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BOOK REVIEWS

Cartesian tensors—an introduction. By G. Temple. Methuen & Co. Ltd., London, and John Wiley & Sons, Inc., New York, 1961. vii + 92 pp. $2.75.

This is an odd, irritating, and excellent book. The oddness is in insistence on modern points of view even when old-fashioned ones may be simpler. Irritation arises from the fact that the book does not appear to have been proof-read. Once through the distracting maze of misprints and errors, however, it becomes clear that there has been a gap in the literature, and that this book, in a second edition, may fill it.

The first four chapters present the basic theorems of tensor algebra and calculus, with scarcely a mention of transformation rules. Through a number of physical and geometrical examples, Professor Temple convinces the reader that tensors are multilinear functions of direction, and defines them as such. The transformation rules for components then appear as a simple and ignorable consequence of the definition. The stubborn and successful refusal to make any use of transformation rules accounts for a good deal of the interest of the book.

Isotropic tensors are derived by studying what in the older terminology would be called orthogonal invariants of vectors. In view of Professor Temple's definition of a tensor, this method of derivation is the natural one, and it is possibly for this reason that no reference is made to the work of Rivlin and Smith, who have used the method extensively. It is unfortunate that the full power of their method is not used, since the reader is left with the impression that derivation of isotropic tensors of any given rank is a separate and complicated job.

The structure of second-rank symmetric tensors is discussed in terms of the spectral theory of operators. The treatment of this subject is exceptionally clear and simple. Also in the line of subjects slanted toward quantum theory, there is a chapter on spinors which I am not qualified to comment on. The book closes with a chapter on orthogonal curvilinear coordinates. There are a few problems, always well-chosen, to prepare the student for material coming later in the book.

Some of the misprints are unnervingly consistent. For example, "Kronecker" is always spelled "Knonecker". The stress tensor is proven symmetric by a calculation which is uniformly wrong. The discussion of invariants of three vectors in three dimensions is misleading, although not literally false. The section on invariants of four vectors is so full of both misprints and real errors as to be useless. In the derivation of strain components in curvilinear coordinates, it is almost never clear, whether the summation convention is in force or not. It is a pity that such a basically good book should be so disfigured.

A. Pipkin

Introduction to the statistical dynamics of automation control systems. By V. V. Solodovnikov. Dover Publications, Inc., New York, 1960. xx + 307 pp. $2.25 (paperbound).

This is an excellent introduction to what might be called the classical theory of control systems. This is to say that it presents a thorough discussion of linear systems with random forcing terms and quadratic criteria for performance. The resulting linear variational equations can then be thoroughly analyzed by means of Laplace transform and complex variable techniques. The results are those of Kolmogorov, Khinchin and Weiner, with the extensions of James, Phillips, Zadeh and others.

The book is particularly useful because of its combination of a large number of illustrative examples and quite reasonable price.

Richard Bellman

Time series analysis. By E. J. Hannon. Methuen & Co. Ltd., London, and John Wiley & Sons, Inc., New York, 1960. viii + 152 pp. $3.50.

This small monograph is one of a series on applied probability theory and statistics. The statistical models discussed in the volume are real-valued processes $y_t = m_t + x_t$ with x_t a stationary residual and m_t a mean value term. The first chapter considers the structure of such processes with the mean value term identically zero. The concept of stationarity is introduced. The spectral representation for

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stationary processes is derived in a heuristic manner. Some of the basic results in the prediction problem for stationary processes are cited but are not derived.

The second chapter considers estimation of the correlogram and parameters of finite parameter schemes. Results on the asymptotic behavior of the statistical distribution of quadratic estimates of covariances of linear processes are given.

Estimation of the spectral density and distribution functions are dealt with in the following chapter. These are again of an asymptotic character. Various spectral weight functions are discussed and some examples are given to illustrate the discussion. Chapter four is concerned with hypothesis testing and confidence intervals. There is some discussion of the problem of resolving jumps in the spectrum. Tests of goodness of fit and confidence regions for the spectral distribution function are obtained.

In the last chapter the condition that $m_t \equiv 0$ is relaxed. The mean value m_t is assumed to be of a regression form with a finite number of unknown regression coefficients. Here the question of estimating the regression coefficients by linear methods is dealt with.

The author has covered many of the recent results in time series analysis, particularly those on spectral analysis. The discussion is heuristic at times but this is inevitable in a book of this size. One can certainly say that it is an interesting addition to this series on probability and statistics.

M. Rosenblatt

Tables of the hypergeometric probability distribution. By Gerald J. Lieberman and Donald B. Owen. Stanford University Press, 1961. vi + 726 pp. $15.00.

This book presents tables of both the point and cumulative probabilities of the hypergeometric distribution

$$p(N, n, k, x) = \binom{n}{x}\binom{N-n}{k-k} \Big/ \binom{N}{N-k}$$

$$\max\,[0, n - k - N] \leq x \leq \min\,[n, k]$$

to six decimals for $N = 1(1)49, 50(10)100$ and $100(100)2000$.

Applications to sequential analysis, distribution of the number of exceedances, Bayesian prediction with a binomial distribution, and to sampling inspection are discussed in the introduction, as are various approximations to the distribution and interpolation procedures for values of N not listed.

W. Freiberger

Tables of Airy functions and special confluent hypergeometric functions. By A. D. Smirnov. Translated from the Russina by D. G. Fry. Mathematical Tables Series: Vol. 7. Pergamon Press Inc., New York, Oxford, London, Paris, 1960. vii + 260 pp. $15.00.

The twenty-four page introduction concerns the definitions and properties of the functions to be tabulated, the calculation of the tables, rules for the use of the tables, and an example of the use of the tables for the solution of a differential equation. The tabulated functions $U(s, 1)$, $U(s, \alpha)$ and $V(s, p_0)$ are linearly independent solutions of the following three ordinary linear differential equations:

the Airy equation $U''(s) + sU(s) = 0$,

the generalized Airy equation $U''(s) + s^{\alpha}U(s) = 0$,

the confluent hypergeometric equation $V''(s) + [p_0(2 - p_0)s^{-1/2} + 4s^{-1}]\, V(s)/4 = 0$.

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In addition the following three integrals of the two linearly independent functions $U_1(s, 1)$ and $U_2(s, 1)$ are tabulated:

$$V_{11}(s) = \int_0^s [U_1^2(s, 1) - (2\pi s^{1/2})^{-1}3^{1/2}\Gamma^2(2/3)]\, ds,$$

$$V_{12}(s) = \int_0^s [U_1(s, 1)U_2(s, 1) - (3s)^{-1/2}/2]\, ds,$$

$$V_{22}(s) = 1 + \int_0^s [U_2^2(s, 1) - (2\pi(3s)^{1/2})^{-1}\Gamma^2(1/3)]\, ds.$$

The tables are divided into three parts. Five-figure values of the functions $U_1(s, 1)$, $U_2(s, 1)$, $U_1'(s, 1)$ and $U_2'(s, 1)$ and their second differences and also the integrals $V_{11}(s, 1)$, $V_{12}(s, 1)$, $V_{22}(s, 1)$ are given in the first part for $s = 0(0.01)10.0$. The same part contains five-figure values of the functions $U_1(s, 1)$, $U_2(s, 1)$, $U_1'(s, 1)$ and $U_2'(s, 1)$ and their second differences for $s = -6.0(0.01)0$. Four-figure values of the functions $U_1(s, \alpha)$ and $U_2(s, \alpha)$ and their derivatives are given in the second part of the tables for $s = 0(0.01)6.0$ for the following values of α: $\pm1/4$, $\pm1/3$, $\pm1/2$, $\pm2/3$, $\pm3/4$, $+5/4$, $+4/3$, $+3/2$, $+5/3$, $+7/4$, $+2$. Four-figure values of the functions $V_1(s, p_0)$ and $V_2(s, p_0)$ and their derivatives are given in the third part of the table for $s = 0(0.01)10.0$ for fixed values of $p_0 = 0.1(0.1)1.0$.

T. H. Hughes

Tables of Thomson functions and their first derivatives. By L. N. Nosova. Pergamon Press, New York, Oxford, London, Paris, 1961. 422 pp. $20.00.

