THE SYNTHESIS OF INFINITE LINES*

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Abstract. The class of nonuniform lines having solutions which exhibit an out-going wave behavior at infinity and which have a rational input admittance is considered. Necessary and sufficient conditions are given for a rational function to be realizable as the input admittance of an infinite line. A closed-form expression is derived by means of which the characteristic impedance $Z_0(x)$ of a line in this class can be constructed from its input admittance. It is shown that this solution to the synthesis problem is unique once the limiting value of $Z_0(x)$ at infinity or at the input is specified. An example in the application of the technique is presented.

1. Introduction. Since World War II there have been a number of references in the literature to the synthesis of nonuniform transmission lines. However, with the exception of the work on stepped lines, these results have been based on, or can be shown to be equivalent to, a perturbation analysis. In the most common formulation for lossless lines the input reflection coefficient as a function of frequency is expressed in terms of the variation in characteristic impedance with distance by means of a Fourier integral. An approximate solution to the synthesis problem, that is, the determination of the characteristic impedance of the line in terms of the input reflection coefficient, is then obtained by inverting the Fourier integral [1].

The synthesis problem for nonuniform lines has its counterpart in other areas of physics. In electromagnetic theory the problem of plane wave propagation in an inhomogeneous stratified media can be expressed in terms of the transmission line equations [2]. The synthesis problem in this case corresponds to the inverse scattering problem wherein it is desired to determine the variation in the index of refraction from the asymptotic behavior of the scattered wave. In quantum mechanics a similar analogy exists. By making a suitable change of variables the transmission line equations can be transformed into the one-dimensional Schrödinger equation [3]

$$\frac{d^2u(x, \omega)}{dx^2} + [\omega^2 - P(x)]u(x, \omega) = 0. \tag{1.1}$$

The inverse problem in this formulation involves the determination of the scattering potential $P(x)$ from a knowledge of the asymptotic scattered field associated with the solution of (1.1). Considerable progress has been made in solving this problem in the case where $P(x)$ is defined over the half line $0 \leq x < \infty$ (see [4]). Using a method based on the work of I. M. Gelfand and B. M. Levitan, Kay [5] has solved the inverse problem for (1.1) over the interval $-\infty < x < \infty$, assuming $P(x) = 0$ for x less than some constant. More recently he has obtained an explicit solution to the same problem whereby

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$P(x)$ can be determined in closed form in terms of elementary functions if the given reflection coefficient is a rational function of frequency [6].

The possibility of applying the inverse scattering theory for (1.1) to the synthesis of nonuniform lines is an attractive one to the circuit designer, since the perturbation assumption required in the present approximate theory ordinarily limits him to structures which are only slightly nonuniform. While Kay's results can be related to the synthesis of nonuniform lines, his formulation is not entirely appropriate when the problem is considered from the circuit point of view. The reason for this is that the circuit designer is not interested in the scattering potential but rather the characteristic impedance. While it is possible to obtain the characteristic impedance in Kay's method from the wave functions of (1.1), this additional step can be a tedious one to carry out in many cases, and one has no assurance that the resulting impedance will be a bounded function over the whole interval.* In the present treatment, which is based on Marchenko's [7] formulation of the inverse scattering problem, these difficulties are avoided by restricting the region of definition for $P(x)$ to the half line, $0 \leq x < \infty$. This approach also leads to a somewhat more general class of realizable reflection coefficients, or equivalently, input admittances, since $P(x)$ is no longer required to be piecewise continuous for all x as assumed by Kay.

The principal result of this paper is the derivation in Sec. 6 of a formula for determining the characteristic impedance of infinite lines which have rational input admittances. It is shown that for the class of lines considered, the characteristic impedance is bounded and positive for all $x \geq 0$.**

2. Formulation of the synthesis problem. The basic assumption in transmission line theory is that there exists a unique voltage $V(z)$ and a current $I(z)$ at each point on the line which are related, in the lossless case, by

$$\frac{dV(z)}{dz} = i\omega L(z)I(z), \tag{2.1a}$$

$$\frac{dI(z)}{dz} = i\omega C(z)V(z), \tag{2.1b}$$

where $L(z)$, $C(z)$, and ω are real. An exp $(-i\omega t)$ time dependence is assumed so that the nomenclature used here will be consistent with that in the mathematical literature. In physical terms $L(z)$ may be considered the variable inductance per unit length and $C(z)$ the variable capacity per unit length. However, we will not be concerned here with the validity of (2.1) for any particular structure. The characteristic impedance at any point z is defined by

$$Z_0(z) = \left(\frac{L(z)}{C(z)}\right)^{1/2}, \tag{2.2}$$

and the local phase coefficient by

*Kay's example in [5] is a case where the characteristic impedance is unbounded at x = 0.

