2.5) reduces to

$$\begin{aligned} \frac{\partial}{\partial R}\left(RH^3 P \frac{\partial P}{\partial R}\right) &+ \frac{1}{R}\frac{\partial}{\partial \theta}\left(H^3 P \frac{\partial P}{\partial \theta}\right) \\ &= \Lambda\left\{\frac{\partial}{\partial R}\left[RPH(U_R + U_R')\right] + \frac{\partial}{\partial \theta}\left[PH(U_\theta + U_\theta')\right]\right\} + R\sigma \frac{\partial(PH)}{\partial T}, \end{aligned} \tag{2.7}$$

in which U_R, U_R', U_θ, U_θ' denote the physical components of surface motion. In the important case of axially symmetric motion, we obtain

$$\frac{\partial}{\partial R}\left(RH^3 P \frac{\partial P}{\partial R}\right) = \Lambda \frac{\partial}{\partial R}\left[RPH(U_R + U_R')\right] + R\sigma \frac{\partial(PH)}{\partial T}; \tag{2.8}$$

when the bearing surfaces are rigid, the Λ-term vanishes.

The character of solutions to Eq. (2.1), and its equivalents (2.5), (2.7), is determined by the magnitudes of Λ and σ. In the remainder of this paper, we consider the special case of zero bearing number. Thus, we focus our attention on the *squeeze-film equation*

$$\frac{\partial}{\partial X_i}\left(H^3 P \frac{\partial P}{\partial X_i}\right) = \sigma \frac{\partial(PH)}{\partial T}, \tag{2.9}$$

which governs the film pressure in the absence of lateral surface motion, and upon its axisymmetric equivalent

$$\frac{\partial}{\partial R}\left(RH^3 P \frac{\partial P}{\partial R}\right) = R\sigma \frac{\partial(PH)}{\partial T}. \tag{2.10}$$

For infinitely long films, Eq. (2.9) reduces to the *one-dimensional squeeze-film equation*

$$\frac{\partial}{\partial X}\left(H^3 P \frac{\partial P}{\partial X}\right) = \sigma \frac{\partial (PH)}{\partial T}. \tag{2.11}$$

3. Exact solutions to the squeeze-film equation. Since the squeeze-film equation (2.9) is, in general, a nonlinear parabolic equation with variable coefficients, present day analytical methods can be expected to provide exact solutions only in the most degenerate cases. Fortunately, one of these cases is of engineering interest: Externally pressurized films, under steady conditions, are governed by Eq. (2.9) with the squeeze number σ set equal to zero:

$$\frac{d}{dX_i}\left(H^3 P \frac{\partial P}{\partial X_i}\right) = 0. \tag{3.1}$$

On source-free segments of the bearing periphery, the pressure is ambient; at supply holes, it equals the inlet pressure.

For an elementary illustration, consider the one-dimensional case, for which Eq. (3.1) reduces to the ordinary differential equation

$$\frac{\partial}{\partial X}\left(H^3 P \frac{\partial P}{\partial X}\right) = 0. \tag{3.2}$$

If the pressure is ambient at $X = 0$ and equal to a supply pressure p_s at $X = 1(x_1 = B)$, the boundary conditions on P are, evidently

$$P(0) = 1, \qquad P(1) = p_s/p_a\ , \tag{3.3}$$

so that

$$P = \sqrt{1 + \frac{p_s^2 - p_a^2}{p_a^2} \frac{\int_0^x dX/H^3}{\int_0^1 dX/H^3}}. \tag{3.4}$$

The axisymmetric equivalent of this result is obtained from Eq. (2.10), with its right side set equal to zero, by an evident calculation.

Less trivial geometries can be treated exactly if H is constant, i.e., if the externally pressurized bearing consists of parallel flat plates, for Eq. (3.1) then reduces to

$$\frac{\partial^2 (P^2)}{\partial X_i\, \partial X_i} = 0, \tag{3.5}$$

and the methods of potential theory become available.

Because of the widespread use of externally pressurized gas bearings, many theoretical investigations of their properties have appeared in the literature. An extensive study of externally pressurized gas bearings under unsteady conditions has recently been presented by Licht and Elrod [6].