Thomson (or Kelvin) functions of the first kind ber x, bei x, the second kind ker x, kei x, and the third kind her x, hei x, are defined by the following equations:

$$\text{ber}_n\, x + i\, \text{bei}_n\, x = J_n(xi\sqrt{i}) = i^n I(x\sqrt{i})$$

$$\text{ker}_n\, x + i\, \text{kei}_n\, x = i^{-n} K_n(x\sqrt{i})$$

$$\text{her}_n\, x + i\, \text{hei}_n\, x = H_n^{(1)}(xi\sqrt{i})$$

where $I_n(t)$ and $K_n(t)$ are Bessel functions of an imaginary argument and $H_n^1(t)$ are Hankel functions of the first kind.

The tables give the numerical values of ber x, bei x, ker x, kei x, and their first derivatives. The values of the functions of the third kind can easily be obtained from the identities

$$\text{her}\, x = \frac{2}{\pi}\text{kei}\, x, \qquad \text{hei}\, x = -\frac{2}{\pi}\text{ker}\, x.$$

The function values are tabulated for $0 \le x \le 10$, for $10 \le x$ 100 ber x and bei x and their first derivatives are multiplied by $e^{-(x/\sqrt{2})}$, ker x and kei x and their first derivatives by $e^{x/\sqrt{2}}$. The tabulations are in intervals 0.01 of the argument.

The tables were prepared on a "Strela" electronic computer.

W. Freiberger

Die Momentenmethode in der angewandten Mathematik, By J. W. Worobjow. VEB Deutscher Verlag der Wissenschaften, Berlin, 1961. viii + 143 pp. $5.47.

The method of successive approximation for the solution of the linear equations arising in problems of mechanics and physics goes back to Liouville and C. Neumann. It has received increased attention in recent years since with the advent of high-speed computers the simplicity of the calculation schemes to which it gives rise have tended to make up for its disadvantages, such as slow convergence.

This work treats the class of those iteration procedures which depend on a variational principle. By the use of the theory of linear operators in Hilbert space, this class is unified in the method of moments. If A is an operator in Hilbert space, and z_0 an element of the space, the sequence

$$z_i = A^i z_0 \qquad (i = 0, 1, \cdots, n)$$

is formed. Now A is replaced by the operator A_n in n-space such that

$$z_i = A_n^i z_0 \qquad (i = 0, 1, \cdots, n - 1)$$

and

$$E_n z_n = A_n^n z_0$$

where E_n projects into the space of $z_0, z_1, \cdots z_{n-1}$. It can be shown that

$$A_n = E_n A E_n .$$

This method thus, for instance, limits the choice of functions ϕ_k into which the required function

$$x = \sum_1^\infty a_k \phi_k$$

is expanded in Galerkin's method.

This general framework is applied to various problems, such as the solution of differential and integral equations, and general convergence theorems are proved. Some knowledge of the spectral theory of self-adjoint operators is required for a deeper understanding of some parts of the book, but the excellence and elegance of the presentation would make the reading of the book a pleasure for any applied mathematician.

W. Freiberger

Handbook of numerical methods for the solution of algebraic and transcendental equations. By V. L. Zaguskin. Translated from the Russian by G. O. Harding. Pergamon Press, New York, 1961. xix + 195 pp. $6.50.

The Russian edition of this text was published in Moscow in 1960. The contents are best indicated by the following selective list of chapter and section headings: Polynomials and transcendental functions (root of a function, basic properties of polynomials, calculational schemes for the multiplication and division of polynomials, Horner's scheme)—Operations with approximate numbers—Approximate determination of roots (Lobachevskii's method, Bernoulli's method, method of iteration, Lin's method, Paluver's method)—Making more accurate roots already found (linear interpolation, Newton's method, Berstoi's method, Belostotskii's method, iteration with quadratic convergence, methods of improving convergence, comparison of methods)—Solution of equations of low order—Solution of simultaneous equations (iteration, Newton's method, steepest descent, comparison of methods of solution of simultaneous non-linear equations, methods of solution of simultaneous linear equations)—Table for the solution of cubic equations.

As these samples indicate, the translation is not always smooth and names of authors are often misspelled (Berstoi for Bairstow, Smil for Smeal, Zeidel for Seidel, etc.).

W. Prager

Analysing qualitative data. By A. E. Maxwell. John Wiley & Sons, Inc., New York, and Methuen & Co., Ltd., London, 1961. 163 pp. $3.00.

This book presents a clear and simple account of modern statistical techniques for the analysis of qualitative data. Little mathematics is presupposed and no attempt is made to explore general principles underlying the methods, which are demonstrated mainly by examples from psychology and psychiatry. There are chapters on goodness of fit tests, rank correlation, item analysis, classification procedures based on Bayes's theorem and decision theory, but the main body of the book is devoted to contingency tables. Each chapter contains a full and up-to-date bibliography and the book should be of excellent service to research workers to whom it is addressed.

W. Freiberger

Einführung in die Kontinuumsmechanik. By William Prager. Birkhäuser Verlag, Basel und Stuttgart, 1961. 228 pp. $7.54.

This is a very interesting book on the fundamentals of continuum mechanics with introductions to several special fields. The author's principal purpose was to provide a unified foundation which should precede the detailed studies of these fields. To this end he adopted a very general point of view and used an adequate uniform mathematical instrument.

Although in the beginning the symbolic notations for vectors and tensors are employed beside the suffix notations with the summation convention in Cartesian coordinates, the emphasis in the main part of the book is placed on the latter. As a matter of fact, the symbolic notation is elegant to use in simple cases like rigid body motion, but becomes cumbersome when applied to complicated types of deformable solids. On the other hand, as the three dimensional space occupied by a continuum is considered to be Euclidian, and as this introduction does not deal with special problems or fields like for instance the shell theory, tensors in curvilinear coordinates were not necessary. These coordinates would only have involved useless complications, rendering the text difficult for the beginner.

The first part is characterized by a clear separation between mathematical preliminaries (Chapter I), statical considerations (Chapter II), kinematical considerations (Chapter III), and the presentation of the basic equations which are common to all types of materials, like the conservation of mass, the equations of motion and the energy theorem (Chapter IV). The basic relations are not derived from infinitesimal elements, but are obtained more conveniently by the use of integral theorems.

The remaining part is devoted to the discussion of a variety of isotropic continua, characterized by different constitutive laws. In general, students learn first the linear theory of elasticity for small deformations and separately the theory of perfect fluids, because these fields are relatively simple and of prime importance in the applications. Studies of more complicated materials are then treated more or less along similar lines. The general standpoint taken here however, including nonlinearity and large deformations of solids, required another, unique and more adequate order placing elasticity at the end. This actually marks the book as a very attractive and original one. First, those constitutive laws are considered, where large deformations do not appear explicitly, because the stresses are connected to the strain rates. By a successive generalization as far as the presence of shear stresses is concerned, there follow after the perfect fluids (Chapter V) the viscous fluids (Chapter VI) and then the visco-plastic solids (Chapter VII). In these chapters some notions on boundary layers and on perfect plasticity are given. Then comes in Chapter VIII the wide class of hypoelastic solids characterized by relations between stress rates and strain rates. After an introduction to finite deformations (Chapter IX) the book ends by a successive specification of the property "elastic" (Chapter X). Ordinary elasticity for small deformations appears as hypoelasticity in the neighborhood of the stress-free state, whereas for large deformations it is termed hyper-elasticity, containing the maximum of elastic properties, in particular adiabatic movement.

The chapter on finite deformations seems to touch to the limit of an introduction to continuum mechanics. At the same time, one feels that the instrument of Cartesian tensors is here at the limit of its ability. For instance, the strain tensors of Almansi and Green contain some metric tensors which would be better treated by the general curvilinear tensor calculus.

The book is written in a clear style, is easy to understand and contains a set of carefully selected exercises at the end. Further, it is based upon a great number of very recent papers as well as outstanding papers of the past going back to the last century. It is therefore difficult for this reviewer to give a full appreciation of the value of the book in these few lines. It represents the author's great experience in research and teaching of continuum mechanics, concentrated in a volume with a relatively small number of pages.

WALTER SCHUMANN

Adaptive control processes: a guided tour. By Richard Bellman. Princeton University Press, New Jersey, 1961. xvi + 225 pp. $6.50.

It has become apparent within the past few years that the theory of automatic control is undergoing a renaissance. The "new automatic control" involves advanced mathematical concepts and a sophistication undreamed of ten or even five years ago and has provided an excellent source of stimulation for mathematicians and theorists on both sides of the iron curtain. In turn now the new theory being developed will aid in the design of physical devices. As yet this new theory has resulted in few if any actual engineering systems. It seems apparent however, that these will soon be forthcoming.