**The author is indebted to the reviewer for bringing to his attention a recent paper by Litvinenko which presents a different approach to the problem considered here [8]. Litvinenko's approach is based on an interesting relationship between the solutions of the Riccati equation which is satisfied by the impedance function (V(x)/I(x)). Since he arrives at a system of differential equations which must be solved recursively, an explicit solution to the synthesis problem is not obtainable by his method.

$$\beta(z) = \omega[L(z)C(z)]^{1/2}. \tag{2.3}$$

With the substitutions [3]

$$x = \frac{1}{\omega}\int_0^z \beta(t)\,dt, \tag{2.4}$$

$$V(z) = u(x)[Z_0(z)]^{1/2}; \qquad I(z) = v(x)[Z_0(z)]^{-1/2}, \tag{2.5}$$

Eqs. (2.1) become

$$\frac{du(x)}{dx} + p(x)u(x) - i\omega v(x) = 0, \tag{2.6a}$$

$$\frac{dv(x)}{dx} - p(x)v(x) - i\omega u(x) = 0, \tag{2.6b}$$

where

$$p(x) = \frac{1}{2}\frac{d \ln Z_0[z(x)]}{dx}. \tag{2.7}$$

Hereafter, we will denote $Z_0[z(x)]$ by $Z_0(x)$. Eliminating $v(x)$ from (2.6) one obtains the one-dimensional Schrödinger wave equation

$$\frac{d^2u(x)}{dx^2} + \omega^2 u(x) = P(x)u(x), \tag{2.8}$$

where the potential function is given by

$$P(x) = p^2(x) - \frac{dp(x)}{dx}. \tag{2.9}$$

Similarly,

$$\frac{d^2v(x)}{dx^2} + \omega^2 v(x) = Q(x)v(x), \tag{2.10}$$

where

$$Q(x) = p^2(x) + \frac{dp(x)}{dx}. \tag{2.11}$$

It will be assumed that $Z_0(x)$ is continuous with a piece-wise continuous derivative for all $x \geq 0$. Then $u(x)$ and $v(x)$ will be continuous over the same interval, and we may define a normalized admittance function for each x by

$$y(x, \omega) = \frac{v(x, \omega)}{u(x, \omega)}, \qquad 0 \leq x < \infty, \tag{2.12}$$

where recognition is now given to the fact that $u(x, \omega)$ and $v(x, \omega)$, as solutions of (2.8) and (2.10), respectively, are also functions of ω. The synthesis problem for the infinite line can now be stated more explicitly: given

$$Y(\omega) \equiv y(0, \omega) \tag{2.13}$$

for all ω, construct $Z_0(x)$ for $x \geq 0$. Before this problem can be solved it will be necessary to impose some additional restrictions on the solutions to (2.8) and (2.10).

3. Properties of the input admittance. Since the range over which $Z_0(x)$ is defined extends to infinity, we are free to restrict our consideration to those solutions of (2.8) and (2.10) which exhibit an out-going wave behavior at infinity. Let $P(x)$ and $Q(x)$ satisfy the conditions

$$\int_0^\infty x \mid P(x) \mid dx < \infty ; \qquad \int_0^\infty x \mid Q(x) \mid dx < \infty . \tag{3.1}$$

Then there exist solutions to (2.8) and (2.10) in the interval $0 \leq x < \infty$ such that

$$\lim_{x \to \infty} u(x, \omega) \exp(-i\omega x) = 1, \tag{3.2a}$$

$$\lim_{x \to \infty} v(x, \omega) \exp(-i\omega x) = 1. \tag{3.2b}$$

Note that these conditions are consistent with (2.6) since (3.1) implies

$$\lim_{x \to \infty} xp^2(x) = 0. \tag{3.3}$$

We shall now list some of the known properties of $u(x, \omega)$ under these restrictions [4]. For all ω,

$$u(x, -\omega) = \bar{u}(x, \omega), \qquad x \geq 0, \tag{3.4}$$

where the bar denotes the complex conjugate. Let $\lambda = \omega + i\sigma$ be a complex frequency variable. Then for each $x \geq 0$, $u(x, \lambda)$ is an analytic function of λ in the upper half-plane, $\sigma > 0$, and continuous down to the real axis. Moreover, for large $\mid \lambda \mid$,