Equation (2.9) with a nonvanishing right side is, in essence, a nonlinear heat equation, which suggests that useful results might be found in Crank's extensive treatise [7]. On page 162, Crank describes an analysis due to Wagner [8] which leads us to an exact solution to Eq. (2.9). Wagner considered diffusion of a solute into a semi-infinite medium, with the diffusivity proportional to the concentration, the surface concentration held constant and the initial concentration equal to zero.

The squeeze-film analog of Wagner's problem is the determination of the pressure field between parallel plates, originally in intimate contact, then suddenly pulled apart. Actually, we can generalize the solution to include an arbitrary step-function change in the gap between semi-infinite parallel plates. Strictly speaking, this problem is outside the scope of squeeze-film theory as reported in this paper because, at the initial instant, dilational stresses and inertia are surely important, and the caloric behavior of the film is probably nearer adiabatic than isothermal (if, indeed, these thermostatic concepts have any meaning at all). It can be hoped, however, that squeeze-film theory provides an accurate picture once the first few instants of time have passed. In any case, the development of a family of exact solutions provides insight to the mathematical, if not to the physical, character of Eq. (2.9).

We assume that the projection of the gas film on the $x_1 - x_2$ plane occupies the half-plane $x_1 > 0$. Since no bearing breadth can be defined, we normalize all spatial variables with respect to the final gap h_0. We suppose that, at time zero, the gap is suddenly changed from αh_0 to h_0, then held at the new value. Thus,

$$H(X, t) = \begin{cases} \alpha, & t \leq 0 \\ 1, & t > 0 \end{cases}. \tag{3.6}$$

We use the lower case t, denoting actual time: since there is no characteristic frequency, the time variable cannot be normalized. For positive values of t, the normalized pressure P is governed by

$$\frac{\partial}{\partial X}\left(P\,\frac{\partial P}{\partial X}\right) = \sigma^*\,\frac{\partial P}{\partial t} \tag{3.7}$$

where σ^* is a characteristic time, defined by

$$\sigma^* = 12\mu/p_a\,. \tag{3.8}$$

Corresponding to the discontinuous change in gap at $t = 0$ there is a discontinuous change in the gas pressure. With the isothermal conditions postulated, Boyle's law requires that

$$P(X, +0) = \alpha, \tag{3.9}$$

$$\lim_{X\to\infty} P(X, t) = \alpha. \tag{3.10}$$

Since the pressure is ambient at the bearing edge,

$$P(0, t) = 1. \tag{3.11}$$

Equation (3.7), subject to the initial condition (3.9) and to the boundary conditions (3.10), (3.11), admits of the self-similar solution

$$P(X, t) = f(\tfrac{1}{2}X\sqrt{\sigma^*/t}), \tag{3.12}$$

where the function $f(y)$ is determined by the ordinary differential equation

$$f\,\frac{d^2f}{dy^2} + \left(\frac{df}{dy}\right)^2 + 2y\,\frac{df}{dy} = 0, \tag{3.13}$$

subject to the boundary conditions

$$f(0) = 1, \qquad \lim_{y\to\infty} f(y) = \alpha. \tag{3.14}$$

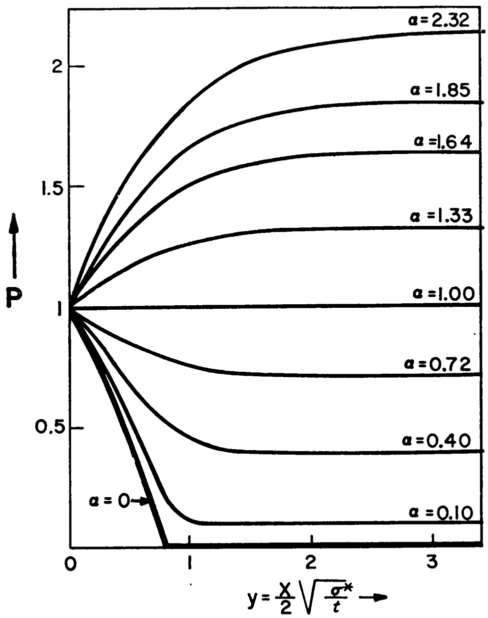

FIG. 3.1. Pressure Profiles in a Semi-Infinite Film.

Numerical integration of Eq. (3.13) subject to the boundary conditions (3.14) yields the family of curves illustrated in Fig. 3.1. As α becomes arbitrarily small, the solution tends uniformly to the curve for $\alpha = 0$ (plates initially in contact). This limiting solution joins the y-axis at about $y = 0.81$, illustrating the curious fact, observed by Wagner [8], that Eq. (3.7)—although parabolic—admits of a solution with a well-defined wave front. Needless to say, the basic assumptions of squeeze-film theory cannot hold true for such an extreme case.