A hierarchy of automatic control might be constructed, with open loop control occupying the lowest place. By far the great majority of industrial control is of this sort. Conventional closed loop control comes next in line. There are a significant number of such systems in practical use today and Wiener has pointed to this fact as heralding the beginning of a new and greater industrial revolution. Above conventional closed loop control may be placed adaptive control since rather than being designed in accordance with its predicted environment and a specific task in mind as is conventional control, the adaptive control is allowed to organize itself, in a restricted sense, so as to yield optimum performance with respect to one or more indices of performance selected by the designer, under a variety conditions perhaps not completely foreseen. There are very few such systems operating under practical conditions today. Finally we might envisage a still higher level of automata that are equipped with a generalized goal along with the ability to profit by repeated experience and which generate their own indices of performance. It might be said that such devices exhibit certain aspects of "learning". This reviewer knows of no examples of such devices in practical use today although a number of more or less trivial laboratory toys have been constructed in the past and a few such generalized devices are under study today. The book at hand is an engaging, speculative and philosophical discourse on various and sundry matters concerned with the two highest of the four forms of automatic control discussed above.

This is an edited and expanded version of a series of lectures on the modern concept of automatic control theory given by Bellman in 1959. Fortunately the editing has not meant the removal of the flavor and wit. This book should not be taken as a formal text or treatise on automatic control in general nor dynamic programming or adaptive control in particular. Rather it is a vigorous, philosophical presentation of a number of mathematical concepts which will be found useful in the study of adaptive control. The reader will find a very stimulating, informal discussion of the calculus of variations vs. dynamic programming as applied to modern optimum control. The principle of optimality, which lies at the heart of Dynamic Programming, provides, in concept, a solution to the classical two-point boundary-value problem. It has been pointed out in recent years that the optimum control problem can be formulated in these terms, thus we see the reason for the interest of engineers in this subject. Unfortunately the computational aspects of the problem as formulated in terms of Dynamic Programming are not trivial. In fact it is difficult if not impossible with the memory capacity of present day digital computers to solve the problem in systems of higher than third order without the imposition of artificial constraints.

There are a number of chapters expounding the philosophy of dynamic programming and sequential analysis. The reader will find discourses on game theory, communication theory and successive approximations. Some of the shorter chapters, like those on Stochastic Control Processes, Markovian Decision Processes, and Stochastic Learning Models for example are little more than a definition of terms and an annotated bibliography. The discussion of adaptive control itself occupies only about fifteen pages late in the text, and is essentially discursive in nature. We must wait until chapter 15 for a definition of adaptive control and then it is not given in one sentence nor is it operative and it is in a chapter with not a single equation. The list of topics in the table of contents is encyclopedic thus the reader can expect only a page or two at most on each. This is obviously inadequate for the serious worker but it is ideal to an interested newcomer who wishes to be oriented on how the various topics are interrelated.

It is amusing for an engineer to hear a mathematician with an excellent classical training excoriating classical mathematics as only Bellman can. The danger lies with the fact that the naive engineer may be induced to give up what he never really learned. Though it may be true that Mathematics and mathematicians badly need new mathematics to solve new problems, we engineers still have much to learn and apply in the classical mathematics.

This book should not be looked upon as a formal, technical effort and it should probably be kept out of the hands of that overly serious, pedantic creature, the Ph.D. candidate. For every one else concerned with modern mathematics and the "new automatic control"; in fact with any modern science it is enthusiastically recommended. It is guaranteed to be almost as much fun to read as it obviously was to write.

JOHN E. GIBSON

Stability in nonlinear control system. By Alexander M. Letov. Princeton University Press, New Jersey, 1961. 316 pp. $8.50.

This book contains an extensive and detailed discussion of the application of the second method of Lyapunov to study of the stability of some important types of control processes. The ingenious procedures of Lur'e, Chetaev and Letov himself are presented in complete, careful and lucid fashion. In order to view these results in proper perspective, let us recall the classical background of Poincaré-Lyapunov stability theory.

A physical system S, specified at any time t by a finite-dimensional state vector $x(t)$ satisfying the vector differential equations $x' = g(x)$, $x(0) = c$, with $g(0) = 0$, is said to be *stable* provided that $\| c \| \leq \delta = \delta(\epsilon)$ ensures that $\| x(t) \| \leq \epsilon$ for $t > 0$. If $g(x)$ possesses a power series expansion of the form Ax + higher order terms, with A constant, the classic result is that all the roots of A must have negative real parts in order that S be a stable system. This is both a necessary and sufficient condition. This useful criterion fails if $g(x)$ does not have the postulated form, a situation which occurs quite frequently in modern control theory where "on-off" or "bang-bang" elements are commonly used.

To study these more general situations, the second method of Lyapunov is utilized. The method is versatile and supple, but definitely requires ingenuity for its successful use. Letov takes the equation

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} a_{ij}x_j + b_i u(t), \qquad i = 1, 2, \cdots, N,$$

$$c_1 \frac{d^2u}{dt^2} + c_2 \frac{du}{dt} + c_3 u = f(v),$$

where

$$v = \sum_{i=1}^{N} d_i x_i - ru,$$

an equation typical of an important class of control systems, and analyzes in detail the work of Lur'e and Chetaev and his own extensions and modifications.

At the end of the book, some further questions are sketched and the need for more powerful methods to provide solutions to the corresponding problems for more general control processes is indicated.

The reader interested in a general discussion of Lyapunov's second method may wish to refer to J. P. LaSalle and S. Lefschetz, *Stability by Lyapunov's Direct Method with Applications*, Academic Press, to appear in 1961; for a discussion of control processes in which $f(v)$ is chosen optimally, he may consult R. Bellman, *Adaptive Control Processes: A Guided Tour*, Princeton University Press, 1961.

The book is recommended for all those studying the theory and application of control processes, and the translator, J. G. Adashko, is to be commended for the flowing version he has produced.

RICHARD BELLMAN

Elementary differential equations. By William Ted Martin and Eric Reissner. Second edition. Addison-Wesley Publishing Co., Inc., 1961. xiii + 331 pp. $6.75.

This is the second edition of an excellent introduction to the theory of differential equations. Particularly good features are a first chapter devoted to the physical and geometric origin of equations of this type, a large number of worked out examples, a chapter on the method of successive approximations, and a chapter on difference equations. The entire book is written in a pleasant expository style and attractively printed.

RICHARD BELLMAN

Mathematical biophysics: physico-mathematical foundations of biology. By N. Rashevsky. Volume I. Dover Publications, Inc., New York, 1960. xxvi + 488 pp. $2.50.

The present 2-Volume edition (950 pages) constitutes the 2nd revision of a book that grew from 340 pages in 1938 to 669 pages in 1948. Rashevsky feels, nevertheless, that the present 3rd edition presents a less adequate picture of contemporary mathematical biology than the 1st edition did two decades ago.

The 1st Volume of the present edition is made up of two parts: approximately 3/4 of the Volume are taken up by a mathematical treatment of the behavior of vegetative cells and cellular aggregates. With the exception of chapters 19 and 28, this first part is almost entirely based on "macroscopic" concepts of classical physics. Whatever molecular biophysics is introduced (in conjunction with some elementary notions of information theory) can hardly do more than alert the reader that there are now extant experimental data and theoretical developments that were clearly not available when Rashevsky laid the foundations of his approach.

The final 1/4 of the 1st Volume gives us Rashevsky's classical calculations on excitation and conduction in peripheral nerves, together with a discussion of Blair's and Hill's theories. The last page of text contains a paragraph that refers to the Hodgkin-Huxley approach but it is again obvious that the experimental substratum of Rashevsky's theorizing does not include the important new discoveries of the period after World War II.

There is a rather sharp break between Volume I, which reflects essentially the book's 1st edition, and Volume II, which is much more contemparary in flavor. Here the concern is with the central nervous system, with such topics as conditioned reflexes and learning, discrimination of relations, the Gestalt problem, and finally with the Boolean algebra of neural nets (starting with the McCulloch-Pitts article in 1943) and properties of random nets (based upon the Shimbel-Rapoport approach).