$$u(x, \lambda) = \exp(i\lambda x) + o[\exp(-\sigma x)], \qquad \sigma \geq 0, \tag{3.5}$$

uniformly for all $x \geq 0$. Equations (3.4) and (3.5) also hold, of course, for $v(x, \lambda)$. From (3.2) we conclude that

$$\lim_{x \to \infty} y(x, \lambda) = 1, \qquad \sigma \geq 0. \tag{3.6}$$

Equation (3.4) and its equivalent for $v(x, \omega)$ imply that

$$\bar{Y}(\omega) = Y(-\omega). \tag{3.7}$$

From (3.5) it follows that in the upper half-plane

$$\lim_{\mid \lambda \mid \to \infty} Y(\lambda) = 1. \tag{3.8}$$

It can be shown that $u(0, \lambda)$ and $v(0, \lambda)$ can have no zeros on the real λ-axis with the possible exception of $\lambda = 0$ and only a finite number of simple zeros on the positive imaginary axis. We will consider only those solutions of (2.8) and (2.10) which, in addition to having the properties previously noted, have the property that $u(0, \lambda) \neq 0$ and $v(0, \lambda) \neq 0$ for any λ in the upper half-plane or on the real axis. These restrictions on $u(x, \lambda)$ and $v(x, \lambda)$ are sufficient to guarantee that the input admittance will be a positive real function of the Laplace variable, $s = -i\lambda$. Before proving this result we shall need the following:

Lemma 1. If $u(x, \omega)$ and $v(x, \omega)$ are solutions of (2.6) which satisfy (3.2), then for all $x \geq 0$,

$$\text{Re}\,[\bar{u}(x, \omega)v(x, \omega)] = 1. \tag{3.9}$$

Proof. Consider the analytic extension of (2.6) in the upper half-plane. Multiply the conjugate of (2.6a) by $v(x, \lambda)$ and (2.6b) by $\bar{u}(x, \lambda)$. Adding and taking the real part, there results, for $\sigma \geq 0$,

$$\frac{\partial}{\partial x} \operatorname{Re} [\bar{u}(x, \lambda) v(x, \lambda)] = -\sigma[| u(x, \lambda) |^2 + | v(x, \lambda) |^2]. \tag{3.10}$$

The lemma follows from (3.2) after integrating (3.10) and allowing σ to approach zero. Lemma 1 is merely a statement of the conservation energy for real ω, since the left side of (3.9) is proportional to the real power flow. We now prove the major result of this section.

Theorem 1. Let $u(x, \omega)$ and $v(x, \omega)$ be solutions of (2.8) and (2.10), respectively, such that conditions (3.1) and (3.2) are satisfied. If $u(0, \lambda)$ has no zeros in the upper half-plane and $u(0, 0) \neq 0$, then the input admittance defined by

$$Y(\lambda) = \frac{v(0, \lambda)}{u(0, \lambda)} \tag{3.11}$$

is a positive real function of the complex variable $s = -i\lambda$.

Proof. Define the input conductance by

$$G(\omega) = \operatorname{Re} [Y(\omega)]. \tag{3.12}$$

Then

$$\operatorname{Re} [\bar{u}(0, \omega) v(0, \omega)] = | u(0, \omega) |^2 G(\omega), \tag{3.13}$$

and by Lemma 1 and (3.4),

$$G(\omega) = \frac{1}{| u(0, \omega) |^2} = \frac{1}{u(0, \omega) u(0, -\omega)}. \tag{3.14}$$

Equation (3.14) implies that for all real ω,

$$G(-\omega) = G(\omega), \tag{3.15a}$$

$$G(\omega) > 0. \tag{3.15b}$$

Under the conditions of the theorem $Y(\lambda)$ will be analytic in the upper half-plane and continuous down to the real axis. Consider a closed contour consisting of a portion of the real axis and a semi-circle in the upper half-plane. For λ within this contour, Cauchy's integral theorem yields

$$Y(\lambda) = \frac{1}{2\pi i} \oint \frac{Y(t)\, dt}{t - \lambda} \tag{3.16}$$

and

$$0 = \frac{1}{2\pi i} \oint \frac{Y(t)\, dt}{t + \lambda}. \tag{3.17}$$

Subtract (3.17) from (3.16) and allow the radius of the semi-circle to approach infinity. In the limit the integral over the semi-circle vanishes and we obtain

$$Y(\lambda) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\lambda}{t^2 - \lambda^2} Y(t)\, dt, \qquad \sigma > 0. \tag{3.18}$$

Using (3.7) this can be written

$$Y(\lambda) = \frac{2}{\pi i} \int_0^\infty \frac{\lambda}{t^2 - \lambda^2} G(t)\, dt, \qquad \sigma > 0. \tag{3.19}$$

It follows from (3.15b) and (3.19) that $Y(is)$ is real for real s and that

$$\text{Re}\, [Y(is)] > 0, \qquad \sigma > 0, \tag{3.20}$$

which completes the proof.