4. Limiting forms of the squeeze-film equation. We began the previous section by considering externally pressurized films at zero squeeze number. Nontrivial solutions result only because of the boundary conditions: The pressure at supply holes differs from the ambient pressure at source-free segments of the boundary. Were this not so, the pressure would be ambient everywhere in the film, as is physically evident.

When the squeeze number is finite but small, we expect that the pressure in a self-acting squeeze film (no external pressurization) will not differ much from ambient: Small squeeze number corresponds to low frequency, so that the bearing has time to "leak." Thus, with σ as a perturbation parameter,

$$P = 1 + \sum_{n=1}^{\infty} \sigma^n P^{(n)}. \tag{4.1}$$

The squeeze-film equation (2.9) then becomes

$$\sigma \frac{\partial}{\partial X_i}\left(H^3 \frac{\partial P^{(1)}}{\partial X_i}\right) + \sum_{n=2}^{\infty} \sigma^n \frac{\partial}{\partial X_i}\left[H^3\left(\frac{\partial P^{(n)}}{\partial X_i} + \sum_{m=1}^{n-1} P^{(m)} \frac{\partial P^{(n-m)}}{\partial X^i}\right)\right]$$
$$= \sigma \frac{\partial H}{\partial T} + \sum_{n=2}^{\infty} \sigma^n \frac{\partial (P^{(n-1)}H)}{\partial T}. \tag{4.2}$$

Collecting equal powers of σ yields

$$\frac{\partial}{\partial X_i}\left(H^3 \frac{\partial P^{(1)}}{\partial X_i}\right) = \frac{\partial H}{\partial T}, \tag{4.3}$$

$$\frac{\partial}{\partial X_i}\left[H^3\left(\frac{\partial P^{(n)}}{\partial X_i} + \sum_{m=1}^{n-1} P^{(m)} \frac{\partial P^{(n-m)}}{\partial X_i}\right)\right] = \frac{\partial(P^{(n-1)}H)}{\partial T} \qquad (n = 2, 3, \cdots). \tag{4.4}$$

Since the pressure is ambient at the bearing periphery, all the $P^{(n)}$ must vanish there.

It is of interest to let

$$1 + \sigma P^{(1)} = \Pi, \tag{4.5}$$

so that, with Eq. (4.1),

$$P = \Pi + O(\sigma^2). \tag{4.6}$$

The boundary conditions require that Π be unity on the bearing periphery, and Eq. (4.3) implies that

$$\frac{\partial}{\partial X_i}\left(H^3 \frac{\partial \Pi}{\partial X_i}\right) = \sigma \frac{\partial H}{\partial T}. \tag{4.7}$$

This is the *incompressible squeeze-film equation,* for which many solutions are available [1]. Thus, with neglect only of terms of the second degree or higher in the squeeze number, an isothermal gas film can be considered incompressible. Physically, at low squeeze numbers the gas leaks out before it is significantly compressed or rarefied.

At very large squeeze number, on the other hand, the gas exhibits almost no tendency to leak. When the frequency of the squeeze motion is sufficiently high, the escape of the gas is effectively blocked by its own viscosity: The bearing tends to behave like a bellows. We use the perturbation parameter σ^{-1} and note that, as the squeeze number becomes indefinitely large, Eq. (2.9) tends to the limiting form

$$\partial(PH)/\partial T = 0. \tag{4.8}$$

There is obviously something wrong with this equation: It predicts that the product PH is a function of the spatial variables only, which leaves no way to satisfy the boundary conditions on the bearing periphery. For example, if

$$H(X_i, T) = \begin{cases} 1 & T \leq 0 \\ \eta(T) & T > 0, \qquad \eta(0) = 1 \end{cases}, \tag{4.9}$$

so that

$$P(X_i, 0) = 1, \tag{4.10}$$

Eq. (4.8) predicts that the pressure satisfies Boyle's law:

$$P(X_i, T) = 1/\eta(T) \qquad (T \geq 0) \tag{4.11}$$

throughout the interior of the film. Thus, the pressure must drop (or jump) discontinuously to ambient at the bearing periphery. Since Eq. (4.8) is derived from Eq. (2.9) simply by setting $\sigma^{-1} = 0$, so that the bearing is *precisely* a bellows, it might be hoped that the next order of approximation predicts a continuous pressure leakage at the periphery, but this is not the case. If we set

$$P(X_i T) = 1/\eta(T) + \sigma^{-1}\varpi(X_i, T), \tag{4.12}$$

substitute Eqs. (4.9) and (4.12) into Eq. (2.9), and neglect terms of the second degree and higher in σ^{-1}, we obtain

$$\partial(\varpi\eta)/\partial T = 0 \tag{4.13}$$

with the same difficulties as before.

The problem, of course, is that perturbation on σ^{-1} is a singular perturbation: Eliminating the term involving σ^{-1} from Eq. (2.9) reduces its order. It would seem, then, that the pressure does indeed obey Boyle's law, except that in narrow boundary layers near the periphery, it changes steeply but continuously to ambient. Even this concept must be treated cautiously, however, for the bearing leakage is a continuing process: The boundary layers tend to diffuse away, as illustrated in Sec. IV by the exact solution for the semi-infinite film.

Mathematically, the problem is twofold. First, passing from Eq. (2.9) to Eq. (4.8) involves, as far as spatial derivatives are concerned, a drop in order of 2, not 1. Also, in (X_1, X_2, T) space, the boundary layers run parallel to the characteristics of the reduced equation. Both difficulties tend to obstruct a singular perturbation approach. The problem invites attention.

5. Small, periodic variation of the gap between infinitely long parallel plates. Consider an infinitely long bearing of uniform width B. We can choose a Cartesian coordinate system so that the projection of this bearing on the $x_3 = 0$ plane is

$$\begin{Bmatrix} -B/2 \le x_1 & \le B/2 \\ -\infty < x_2 & < \infty \end{Bmatrix}. \tag{5.1}$$

The one-dimensional squeeze-film equation (2.11) is then applicable.

Assume that the surfaces of the bearing are parallel plates and that the gap between them varies according to

$$h = h_0(1 + \delta \cos \omega t), \tag{5.2}$$

where the constant δ is small compared with unity. We then expect that the deviation of pressure from ambient will be of order δ. Thus,

$$p = p_a[1 + \delta\Pi + O(\delta^2)]. \tag{5.3}$$

We now substitute Eqs. (5.2) and (5.3) into Eq. (2.11) and neglect terms of the second degree or higher in δ. In terms of the normalized variables, we obtain

$$\frac{\partial^2 \Pi}{\partial X^2} = \sigma \frac{\partial \Pi}{\partial T} - \sigma \sin T. \tag{5.4}$$

Since the pressure is ambient at the bearing edges,

$$\Pi(\pm 1/2, T) = 0. \tag{5.5}$$

We seek a steady-state solution to Eq. (5.4) subject to Eq. (5.5) by assuming

$$\Pi(X, T) = \Pi_1(X) \cos T + \Pi_2(X) \sin T. \tag{5.6}$$

Substituting into Eq. (5.4), we obtain

$$\left(\sigma \Pi_2 - \frac{d^2 \Pi_1}{dX^2}\right) \cos T = \left(\sigma + \sigma \Pi_1 + \frac{d^2 \Pi_2}{dX^2}\right) \sin T. \tag{5.7}$$

If this equation is to hold for all values of T, both sides must vanish identically. Similarly, the boundary conditions expressed by Eq. (5.5) are satisfied if and only if both Π_1 and Π_2 vanish at $X = \pm 1/2$. Thus Π_1 and Π_2 are determined by the pair of differential equations

$$\frac{d^2\Pi_1}{dX^2} - \sigma\Pi_2 = 0, \tag{5.8}$$

$$\frac{d^2\Pi_2}{dX^2} + \sigma\Pi_1 + \sigma = 0 \tag{5.9}$$

subject to

$$\Pi_1(\pm 1/2) = \Pi_2(\pm 1/2) = 0. \tag{5.10}$$

A variety of elementary methods exist for solving this system. The route entailing the least manipulation probably results from combining (5.8) and (5.9) into one second order equation for the complex variable $(\Pi_1 - i\Pi_2)$. Carrying out the details leads us to