The last 16 chapters of this Volume are grouped into a section entitled *General Mathematical Principles of Biology*. A discussion of form in relation to structure and function, such as locomotion, leads to the formulation of a principle of optimal design. Set-theoretical and combinatorial (graph) approaches to a topological biology are compared and there is also a chapter in which the author speculates on a possible geometrization of biology.

From what precedes, it should be clear that these two Volumes constitute a monumental and pioneering opus with respect to the subject matter of the life sciences. Elsewhere, Rashevsky has tried to extend this approach to the behavioral sciences. Rashevsky has shown that there is very little in classical biology that has remained foreign to him. He has also—and this is probably more important—introduced, often through his many capable disciples and associates, mathematical thinking into areas that have seemed immune from it for decades, if not centuries. Whether Rashevsky's mathematical approach will ultimately turn out to have been the correct one, becomes thus almost irrelevant.

W. A. ROSENBLITH

Introduction to mechanics of continua. By William Prager. Ginn and Company, Boston, New York, Chicago, 1961. x + 230 pp. $8.00.

Professor Prager's newly published book is a highly remarkable unified presentation of mechanics of continua. As far as applied mathematics are concerned, it approaches perfection: judiciously organized in ten chapters, it combines preciseness, conciseness, and clarity. The very well presented book has plenty of text to assist formulas every time it proves necessary, and contains also numerous exercises, giving to the reader the opportunity to test and deepen his knowledge of the subject.

Chapter I deals with the geometrical foundations of the theory. The author succeeds in showing clearly the respective merits of symbolic and tensor notations, using both in a proportion which appears to form naturally, and establishing their connections all along the chapter. This balanced point of view is maintained in the remainder of the book, where tensor notation of course predominates but without complete disappearance of symbolic notation.

We have then one chapter on the state of stress, one chapter on the instantaneous motion, and one chapter on fundamental laws applying to all kinds of continua.

To follow, chapters V to VII are devoted to special types of continua: perfect fluids, viscous fluids, visco-plastic and perfectly plastic materials.

Chapters VIII and X deal with a continua presenting properties of elasticity. The author introduces there a new terminology to distinguish three types of continua he calls hypo-elastic, elastic and hyper-elastic, due to their increasing "elasticity". This new terminology seems questionable (see R. HILL, Jour. of Mech. Phys. Solids, 9, 3 July 1961, pp. 210-211).

The reviewer especially appreciated chapter IX, included between the two chapters devoted to elasticity, in which the analysis of finite strains is developed.

Except for the concept of tensor, which is introduced inductively from the simpler concepts of scalar and vector, the presentation is very generally deductive, particular cases being treated as simplifications of the more general case considered before. Moreover, because of its conciseness, this book could not be assimilated by, say, normally trained students in engineering of the European type, without a large amount of effort. To get acquainted for the first time with mechanics of continua, the reviewer would personnally prefer an inductive presentation, with as much emphasis as possible on physical aspects of the phenomena. For instance, to prove the symmetry of the stress tensor, he would prefer using the equations of equilibrium of rotation of an elementary parallelepiped to showing that the dual vector of the stress tensor vanishes.

On the other hand, the reviewer is convinced that Professor Prager's book will prove an exceptionnally valuable tool of synthesis for those who want either to clarify and unify their ideas on mechanics of continua, or to start from a more solid basis to progress in that field.

M. SAVE

Markov learning models for multiperson interactions. By Patrick Suppes and Richard C. Atkinson. Stanford University Press, California, 1960. xii + 296 pp. $8.25.

The authors describe the purpose and subject matter of their book as follows:

> "The common part of three disciplines, all actively developed within the last two decades, forms the subject matter of this book. One of these disciplines is the study by social psychologists of individuals interacting in small groups; the second is the study of learning as a stochastic process; and the third is the study of games from the standpoint of a mathematical theory of strategy. Our primary concern is with the application of learning theory to small group experiments that closely resemble game situations." (p. 1)

In general, the book is a description of a highly formalized and rigorously conceived learning theory, exactly applied and systematically tested in a number of well-structured experimental routines. The authors concentrate on learning theory, with game theory and social interactions serving mainly to determine the structure of the experiments.The analyses of data and the theory developed to describe the data are in the tradition of learning theory and in particular follow statistical learning theory as developed by Drs. Estes and Burke. Although learning theory of the stimulus-response type predominates, the empirical results of the game-theory and social-interaction experiments will be of interest to game theoreticians and social psychologists.

Some historical perspective is necessary to fully appreciate the importance of this book to learning theory. Since 1950, the trend in learning theory has been toward more quantitative and more limited psychological theories; this book represents a continuation of the trend. Stochastic models have provided the foundations for most of these theories. Two theories, basically different but converging in manner of application, have predominated. One theory describes the changes in behavior (the probability of a response) by linear operators. This is the approach taken by Bush and Mosteller in their book *Stochastic Models for Learning.* A second approach is statistical learning theory (SLT) which has been described in a number of papers by Estes and Burke.

Like SLT, the theory considered by Suppes and Atkinson—stimulus-sampling theory (SST)—developed from considering stimulus-sampling processes in which the number of elements available for sampling was relatively small. Two subdivisions of the theory have been considered. One is similar to the original formulation of SLT in which a variable number of elements are sampled on each trial. The other assumes a fixed sample size, the simplest case being one element sampled on each trial. This latter case, one element sampled on each trial, is the type of theory considered by Suppes and Atkinson.

The book begins with an introduction to stimulus-sampling theory in which the axioms are discussed and clearly stated. Two models are then derived in detail in order to illustrate the technique for deriving specific models from the general theory. One of the models is for the simple noncontingent verbal conditioning experiment, the other for the zero-sum, two-person interaction situation.

The second chapter is concerned with methods of analyzing and testing the theory. For Markov models in which the states are observable, a maximum-likelihood method of estimation of parameters is derived. For Markov chains in which the states are not observable, a pseudomaximum-likelihood estimation procedure is derived. Some statistical tests for Markov chains are presented but not derived since their derivations are available elsewhere.

The remainder of the book is devoted primarily to the analysis of experiments. A number of estimation procedures and goodness-of-fit tests are utilized in a thorough analysis of experimental results. Therefore, the reader can gain a clear understanding of the application of theory to data. Furthermore, because many of the statistical tests run upon the data are independent of any theoretical orientation, they can be used by other researchers with different theoretical bents.

Chapters 3 through 11 present a wealth of experimental material as well as a number of theoretical ideas. A standard experimental routine—two-choice verbal conditioning—which has been fairly adequately described by SLT forms the basis for almost all of the experiments run. In the simple noncontingent verbal-conditioning experiment, a subject is run through a number of trials. Each trial begins with a signal for the subject to predict which one of two events will occur on that trial. Following the subject's prediction, one of the two events occurs, reinforcing the corresponding response. Of crucial importance is the rule for presenting the event outcomes. Usually each event is presented with a probability that remains fixed over trials, i.e., Bernoulli trials. Sometimes the event probabilities are made contingent upon the response of the subject.

Since results from such an experimental routine have been extensively studied and are fairly well understood, Suppes and Atkinson extended the routines to two-person interaction experiments. (Burke has considered the application of the Linear Models to the same routine.) In the simple interaction experiment, subjects make their choice independently of each other. The interaction of the subjects is due to the contingency of the reinforcing events upon both the subjects' responses. Thus, in the zero-sum, two-person game experiment, two players, A and B, each make one of two responses, $A1$ or $A2$ and $B1$ or $B2$, respectively, on each trial. For each of the four possible combinations of responses, a probability, a_i ($i = 1, 2, 3, 4$), that subject A will be correct and B incorrect is stated. These probabilities form a payoff matrix, where

> "In thinking of each trial as the play of a simple 2×2 game, it is to be noted that the payoff is not a matter of being correct or incorrect, but the probability a_i of being correct. The experimenter's selection of actual reinforcements on the basis of the a_i's corresponds to a chance move by the referee in a game." (p. 23)

The authors proceeded systematically in their experimentation.