In connection with Theorem 1, note that if $u(0, 0) \neq 0$, then $v(0, 0) \neq 0$. The contrary statement, that $u(0, 0) \neq 0$ and $v(0, 0) = 0$, implies $G(0) = 0$ which, from (3.15b), is impossible. Also, if $u(0, \lambda)$ has no zeros in the upper half-plane, neither does $v(0, \lambda)$ since the reciprocal of a positive real function is also positive real. These observations can be summarized as follows:

Corollary. If $u(0, \lambda)$ satisfies the conditions of Theorem 1, then $v(0, \lambda)$ also satisfies these conditions, and conversely.

In order to obtain a closed-form solution to the synthesis problem we will find it necessary to restrict $u(0, \omega)$ and $v(0, \omega)$ to the class of rational functions. In this case the determination of $u(0, \omega)$ or $v(0, \omega)$ from $Y(\omega)$ is straight forward, and we can establish sufficient as well as necessary conditions for the realizability of $Y(\lambda)$ as the input admittance of an infinite line.

Theorem 2. Let $u(0, \omega)$ and $v(0, \omega)$ be rational functions of ω which satisfy the conditions of Theorem 1 and which have only simple poles. In order that $Y(\lambda)$ be an admittance function as defined in (3.11), it is necessary and sufficient that:

a) $Y(\lambda)$ be a positive real function of $s = -i\lambda$,
b) $Y(\lambda)$ be a rational function of λ having no poles or zeros on the real λ-axis,
c) the zeros of $Ev[Y(\lambda)]$ be simple, and
d) $\lim_{\lambda \to \infty} Y(\lambda) = 1$.

Furthermore, given any one of the three quantities in (3.11), the other two are uniquely determined.

Proof. Since a positive real function can have no zeros or poles in the upper half of the λ-plane, $Y(\omega)$ must have the form

$$Y(\omega) = \frac{\prod_{i=1}^{n} (\omega - \mu_i^*)}{\prod_{i=1}^{n} (\omega - \mu_i)}, \tag{3.21}$$

where the μ_i and μ_i^* are complex constants such that

$$\text{Im}\, [\mu_i] < 0, \qquad \text{Im}\, [\mu_i^*] < 0, \qquad i = 1, 2, \cdots, n,$$

and n is an integer. Using (3.7) the real part of (3.21) can be written

$$G(\omega) = \text{Ev}\, [Y(\lambda)]_{\lambda=\omega} = \frac{\prod_{i=1}^{n} (\omega + \kappa_i) \prod_{i=1}^{n} (\omega - \kappa_i)}{\prod_{i=1}^{n} (\omega - \mu_i) \prod_{i=1}^{n} (\omega + \mu_i)}, \tag{3.22}$$

where the κ_i are defined as the zeros of $G(\lambda)$ in the upper half-plane. The right side of (3.22) will be real if and only if for each μ_i and κ_i there is another which is its reflection in the imaginary axis. Therefore, any zeros of $G(\lambda)$ on the real axis must be of even order. We conclude from condition (c) that the κ_i are distinct and satisfy

$$\mathrm{Im}\,[\kappa_i] > 0, \qquad i = 1, 2, \cdots, n. \tag{3.23}$$

With these restrictions on μ_i and κ_i it is clear that (3.22) satisfies the conditions given in (3.15). Furthermore, we can always factor $G(\omega)$ according to (3.14) to obtain the function

$$u(0, \omega) = \frac{\prod_{i=1}^{n} (\omega - \mu_i)}{\prod_{i=1}^{n} (\omega + \kappa_i)}, \tag{3.24}$$

which satisfies (3.4) and (3.5) at $x = 0$ and which has no poles or zeros in the upper half-plane. It is evident that this factorization can be carried out in only one way if the conditions of the theorem are to be satisfied. Since $Z(\lambda) = Y(\lambda)^{-1}$ satisfies the same conditions given in the theorem for $Y(\lambda)$, we can show in the same manner that $Y(\lambda)$ also determines uniquely the function

$$v(0, \omega) = \frac{\prod_{i=1}^{n} (\omega - \mu_i^*)}{\prod_{i=1}^{n} (\omega + \kappa_i)}. \tag{3.25}$$

In Sec. 5 it will be shown that the $u(0, \omega)$ and $v(0, \omega)$ given in (3.24) and (3.25) determine solutions of (2.8) and (2.10) which satisfy the conditions of Theorem 1. This completes the proof of the sufficiency of conditions (a) through (d). The necessity follows immediately after noting that $G(\omega)$ determines $Y(\lambda)$ uniquely via (3.19).