$$\Pi_1(X) = \frac{2}{\cosh k + \cos k}\left(\cosh\frac{k}{2}\cos\frac{k}{2}\cosh kX\cos kX + \sinh\frac{k}{2}\sin\frac{k}{2}\sinh kX\sin kX\right) - 1, \tag{5.11}$$

$$\Pi_2(X) = \frac{2}{\cosh k + \cos k}\left(\sinh\frac{k}{2}\sin\frac{k}{2}\cosh kX\cos kX - \cosh\frac{k}{2}\cos\frac{k}{2}\sinh kX\sin kX\right), \tag{5.12}$$

where

$$k = (\sigma/2)^{1/2}. \tag{5.13}$$

The squeeze film force is given by

$$F = \int_{-1/2}^{1/2} (P - 1)\, dX. \tag{5.14}$$

where, at any specific time, the magnitude of F is the average loading pressure (gage) in atmospheres. Thus, with Eqs. (5.3) and (5.6),

$$F/\delta = \cos T \int_{-1/2}^{1/2} \Pi_1(X)\, dX + \sin T \int_{-1/2}^{1/2} \Pi_2(X)\, dX. \tag{5.15}$$

With (5.11) and (5.12),

$$F/\delta = -f_1(\sigma)\cos T + f_2(\sigma)\sin T, \tag{5.16}$$

where

$$f_1(\sigma) = 1 - \frac{1}{k}\frac{\sinh k + \sin k}{\cosh k + \cos k}, \tag{5.17}$$

$$f_2(\sigma) = \frac{1}{k}\frac{\sinh k - \sin k}{\cosh k + \cos k}. \tag{5.18}$$

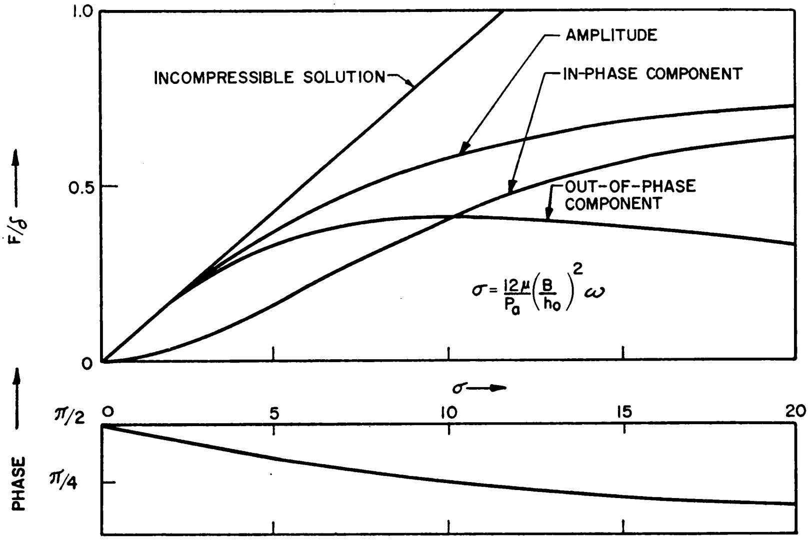

FIG. 5.1. Gas Squeeze-Film Force Amplitude and Phase for Infinitely Long Plane Parallel Plates, $0 \leq \sigma \leq 20$.

Equivalently,

$$F/\delta = -A(\sigma) \cos [T + \varphi(\sigma)], \tag{5.19}$$

where

$$A = (f_1^2 + f_2^2)^{1/2}, \tag{5.20}$$

$$\varphi = \arctan (f_2/f_1). \tag{5.21}$$

For small values of σ, the in-phase and out-of-phase components of the force are given by

$$f_1(\sigma) = \sigma^2/120 + \cdots , \tag{5.22}$$

$$f_2(\sigma) = \sigma/12 + \cdots . \tag{5.23}$$

Thus, the amplitude of F/δ for small σ is very nearly equal to $f_2(\sigma)$ and the force leads the motion by nearly 90°.

For very large σ,

$$f_1(\sigma) \sim 1 - (2/\sigma)^{1/2}, \tag{5.24}$$

$$f_2(\sigma) \sim (2/\sigma)^{1/2}, \tag{5.25}$$

so that the force and the motion are almost in phase.

Figure 5.1 illustrates the force-frequency curve in the range where the most significant behavior is observed; the solution for an incompressible film is included. Figure 5.2 illustrates the force-frequency curve over the full range.