In the first experiment:

> "The interaction of the two subjects forming a pair was limited in at least three ways: (1) neither subject was shown the payoff matrix; (2) neither subject was directly informed of the responses of the other; (3) subjects were in fact told that they were not interacting at all, but were being run in pairs only as a matter of experimental convenience. Each of these restrictions was removed in later experiments, . . ." (p. 78)

The second set of experiments was non-zero-sum, that is, both players could be correct or incorrect on a trial, and the players were informed that they were interacting. In the third set of experiments. one of the players was informed of the other's responses. This was done in two different ways. In one experiment, subject B was informed of A's response before he made his own response. In another experiment, subject B was informed of A's response only after he had made his own response. A fourth set of experiments involved three persons in a simple majority game in which no subject was given information concerning the responses of the other two subjects. The rule for presenting reinforcements depended upon what the majority, i.e., two subjects, did on a trial. A set of experiments was then analyzed in which the subjects were shown the payoff matrix.

Chapter 10 is a truly exciting chapter in which the one-element model, multi-element model, generalized conditioning model, linear model, hypothesis model, and a memory model are all applied to the data of a simple noncontingent verbal conditioning experiment with pairs of subjects run in a statistically dependent manner and with monetary pay-offs provided for the correct response. The last experiment, described in Chapter 11, tested the derivations of utility functions from the basic learning axioms.

In the concluding chapter of the book, some extensions of theory to other experimental situations are suggested although not tested. However, tests of the extensions have since been published. The first extension considered by Suppes and Atkinson is to a social-type situation previously experimentally investigated by Solomon Asch. The second extension is to a simple oligopoly situation. The third is to what the authors call a continuous game, in which a continuum of responses is considered instead of a finite number of response classes.

For the psychologist interested in learning theory, this is an important book. It is an intensive application of a particular theoretical approach—stimulus sampling theory—which is being applied at present to a number of diverse learning situations, from t-maze behavior of the rat to paired-associate and concept learning in the child to simple oligopolistic situations. For those interested primarily in game theory or social processes, this book has a limited offering. However, if the game theorist is particularly interested in the ability of game theory to describe and not proscribe behavior, then the book may be valuable since almost all the experiments were analyzed from the standpoint of game theory. The social psychologist may find that the variables treated are somewhat limited and not the variables of most interest to him.

Perhaps because of the similarity of my interests and those of the authors and because of my respect for empirically orientated theorizing, I find no serious issues on which to challenge the authors. To the anti-theoretical, to the entirely empirical, or to the biologically orientated student of learning, the results and analyses of the book may seem trivial. But if any descriptive theory of behavior is to be established, this book certainly represents a contribution toward that end.

RICHARD MILLWARD

Dynamic optimization and control—a variational approach. By Walerian Kipiniak. M. I. T. Press, Cambridge, and John Wiley & Sons, Inc., New York, London, 1961. $x + 233$ pp. $4.95.

This book is devoted to the study of important classes of feedback control processes, of both deterministic and stochastic type. It contains interesting analytic and computational results in connection with the determination of optimal trajectories and with the solution of two-point boundary-value problems in general.

Unfortunately, its value is almost completely destroyed by a barbarous and impenetrable notation. Since the monograph was originally a Ph.D. thesis and thus represented a first attempt at sustained scientific writing, the author should not be unduly reproached for the profusion of new and unnecessary symbols and for equations where particular quantities bear simultaneously *four* iterated superscripts and *three* iterated subscripts. But certainly a press such as Wiley which contributes so many excellent books to modern mathematics, and to control engineering in particular, should be reproached for allowing this unedited version of a worthwhile research effort to appear.

RICHARD BELLMAN

Mathematical models and methods in marketing. Edited by F. M. Bass, M. R. Greene, E. A. Pessemier, A. Shuchman, R. D. Buzzell, W. Lazer, D. L. Shawver, C. A. Theodore G. W. Wilson. Richard D. Irwin, Inc., Homewood, Illinois, 1961. *xi* + 545 pp. $8.50.

An interesting experiment in textbook writing, this volume consists of 18 articles reprinted from *Operations Research* and some economic journals giving representative but otherwise unrelated examples of model building in the theory of marketing. These articles are usually preceded by an introduction which motivates and paraphrases the mathematical model; they are frequently followed by a mathematical appendix which elaborates some of the points, fills in algebraic steps, and occasionally suggests alternative approaches. The models range from simple break-even charts to more sophisticated concepts such as a differential equation expressing the response of sales to advertising, a Markov chain to represent consumers' propensities for brand switching and two-person zero-sum games in the allocation of advertising funds among regional markets. The articles are arranged by subjects in marketing not by the mathematical tools used. A joint bibliography would have been desirable. This is a useful book of readings for Operations Research courses. It also shows that there are many unexplored possibilities for mathematical model building in the study of markets.

Martin J. Beckmann

Statistical analysis and optimization of systems. By E. L. Peterson. John Wiley & Sons, Inc., New York, London, 1961. xi + 190 pp. $9.75.

This book affords an excellent chance for the modern mathematically oriented engineer and economist to view in one compact package the fundamental concepts, problems and methods of solution of the newly emerging theories of deterministic and stochastic control processes. This volume is a fine amalgam of classical and modern techniques.

The first few chapters are devoted to a discussion of random variables and the response of linear systems to random inputs. This leads naturally in the work of Wiener, Zadeh, Shinbrot and others. To the results and references cited by the author should be added the recent important work of Kalman. An expository account of this will be found in the Proceedings of the Symposium on Optimization, held at RAND in 1960. From the variational problems associated with the foregoing, involving linear equations and quadratic criteria, the author turns to the study of general nonlinear control processes. He presents the dynamic programming approach as well as that of the classical calculus of variations and a sketch of the Pontrjagin maximum principle. The dimensionality difficulties lead to the introduction of various approximation methods due to Bellman and Kalaba, and Merriam. The discussion here is rather brief and it is to be hoped that in a subsequent edition, it will be expanded.

Richard Bellman

The mathematical theory of linear systems. By B. M. Brown. John Wiley & Sons, Inc., New York, London, 1961. xv + 267 pp. $8.00.

This well written and well produced text, which is intended for students of engineering, can best be described by stating that it is one of differences and contrasts. It is part of a series on Automation and Control, and yet its title—and contents—refer to linear systems as a whole, with only a sketchy and inadequate presentation of the special mathematical techniques applicable to feedback (control) systems. Further, the theory of complex variables, which today is generally regarded as a central mathematical tool for the treatment of linear systems (or equations), is not introduced at all. Thus the approach is almost entirely through the classical differential and difference operators, with the Laplace, Fourier and z-transforms playing only a minor role. The author rightly suggests that the classical and transform techniques are complementary, but their relative importance is interpreted quite differently amongst the engineers of this country, at least. Finally, the bi-lateral or double-sided Laplace transformation is introduced as fundamental. This may be true, but in most engineering problems the single-sided transform either must be used—to ensure convergence—or at least is adequate and more convenient.

In general then, the reviewer feels that differential equations, per se, would normally be treated—

probably classically—in a strictly mathematical course, and that the engineering student should then be introduced to linear systems by way of complex variables and transformation methods; the methods he will find most generally used in practice.

The level of treatment is somewhat uneven. Thus the reviewer found the introductory chapters on differential and difference operators to be clearly and carefully written. This was also true of the treatment of the singularity functions, where, admittedly, the bi-lateral Laplace transformation comes into its own. However, the two chapters on stability and feedback, and to a lesser extent those on statistical methods and statistical optimization, were too brief to be satisfying.

In summary, the reviewer considers that this book provides an interesting and useful exposition of the classical treatment of linear systems; that it is well worth reading if one wishes to compare classical and transform methods, or the single-sided and double-sided Laplace transforms; but that its orientation makes it less suitable as a basic text for engineering students.

J. R. Ward

Progress in optics. Volume I. Edited by E. Wolf. North Holland Publishing Co., Amsterdam, and Interscience Publishers, Inc., New York, London, 1961. xii + 342 pp. $12.00.

The editor is to be congratulated for having obtained the help of a staff of young scientists who bring a fresh outlook and are masters of the latest techniques applied to optical problems. However, some of the young authors seem not to have sufficient knowledge of the historical development in the field. Since this book and the following volumes are to serve as reference works for the future, describing progress in optics, certain precautions might be advisable.

For instance the article by Mr. Pegis, which is entitled, "The Modern Development of Hamiltonian Optics," only discusses the work of a single author, though significant contributions to Hamiltonian optics have been made in Sweden, Germany, Russia, Italy, Argentina, and the United States. However, T. Smith, the author discussed, has made the most important contribution to the subject and Mr. Pegis' lucid discussion, which clears up many difficult points in the papers of this great scientist, will be of great benefit to the reader.