Condition (c) in Theorem 2 and the corresponding condition on the poles of $u(0, \lambda)$ is somewhat more restrictive than necessary. However, by assuming the constants κ_i are all distinct, we will be able to obtain a closed-form solution to the synthesis problem for this class of input admittances.

From the proof of Theorem 2 it is evident that if $u(0, \lambda)$ were permitted to have zeros in the upper half-plane it would be impossible to establish a unique correspondence between $G(\omega)$ and $u(0, \omega)$ since the factorization could then be accomplished in at least two different ways. It would also be possible to multiply $u(0, \omega)$ by a factor of the form $\prod_{j=1}^{m} (\omega - i\alpha_j)/\prod_{j=1}^{m} (\omega + i\alpha_j)$, where the α_j are real and positive, without changing either $G(\omega)$ or $Y(\lambda)$. This would result in a multiplicity of lines each having the same input admittance. A similar situation occurs in the quantum mechanical problem where a lack of uniqueness in the construction of the potential function can be attributed to the existence of a discrete spectrum for the Schrödinger operator. In fact, the zeros of $u(0, \lambda)$ in the upper half-plane constitute the discrete spectrum giving rise to what are known in this case as bound energy states. [4] By restricting the zeros of $u(0, \lambda)$ to the lower half-plane we will be able to show that to each admittance function which satisfies the conditions of Theorem 2 there is a unique solution to (2.8) and (2.10).

4. Some implications of Marchenko's theory. Let $P(x)$ satisfy condition (3.1).

Then it can be shown that the solutions of (2.8) which satisfy (3.2a) have the representation [4]

$$u(x, \lambda) = \exp(i\lambda x) + \int_x^\infty A(x, y) \exp(i\lambda y)\, dy, \qquad \sigma \geq 0, \tag{4.1}$$

where $A(x, y)$ is square-integrable in y for $x \neq 0$. Similarly, there exists a function $A^*(x, y)$ such that

$$v(x, \lambda) = \exp(i\lambda x) + \int_x^\infty A^*(x, y) \exp(i\lambda y)\, dy, \qquad \sigma \geq 0. \tag{4.2}$$

Marchenko [7] has shown that when $u(0, \lambda)$ has no zeros in the upper half-plane and $u(0, 0) \neq 0$, $A(x, y)$ satisfies the integral equation

$$F(x + y) + A(x, y) + \int_x^\infty A(x, t)F(t + y)\, dt = 0, \qquad x < y, \tag{4.3}$$

where

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^\infty [1 - S(\omega)] \exp(i\omega t)\, d\omega, \tag{4.4}$$

and

$$S(\omega) = \frac{u(0, -\omega)}{u(0, \omega)}. \tag{4.5}$$

These results provide the basic motivation for the theory which follows.

Marchenko's equation (4.3) reduces to a system of algebraic equations when $S(\omega)$ is rational. Assume $u(0, \omega)$ satisfies the conditions of Theorem 2. Then, from (3.24), $1 - S(\omega)$ can be expanded in the form

$$1 - S(\omega) = \sum_{\nu=1}^n \frac{\rho_\nu}{\omega - \kappa_\nu} + \sum_{\nu=1}^n \frac{\sigma_\nu}{\omega - \mu_\nu}, \tag{4.6}$$

where the residues ρ_ν and σ_ν are either imaginary constants or complex constants such that to every ρ_ν or σ_ν there is another which is its reflection in the imaginary axis. Similarly, we can associate with $A^*(x, y)$ in (4.2) the function

$$S^*(\omega) = \frac{v(0, -\omega)}{v(0, \omega)}, \tag{4.7}$$

which, from (3.25), has the expansion

$$1 - S^*(\omega) = \sum_{\nu=1}^n \frac{\rho_\nu^*}{\omega - \kappa_\nu} + \sum_{\nu=1}^n \frac{\sigma_\nu^*}{\omega - \mu_\nu^*}. \tag{4.8}$$

It is clear that $S(\omega)$ and $S^*(\omega)$ are not unrelated. The following lemma establishes the connection between them.