6. Small, periodic variation of the gap between parallel disks. We now turn our attention to the squeeze film between two parallel, coaxial disks of radius B. It is clearly most convenient to use the polar coordinate system introduced in Sec. II. The projection of the bearing on the $X_3 = 0$ plane is

$$R \leq 1. \tag{6.1}$$

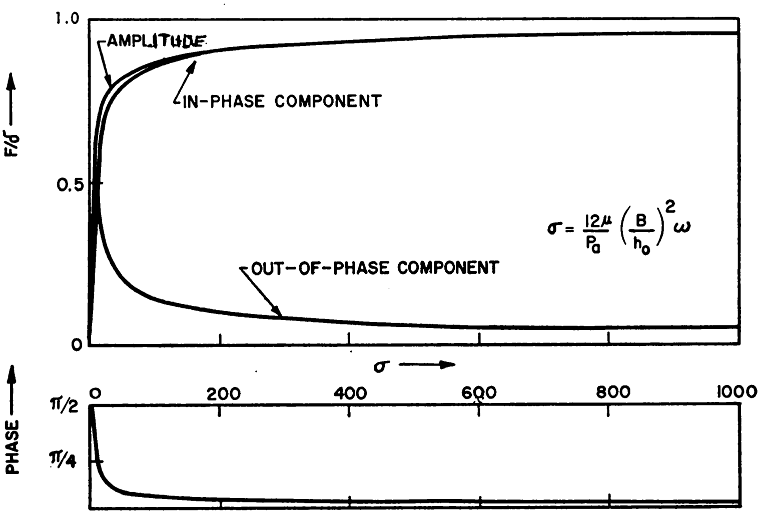

FIG. 5.2. Gas Squeeze-Film Force Amplitude and Phase for Infinitely Long Plane Parallel Plates, $0 \leq \sigma \leq 1000$.

Because of the axial symmetry, the pressure between the plates is governed by the form of the squeeze-film equation given in Eq. (2.10), subject to the boundary condition

$$P(1, T) = 1, \tag{6.2}$$

and to the restriction

$$P(0, T) \quad \text{finite.} \tag{6.3}$$

We note in passing that the pressure is governed by this same equation and these same boundary conditions when either or both of the disks rotate about their mutual axis. This can be seen from Eq. (2.8): For such rotation, steady or time-dependent, U_R and U_R' vanish. Because of this fact, the solution to Eqs. (2.10), (6.2) and (6.3) for small, sinusoidal variation of H has already appeared in the literature. At the Ninth International Congress of Applied Mechanics (Brussels) 1956, Professor M. Reiner of the Israel Institute of Technology demonstrated an instrument for measuring the non-Newtonian properties of air. A disk 6.7 cm in diameter was spun at 7,000 rpm opposite a fixed stator disk 0.02 mm away. A manometer connected to a 4-mm hole in the center of the stator recorded pressures which differed qualitatively from the predictions of Newtonian flow theory. Taylor and Saffman [9] investigated the possible effects of engineering imperfections in Reiner's instrument. They calculated the pressure perturbation which would result if the disks were not quite parallel, or if they were vibrating, so that the gap between them varied sinusoidally with time.

If the gap between the disks varies according to

$$h = h_0(1 + \delta \cos \omega t), \tag{6.4}$$

we expect that

$$p = p_a[1 + \delta \Pi + O(\delta^2)]. \tag{6.5}$$

We now substitute Eqs. (6.4) and (6.5) into Eq. (2.10), neglecting the terms of the second degree or higher in δ, and obtain

$$\frac{\partial}{\partial R}\left(R\,\frac{\partial \Pi}{\partial R}\right) = R\sigma\left(\frac{\partial \Pi}{\partial T} - \sin T\right). \tag{6.6}$$

Taylor and Saffman [9] showed that a periodic solution to Eq. (6.6), subject to Eqs. (5.2) and (5.3), is provided by

$$\Pi(R, T) = \Pi_1(R) \cos T + \Pi_2(R) \sin T, \tag{6.7}$$

where

$$\Pi_1(R) = \frac{\operatorname{ber} \sqrt{\sigma} \operatorname{ber} (\sqrt{\sigma R}) + \operatorname{bei} \sqrt{\sigma} \operatorname{bei} (\sqrt{\sigma R})}{(\operatorname{ber} \sqrt{\sigma})^2 + (\operatorname{bei} \sqrt{\sigma})^2} - 1 \tag{6.8}$$

and*

$$\Pi_2(R) = \frac{\operatorname{bei} \sqrt{\sigma} \operatorname{ber} (\sqrt{\sigma R}) - \operatorname{ber} \sqrt{\sigma} \operatorname{bei} (\sqrt{\sigma R})}{(\operatorname{ber} \sqrt{\sigma})^2 + (\operatorname{bei} \sqrt{\sigma})^2}. \tag{6.9}$$