The same criticism, though in a smaller way, applies to the article by Miyamoto on wave optics and geometrical optics in optical design. The work by R. Straubel, Ignatowsky, Strehl, and Jentzsch is not even mentioned, and the contributions of Debye and Picht are certainly not given their right evaluation. On the other hand, the article is an important contribution, containing some of the ideas of the author with respect to the connection between spot diagrams and diffraction patterns.

The article by Barakat on the intensity distribution in diffraction images contains a much more extensive (though still incomplete) bibliography. It has the merit that it brings some of the ideas of R. Luneberg (who, alas, died prematurely) to the attention of a larger public.

"Light and Information" by D. Gabor relates optics more closely to physics and the second law of thermodynamics, applying to optics the concept of entropy, which forms the basis of a new branch of mathematics called information theory.

H. Wolter's interesting contributions point out the analogies and differences between optics and electronics.

H. Kubota discusses the progress made in the study of multilayer films and their uses in many practical processes. Fiorentini speaks of selected interesting problems in vision.

The book ends with a delightful chapter by A. W. van Heel on alignment devices, which shows how an idea can be put to good use if an ingenious scientist gets hold of it.

M. Herzberger

Plasticity. Proceedings of the Second Symposium on Naval Structural Mechanics. Edited by E. H. Lee and P. S. Symonds. Pergamon Press, Oxford, London, New York, Paris, 1960. 611 pp. $10.00.

The book contains 32 reports read at a Symposium on the theory of plasticity held at Brown University, April 5–7, 1960. The Symposium was sponsored jointly by this university and the Office of Naval Research of the U. S. Navy. Leading specialists from the United States, the Soviet Union, Great Britain and Roumania were invited to participate in the Symposium. The reports were devoted to six areas of plasticity. A survey report covering each of the areas was followed by reports on current particular topics. Specific problems in design of naval vessels were included in one of the areas (sixth). The following is an enumeration of areas and reports.

I. Atomic theory of plastic flow and fracture.

J. J. Gilman, "Physical nature of plastic flow and fracture" (survey: the analysis of the above processes in terms of the progress in the theory of dislocations and the theory of crack propagation). R. S. Davis and K. A. Jackson, "On the deformation associated with compression shocks in crystalline solids" (rough estimations of stresses and strains and considerations of the accompanying dislocation changes in compression shock waves). C. Elbaum, "The relation between the plastic deformation of single crystals and of polycrystals" (on the basis of experiments carried out on aluminum the role played by grain boundaries in the plastic deformation as well as the relation between the behavior of single crystals that deform by slip on many slip systems and the deformation of polycrystals are studied).

II. Stress-strain relations including theromplasticity and creep.

P. M. Naghdi, "Stress-strain relations in plasticity and thermoplasticity" (a survey devoted to small isothermal and nonisothermal deformations, especial attention being given to thermodynamical aspects of the latter problem). D. C. Drucker, "Extension of the stability postulate with emphasis on temperature changes" (introduction of adiabatic loading surfaces, analysis of effects of variation in tensile yield stress with increase in temperature; one of the conclusions is that it is necessary to use uncoupled thermal and stress equations when the yield stress decreases with temperature). R. M. Haythornthwaite, "Stress and strain in soils" (preliminary results of author's experiments according to which the plastic potential function does not coincide with the yield function for soils (mixture of 30 per cent of clay with 70 per cent of quartz flour)). L. W. Hu, "Plastic stress-strain relations and hydrostatic stress" (the significance of the inclusion of hydrostatic stress in plasticity theory is shown). Aris Phillips, "Pointed vertices in plasticity" (considerations based on author's experiments in favor of the existence of pointed vertices at loading surfaces).

III. Basic theory including stability and uniqueness.

V. V. Sokolovsky, "Axial plastic flow between non-circular cylinders" (the method of transforming the basic equations which allows the solution of a given non-linear problem to be found when the solution of the corresponding linear problem is known, the flow between two confocal elliptical cylinders is treated as an example). E. T. Onat, "The influence of geometry changes on the load-deformation behavior of plastic solids" (a survey covering the theory of stability and uniqueness of solution of boundary value problems for work-hardening materials taking into account geometry changes developed for the case of a plane circular built-in arch). Bernard Budiansky, Zvi Hashin and J. Lyell Sanders, Jr., "The stress field of a slipped crystal and the early plastic behavior of polycrystalline materials" (the behavior of polycrystals is analysed on the basis of studying the stress field associated with a single slipped crystal of spherical shape in an infinite isotropic elastic matrix). V. S. Lensky, "Analysis of plastic behavior of metals under complex loading" (the results of author's experiments (carried out on a machine provided with a programming device) supporting and developing the fundamentals of A. A. Ilyushin's theory). H. J. Greenberg, W. S. Dorn and E. H. Wetherell, "A comparison of flow and deformation theories in plastic torsion of a square cylinder" (numerical solution of a two-dimensional boundary value problem for a strain-hardening material indicating (in accord with the known criterion given by B. Budiansky) the coincidence of the results furnished by the two theories).

IV. Boundary value problems including plates and shells.

Philip G. Hodge, Jr., "Boundary value problems in plasticity" (a survey primarily devoted to the piecewise linear theory of plasticity being developed by the author). Y. N. Rabotnov, "The theory of creep and its applications" (a survey containing the justification of the quasi-static criterion of stability in creep). M. E. Lunchick, "Plasticity research on submarine pressure hulls conducted at the David Taylor Model Basin" (the results of experimental investigations on buckling of stiffened cylindrical shells made of elastic-perfectly plastic or strain-hardening materials). J. H. Weiner and H. G. Landau, "Thermal stresses in elasto-plastic bodies" (one-dimensional problems for viscoelastic-perfectly plastic bodies with a temperature-dependent yield condition).

V. Dynamic loading and plastic waves.

N. Cristescu, "European contributions to dynamic loading and plastic waves" (a survey on the propagation of perturbations in bars, strings and disks of materials possessing various rheological or plastic properties). T. H. H. Pian, "Dynamic response of thin shell structures" (an approximate solution by the energy method of problems of large plastic deformation and postbuckling behavior of multi-cell shell beam under dynamic loading with account of experiments). Two reports on the topic "Current research on plastic wave propagation at the University of Texas": a) H. J. Plass, Jr., "A theory of longitudinal plastic waves in rods of strain-rate dependent material, including effects of lateral inertia and shear" (the solution of this problem is proposed by means of the method of characteristics); b) E. A. Ripperger, "Experimental studies of plastic wave propagation in bars" (the results of experiments with copper and aluminum bars are compared with theoretical predictions (with no consideration of the results of the preceding report)). S. R. Bodner and P. S. Symonds, "Plastic deformations in impact and impulsive loading of beams" (the results of author's experiments indicated that it is necessary to consider the space and time variation of the yield moment in a rigid-plastic analysis). N. Cristescu, "Some observations on the propagation of plastic waves in plates" (the solution of the problem of the transverse impact on a plate by a rotating cylinder based on the deformation theory).

VI. Developments in design.

Two survey reports on problems of plasticity in naval structures: a) John Vasta, "Static loading" and b) A. H. Keil, "Explosive and impact loading" (possibilities of application of the methods of limit design, dynamics of rigid-plastic structures and some rheological models for evaluating the behavior of complex naval ship structures are discussed). Jacques Heyman, "Progress in plastic design" (a survey covering works (chiefly English) on analysis and design of reinforced concrete and steel civil structures with examples of erected buildings). Lynn S. Beedle, "On the application of plastic design" (a survey of the development and application of limit design methods in civil engineering and shipbuilding in the United States). Wm. Cohen, "Rheological problems of solid-propellant rocketry" (it is emphasized that the rational use of solid propellant for Polaris, Vanguard and other types of ballistic missiles requires the knowledge of the physical properties of the propellant). R. T. Shield, "Optimum design methods for structures" (a review of the methods for the optimum design for perfectly plastic materials and their extension to the case of other materials). S. Kendrick, "A review of research carried out at the Naval Construction Research Establishment into structural behavior beyond the elastic limit" (the results of the theoretical and experimental work carried out in England for studying the behavior of plated grillages and networks of beams under static loading).

The work of the Symposium was summed up in the concluding remarks by W. Prager.

The book is of great value both for investigators and those interested in applications.

G. S. Shapiro

Elements of linear algebra. By Lowell J. Paige and J. Dean Swift. Ginn and Co., Boston, New York, Chicago, 1961. xiii + 348 pp. $7.25.