The net force W acting to keep the disks apart is given by

$$W = 2\pi B^2 \int_0^1 R(p - p_a)\, dR. \tag{6.10}$$

From Eqs. (6.5) and (6.7),

$$W' = W/\Pi B^2 p_a = 2\,\delta \cos T \int_0^1 R\Pi_1(R)\, dR + 2\,\delta \sin T \int_0^1 R\Pi_2(R)\, dR. \tag{6.11}$$

To evaluate the integrals appearing in Eq. (6.11), we make use of certain identities involving Bessel functions, summarized here for convenience:

$$\int_0^x x \operatorname{ber} x\, dx = x \frac{d}{dx} (\operatorname{bei} x), \tag{6.12}$$

$$\int_0^x x \operatorname{bei} x\, dx = -x \frac{d}{dx} (\operatorname{ber} x), \tag{6.13}$$

$$\operatorname{ber} x + i \operatorname{bei} x = J_0(i^{3/2}x), \tag{6.14}$$

$$\frac{d}{dx} J_0(x) = -J_1(x), \tag{6.15}$$

$$J_1(i^{3/2}x) = \operatorname{ber}_1 x + i \operatorname{bei}_1 x, \tag{6.16}$$

$$\int_0^x x \operatorname{ber} x\, dx = \frac{x}{\sqrt{2}} (\operatorname{bei}_1 x - \operatorname{ber}_1 x), \tag{6.17}$$

$$\int_0^x x \operatorname{bei} x\, dx = -\frac{x}{\sqrt{2}} (\operatorname{ber}_1 x + \operatorname{bei}_1 x). \tag{6.18}$$

Eqs. (6.17) and (6.18), which are the ones of interest to us, are derivable from Eqs. (6.12) to (6.16).

*A misprint occurs in the Taylor and Saffman paper at this point. The second of their equations (15), which corresponds to our Eq. (6.9), should have one side or the other multiplied by -1.

With Eqs. (6.8) and (6.9), Eq. (6.11) becomes

$$W'/\delta = -g_1(\sigma) \cos T + g_2(\sigma) \sin T, \tag{6.19}$$

where

$$g_1(\sigma) = 1 - \sqrt{\frac{2}{\sigma}} \cdot \frac{\operatorname{ber} \sqrt{\sigma} (\operatorname{bei}_1 \sqrt{\sigma} - \operatorname{ber}_1 \sqrt{\sigma}) - \operatorname{bei} \sqrt{\sigma} (\operatorname{ber}_1 \sqrt{\sigma} + \operatorname{bei}_1 \sqrt{\sigma})}{(\operatorname{ber} \sqrt{\sigma})^2 + (\operatorname{bei} \sqrt{\sigma})^2}, \tag{6.20}$$

$$g_2(\sigma) = \sqrt{\frac{2}{\sigma}} \frac{\operatorname{ber} \sqrt{\sigma} (\operatorname{ber}_1 \sqrt{\sigma} + \operatorname{bei}_1 \sqrt{\sigma}) + \operatorname{bei} \sqrt{\sigma} (\operatorname{bei}_1 \sqrt{\sigma} - \operatorname{ber}_1 \sqrt{\sigma})}{(\operatorname{ber} \sqrt{\sigma})^2 + (\operatorname{bei} \sqrt{\sigma})^2}. \tag{6.21}$$

Equivalently,

$$W'/\delta = a(\sigma) \cos [T + \psi(\sigma)], \tag{6.22}$$

where

$$a = (g_1^2 + g_2^2)^{1/2}, \tag{6.23}$$

$$\psi = \arctan (g_2/g_1), \tag{6.24}$$

The behavior of g_1 and g_2 for small values of σ can be ascertained with the help of Eqs. (6.14) and (6.16), together with Maclaurin series