There is an increasing recognition of the fact that linear algebra, besides being important in its own right, provides an excellent introduction to the power and elegance of abstract methods. More and more colleges are making it the subject of their first course in abstract algebra. This text is specifically designed for such a course and its outstanding feature is a skillful weaving together of the abstract and the concrete.

The vector space, $V_n(R)$, of ordered n-tuples of real numbers, is constantly kept in the foreground, but the definitions and proofs are those for an arbitrary real vector space. For almost every step there is a numerical example to illustrate the abstract concept. Although it is done well, this emphasis on illustrative examples is overdone. The teacher will need to devote much class time to separating the wheat from the chaff, to pointing up the crucial ideas, and to making sure that the student does not expend all his energy on the computations while losing sight of the theorems which they illustrate.

The subjects covered include all those that are likely to be treated in a one semester course, but inner product spaces do not get their due, while Hermitian forms and matrices are left as an exercise for the reader. Linear functionals and the dual space receive bare mention.

After a brief miscellany—sets, fields, functions, and proofs—the book proper starts with Chapter 2. The preface suggests that this chapter may be skipped, but for most beginners this introduction to vector spaces via the geometry of 3-space will be essential. Unfortunately, needless complications arise because the authors insist on a pedantic distinction between coordinatized Euclidean 3-space and $V_3(R)$.

Chapter 3 begins with a discussion of $V_n(R)$ and the definition of real vector spaces. Next come the standard results about linear dependence, bases, dimension, and subspaces. For no discernable reason, only passing mention is made of inner products in general. The Gram-Schmidt process and orthogonal complementation are presented as if they applied only to subspaces of $V_n(R)$ with the usual inner product.

Except for the Cauchy-Schwarz inequality the n-dimensional analytic geometry of Chapter 4 may be omitted without interrupting the continuity.

Determinants are defined as alternating n-forms on $V_n(R)$, but the development is essentially classical. Matrices are here introduced only as a terminological device so the theorem on the product of determinants must be deferred to Chapter 7.

The motivation for the study of linear transformations is weak, but the treatment is clear and the essential theorems are proved. Matrices are now introduced properly, as a way of describing linear transformations. Changes of basis are treated elegantly. The concept of similarity of matrices makes its appearance and leads to a general discussion of equivalence relations.

The usual material on elementary row transformations, rank, bilinear forms, and quadratic forms is developed in Chapters 7 and 8. Two sections, "Groups" and "The Orthogonal Group" seem out of place. From these sections only the notions of orthogonal transformation and orthogonal matrix are later exploited.

Skipping Chapter 9 for the moment, we find in Chapter 10 a pitifully brief account of characteristic values and characteristic vectors. The book is intended for physicists, engineers, and chemists, as well as for mathematicians and economists, but there is no hint of the immense range of the applications of the concepts introduced here. Symmetric transformations do not appear—only symmetric matrices. The fact that the eigenvectors of a symmetric matrix span the space can be found, slightly camouflaged. There is no mention of orthogonal projections.

Chapters 9 and 11 belong together and are far more sophisticated than the rest of the book. One cannot blame the authors for wanting to include the beautiful theory of canonical forms for transformations on a vector space over an arbitrary field, but the subject is not one for beginners. There is one unfortunate choice of terminology. Six books readily available to the reviewer agree with each other and disagree with this one as to what constitutes Jordan form.

Teachers contemplating adoption of this text will naturally compare it with that by Hoffman and Kunze. It demands far less sophistication on the part of the student than the latter, but the crucial ideas both of concept and of proof are presented less sharply. Its great merit is its use of concrete examples leading naturally to their abstract counterparts.

There is a tremendous selection of routine computational exercises, a very good supply of significant numerical problems, and an adequate number of theoretical problems at all levels of difficulty.

F. M. Stewart

An introduction to the theory of vibrating systems. By W. G. Bickley and A. Talbot. Clarendon Press, Oxford, 1961. xii + 238 pp. $4.80.

This is a well written introduction to classical vibration theory that appears suitable for the advanced undergraduate level. Its special merits in comparison with other texts on the subject are the care with which the physical concepts are introduced and the use of a mathematical level appropriate to the problems. Both finite and continuous systems are treated and particular attention is given to energy methods. The presentation is completely theoretical and, except for a discussion of analogies between electrical and mechanical systems, all applications and examples are relegated to exercises at the ends of the chapters. For the relatively small size of the book, the authors have done exceedingly well in presenting the basic concepts of vibration theory in a lucid and rigorous manner.

S. R. Bodner

Cours de mathématiques. By J. Bass. 2nd edition. Masson & Co., Paris, 1961. Vol. I, xiii + 576 pp.; Vol. II, 438 pp. $19.64.

The first edition of this treatise was reviewed in this Quarterly **16**, 202 (1958). In the present edition numerous minor revisions and additions have been made. The replacement of the single unwieldy volume of the first edition by two volumes of normal size, will doubtless be welcomed by many users.

W. Prager

Probleme plane in teoria elasticităţii. By P. P. Teodorescu. Vol. I. Editura Academiei Republicii Populare Romine, Bucharest, 1961. 995 pp.

This is the first volume of a projected two-volume treatise on plane elastostatics. The book is evidently addressed to engineers and particular emphasis is placed on applications of interest to structural engineers. Following a somewhat cursory review of the three-dimensional equilibrium theory, the author turns to the plane problem and proceeds directly to the two-dimensional (plane strain or generalized plane stress) treatment of this space problem. A subsequent major chapter deals with various exact and approximate methods for coping with the ensuing plane boundary-value problems. The remaining chapters are devoted to a detailed exposition of a host of solutions to specific problems associated with rectangular and strip-shaped domains, with the entire plane, the half-plane, and the quarter-plane. Applications which require the use of curvilinear coordinates are to be taken up in the forthcoming second volume. The present volume contains nine appendices, in which the author has compiled a catalogue of series and integral representations appropriate to technologically significant loading cases that arise in connection with the particular problems considered earlier. The book concludes with a set of numerical tables of certain relevant special functions. For the benefit of readers who are not familiar with Rumanian, English and Russian versions of the table of content, as well as an abstract of the treatise in these two languages, have been included.

ELI STERNBERG

Turbulence—classic papers on statistical theory. Edited by S. K. Friedlander and Leonard Topper. Interscience Publishers, Inc., New York, London, 1961. ix + 187 pp. $6.00.

This somewhat random collection of papers on turbulence theory is perhaps more remarkable for what has been left out than for what has been included. Not that the papers which the editors chose to reprint are unworthy of the honor—quite the contrary. There is the classic work of Taylor on diffusion and energy decay in isotropic turbulence, and the paper by von Kármán and Howarth on the behavior of velocity correlations; Kolmogoroff's three papers on the similarity hypothesis are presented in awkward but readable translation from the original Russian, and there are further contributions by Lin and von Kármán on this subject. Of rather less interest, at least in an anthology of this sort, are two reviews by Dryden and von Kármán, which to a large extent repeat and summarize the contents of articles already included in the collection.

It is undeniably convenient to have all this basic literature available in one volume. Nevertheless, it must be pointed out that much important work has been omitted: there is not a single article by Batchelor or by Townsend; Chandrasekhar's work goes completely undiscussed, and Heisenberg's quasi-equilibrium theory is barely mentioned. It is hard to escape the conclusion that a little more thoroughness on the part of the editors could have made this volume a good deal more valuable than it is.

STEPHEN PRAGER

Praktische Mathematik für Ingenieure und Physiker. By R. Zurmühl. 3rd edition. Springer-Verlag, Berlin, 1961. xiv + 548 pp. $7.40.

The second edition of this well-known text appeared in 1957. The present edition contains numerous revisions and additions, especially in the chapters on linear equations and matrices, statistics, and differential equations.

Mechanics of solids and fluids. By Robert R. Long. Prentice-Hall, Inc., New Jersey, 1961. x + 156 pp. $6.75.

This is a textbook for a third-year undergraduate course. The chapters, in order, are on vectors and tensors, stress, deformation and flow, mechanical properties of real materials, elasticity, with problems, and fluid mechanics, with problems. The treatment of these topics is generally suitable for the intended audience, although misleading in some places. It is good to see a discussion of real materials in such a course, and to see tensor notation coming into general use at the undergraduate level.

ALLEN PIPKIN

Proceedings of the Second International Conference on Operational Research. (Aix-en-Provence 1960). John Wiley & Sons, Inc., New York, 1961. xx + 810 pp. $15.00.

The conference was organized by the International Federation of Operational Research Societies. Part I of the Proceedings reports on seven sessions devoted to the following topics: some methodological aspects of operations research (5 papers), computers and operational research (5 papers), the measurements of human factors (4 papers), new mathematical methods (6 papers), control of production (4 papers), evaluation of inventory control (review session: 1 introductory paper and 6 abstracts), and new methods in mathematical programming (review session: 1 paper and 8 abstracts). Part II reports on eight sessions that were concerned with the application of operational research to the following fields: steel industry (4 papers), oil industry (4 papers), local government (3 papers), atomic and electric power (6 papers), national decisions (7 papers), military problems (4 papers), mining (3 papers), and transport (6 papers).

Symposium on the numerical treatment of ordinary differential equations, integral and integro-differential equations. Proceedings of the Rome Symposium (Sept. 20–24, 1960) organized by the Provisional International Computation Center. Birkhäuser Verlag, Basel, 1960. 679 pp. $8.40.

The interesting volume contains the following contributions. ALTMAN, M., Iterative methods for the numerical solutions of ordinary differential equations.—ANTOSIEWICZ, H. A., Lyapunov-like functions and approximate solutions of ordinary differential equations.—ARTEMIADIS, N., Fourier transforms and analytical functions.—BLANC, CH., Estimation d'erreurs dans la résolution de problèmes aux limites.—BOCHENEK, K., Les solutions asymptotiques des équations différentielles et l'évaluation des restes.—BUCKINGHAM, R. A., The solution of integro-differential equations occurring in nuclear collision problems.—BUECKNER, H. F., An iterative method for solving nonlinear integral equations.—BUKOVICS, E., Natürliche Eigenwertprobleme bei gewöhnlichen Differentialgleichungen.—CAPRIZ, G., A numerical approach to some problems of continuum mechanics and related questions of stability.—CARTERON, J. & MEALLIER, A., Intégration par une méthode de différence des équations intrinsèques d'un fil raide pesant, avec conditions aux limites. Application à la pose d'une conduite.—CESCHINO, F., Sur certaines applications de l'intégration approchée.—CLENSHAW, C. W., The numerical solution of ordinary differential equations in Chebyshev series.—COURANT, R., General problems confronting computing centers.—DELSARTE, J., Sur quelques équations de type recurrentiel et sur leurs applications numériques aux équations différentielles et intégro-différentielles.—DIAZ, J. B., On existence, uniqueness, and numerical evaluation of solutions of ordinary and hyperbolic differential equations.—DOUGLAS, A. S., and ASHTON, W. D., A new difference method for solution of certain second order non-linear differential equations.—DOUGLAS, JR., J., Mathematical programming and integral equations.—FENYÖ, S., Bemerkung über eine Lösungsmethode der Differentialgleichungssysteme.—FIEDLER, M., Some estimates of spectra of matrices.—FREY, T., Über ein neues Iterationsverfahren zur Lösung von Integral- und Intergrodifferentialgleichungen.—GENUYS, F., Rapport général sur le traitement numérique des équations différentielles.—GIESE, J. H., On the asymptotic behavior of certain one-dimensional flows.—GILLES, D. C., Some numerical solutions of two-point eigenvalue equations with one boundary at infinity.—GOTO, M., On the general solution of some non-linear differential equation.—GRAFFI, D., Sull'integrazione approssimata delle equazioni della meccanica nonlineare.—HAIN, K., and HERTWECK, F., Numerical integration of ordinary differential equations by difference methods with automatic determination of step-length.—HENRICI, P., The progagation of round-off error in the numerical solution of initial value problems involving ordinary differential equations of the second order.—ISAACSON, E., and GROSSMAN, K., The numerical solution of a singular integral equation.—KELLER, H. B., Convergence of the discrete ordinate method for the anisotropic scattering transport equation.—KRÜCKEBERG, F., and UNGER, H., On the numerical integration of ordinary differential equations and the determination of error bounds.—LANCE, G. N., and ROGERS, M. H., The numerical solution of the differential equations governing the motion of viscous fluid between two rotating discs.—LANCZOS, C., Solution of ordinary differential equations by trigonometric interpolation.—LATTÈS, R., and LIONS, J. L., Sur une classe de problèmes aux limites intervenant en physique de réacteurs.—LAUDET. M., and OULÈS, H., Sur l'intégration numérique des équations intégrales du type de Volterra.—LESKY, P., Lösung des dreidimensionalen Vierkörperproblems: Sonne, Erde, Mond und Raumschiff.—LUKASZEWICZ, L., Accumulation of errors in approximate calculations.—MAKINSON, G. J., and YOUNG, A., The stability of solutions of differential and integral equations,—McCarroll, R. Solution of integro-differential equations arising in the theory of atomic scattering.—Martensen, E., Numerische Auflösung der Integralgleichung des Robinschen Problems für eine torusartige Berandung.—Moriguti, S., Theory of numerical convergence of iterative processes with applications to differential equations.—Moser, J., Bistable systems of differential equations.—Nicolovius, R., Iterative solution of boundary value problems using remainder terms of Taylor expansions.—Noble, B., The numerical solution of the singular integral equation for the charge distribution on a flat rectangular lamina.—Pekeris, C. L., Alerman, Z., and Finkelstein, L., Solution of the Boltzmann Hilbert integral equation. Propagation of sound in a rarefied gas.—Pennacchi, R., Su un metodo numerico per la risoluzione della equazione integrale di Volterra.—Popoviciu, T., Sur la délimitation du reste dans les formules d'approximation linéaire de l'analyse.—Pouzet, P., Méthode d'intégration numérique des équations intégrales et intégro-différentielles du type de Volterra de seconde espèce. Formules de Runge-Kutta.—Pucci, C., An intero-differential equation.—Redish, K. A., Application of computing techniques to nuclear physics.—Seegmüller, G., Error analysis for the numerical solution of certain differential equations in gasdynamics. —Sneider, M. A., Applicazione sui calcolatori elettronici tipo 650 IBM e 1620 IBM.—Teodorescu, N., Fonctions holomorphes (α) et leur approximation par polynômes aréolaires.—Todd, J., The condition of certain integral equations.—Underhill, L. H., A self-consistent one-group method for the Boltzmann transport equation in neutronics.—Van Norton, R., On the spectrum of a mono-energetic neutron transport operator.—Vejvoda, O., Perturbed boundary-value problems and their approximate solution. —Walther, A., and Dejon, B., General report on the numerical treatment of integral and integro-differential equations.

Cybernetics. By Norbert Wiener. Second edition. The M. I. T. Press, Cambridge, and John Wiley & Sons, Inc., New York, London, 1961. xvi + 212 pp. $650.

This outstanding book was first published in 1948 and widely reviewed at that time. Consequently, there is little point in discussing it in detail here. The second edition contains two additional chapters, X, "On Learning and Self-Reproducing Machines," and XI, "Brain Waves and Self-Organizing Systems," and an additional Preface.

Despite the fact that most of the material is thirteen years old, or older, the book is required reading for all those interested in both the many purely mathematical questions posed by Wiener, and in applying mathematical techniques to the fields of engineering, economics and biology. Whether one agrees or disagrees with the text, the task of analyzing Wiener's ideas is extremely challenging and rewarding.

The type of mathematical analysis used has probably reached its highwater mark in applications and will in all likelihood assume a position of less and less importance henceforth. But the many problems he raises are still fresh and almost untouched, and they will increase in importance over time.

Richard Bellman

ANNOUNCEMENT

The Committee on the Undergraduate Program in Mathematics announces the availability of its "Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists." Reflecting the thoughts of many educators in the mathematical and physical sciences, this booklet discusses the background against which the Recommendations were formulated, describes the details of the mathematics part of curricula for undergraduate engineering and physics students, and discusses the content of suggested courses, including references. These Recommendations appear in recent or current issues of *The American Mathematical Monthly*, *The Journal of Engineering Education*, and the *American Journal of Physics*. You may, however, receive the complete report in separate cover upon request from:

Professor Robert J. Wisner, Executive Director
Committee on the Undergraduate Program in Mathematics
Michigan State University Oakland
Rochester, Michigan