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^4(2!)^2} - \frac{z^6}{2^6(3!)^2} + \cdots, \tag{6.25}$$

$$J_1(z) = \frac{z}{2}\left[1 - \frac{z^2}{2^2 2!} + \frac{z^4}{2^4 2!\, 3!} - \frac{z^6}{2^6 3!\, 4!} + \cdots\right]. \tag{6.26}$$

By setting $z = i^{3/2}x$, we obtain

$$\operatorname{ber} x = 1 - \frac{x^4}{64} + O(x^8), \tag{6.27}$$

$$\operatorname{bei} x = \frac{x^2}{4} - \frac{x^6}{384} + O(x^{10}), \tag{6.28}$$

$$\operatorname{ber}_1 x = -\frac{x}{2\sqrt{2}} - \frac{x^3}{16\sqrt{2}} + \frac{x^5}{384\sqrt{2}} + O(x^7), \tag{6.29}$$

$$\operatorname{bei}_1 x = \frac{x}{2\sqrt{2}} - \frac{x^3}{16\sqrt{2}} - \frac{x^5}{384\sqrt{2}} + O(x^7). \tag{6.30}$$

Equations (6.20) and (6.21) then give us

$$g_1(\sigma) = \sigma^2/48 + \cdots, \tag{6.31}$$

$$g_2(\sigma) = \sigma/8 + \cdots, \tag{6.32}$$

To investigate the behavior of g_1 and g_2 for large values of σ, we employ the asymptotic formulas

$$J_n(z) \sim (2/\pi z)^{1/2} \cos (z - n\pi/2 - \pi/4), \qquad |\arg z| < \pi. \tag{6.33}$$

With Eqs. (6.14) and (6.16), we find, after some lengthy but straightforward calculations,

$$\operatorname{ber} x \sim (2\pi x)^{-1/2} e^{\sqrt{x}/2} \cos (x/2^{1/2} - \pi/8), \tag{6.34}$$

$$\operatorname{bei} x \sim (2\pi x)^{-1/2} e^{\sqrt{x}/2} \sin (x/2^{1/2} - \pi/8), \tag{6.35}$$

$$\operatorname{ber}_1 x \sim -(2\pi x)^{-1/2} e^{\sqrt{x}/2} \sin (x/2^{1/2} - \pi/8), \tag{6.36}$$

$$\operatorname{bei}_1 x \sim (2\pi x)^{-1/2} e^{\sqrt{x}/2} \cos (x/2^{1/2} - \pi/8). \tag{6.37}$$

Equations (6.20) and (6.21) then yield, for large σ,

$$g_1(\sigma) \sim 1 - (2/\sigma)^{1/2}, \tag{6.38}$$

$$g_2(\sigma) \sim (2/\sigma)^{1/2}. \tag{6.39}$$

It is not surprising that, as σ approaches infinity, so that the effect of leakage becomes less and less, the bearing force has the same functional form for the circular disks as it did for the infinitely long bearing. That the coefficients agree is coincidence: If, for the breadth B, we were to select the disk diameter instead of the radius, Eqs. (6.38) and (6.39) would undergo an obvious modification.

References

1. Gross, W. A., *Unsteady bearing films and bearing systems*, IBM Research Report RJ-117-8 (1960)
2. Tipei, N., *Ecuaţiile Lubrificaţiei cu Gaze*, Communicările Acad. R. P. Romîne, **4**, 699 (1954)
3. Elrod, H. G., *A derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties*, Q. Appl. Math. **17**, 349-59 (Jan. 1960)
4. Review No. 3206, Appl. Mech. Rev. **13**, 457 (1960)
5. Comolet, R., *Ecoulement d'un fluide entre deux plans paralleles: Contribution a l'étude des buteés d'air*, Publications Scientifiques et Techniques, No. 334 (1957)
6. Licht, L. and Elrod, H. G., *An Analytical and Experimental Study of the Stability of Externally Pressurized, Gas-Lubricated Thrust Bearings*. Interim Report I-A2049-12, The Franklin Institute (1961)
7. Crank, J., *The mathematics of diffusion*, Oxford (1956)
8. Wagner, C., *Diffusion of lead chloride dissolved in solid silver chloride*, J. Chem. Phys. **18**, 1229 (1950)
9. Taylor, G. I., and Saffman, P. G., *Effects of compressibility at low Reynolds number*, J. Aero. Sci. **24**, 553 (1957)