Lemma 2. If $u(0, \lambda)$ and $v(0, \lambda)$ satisfy the conditions of Theorem 2, then with reference to (4.6) and (4.8), $S(\lambda)$ and $S^*(\lambda)$ have the same poles in the upper half-plane, which are the κ_i defined in (3.22), and

$$\rho_\nu^* = -\rho_\nu, \qquad \nu = 1, 2, \cdots, n. \tag{4.9}$$

Proof. The fact that $S(\lambda)$ and $S^*(\lambda)$ have common poles in the upper half-plane is already apparent. Consider the sum $S(\omega) + S^*(\omega)$. By (3.4) and Lemma 1 this can be written

$$S(\omega) + S^*(\omega) = \frac{\bar{u}(0, \omega)}{u(0, \omega)} + \frac{\bar{v}(0, \omega)}{v(0, \omega)} = \frac{2}{u(0, \omega)v(0, \omega)}. \tag{4.10}$$

Therefore, $S(\lambda) + S^*(\lambda)$ is analytic in the upper half-plane, and (4.9) follows from (4.6) and (4.8).

The integral in (4.4) can be evaluated using contour integration. By closing the contour in the upper half-plane, one obtains from (4.6)

$$F(t) = i \sum_{\nu=1}^{n} \rho_\nu \exp(i\kappa_\nu t), \qquad t \geq 0. \tag{4.11}$$

It is readily shown that the κ_ν are the roots of the characteristic equation for (4.3). Therefore, the solution of Marchenko's equation in the case of rational $S(\omega)$ has the form

$$A(x, t) = \sum_{\nu=1}^{n} f_\nu(x) \exp(i\kappa_\nu t), \qquad t \geq x \geq 0, \tag{4.12}$$

where the $f_\nu(x)$ are continuous functions of x with continuous first and second derivatives for $x > 0$. By Lemma 2 the dual function $A^*(x, t)$ can be written

$$A^*(x, t) = \sum_{\nu=1}^{n} f_\nu^*(x) \exp(i\kappa_\nu t), \qquad t \geq x \geq 0. \tag{4.13}$$

The solutions to (2.8) and (2.10) can now be expressed in terms of the $f_\nu(x)$ and $(f_\nu^* x)$. From (4.1) and (4.2) we obtain

$$u(x, \omega) = \exp(i\omega x) - \sum_{\nu=1}^{n} f_\nu(x) \frac{\exp[i(\kappa_\nu + \omega)x]}{i(\kappa_\nu + \omega)}, \qquad x \geq 0 \tag{4.14}$$

and

$$v(x, \omega) = \exp(i\omega x) - \sum_{\nu=1}^{n} f_\nu^*(x) \frac{\exp[i(\kappa_\nu + \omega)x]}{i(\kappa_\nu + \omega)}, \qquad x \geq 0. \tag{4.15}$$

By substituting (4.11) and (4.12) in (4.3) the unknown $f_\nu(x)$ can be shown to satisfy the system

$$-\rho_\nu \sum_{\mu=1}^{n} f_\mu(x) \frac{\exp[i(\kappa_\mu + \kappa_\nu)x]}{\kappa_\mu + \kappa_\nu} + f_\nu(x) + i\rho_\nu \exp(i\kappa_\nu x) = 0, \qquad \nu = 1, 2, \cdots, n. \tag{4.16}$$

By Lemma 2 the $f_\nu^*(x)$ must satisfy the dual system

$$\rho_\nu \sum_{\mu=1}^{n} f_\mu^*(x) \frac{\exp[i(\kappa_\mu + \kappa_\nu)x]}{\kappa_\mu + \kappa_\nu} + f_\nu^*(x) - i\rho_\nu \exp(i\kappa_\nu x) = 0, \qquad \nu = 1, 2, \cdots, n. \tag{4.17}$$

We shall now show that the system of equations (4.16) has a unique solution for the $f_\nu(x)$. This can be demonstrated most readily by proving that the operator kernel in (4.3), $\delta(x - y) + F(x + y)$, is positive definite. Write (4.4) as

$$F(t + y) = \frac{1}{2\pi} \int_0^\infty [e^{i\omega(t+y)} + e^{-i\omega(t+y)} - e^{i\omega(t+y)}S(\omega) - e^{-i\omega(t+y)}S(-\omega)]\, d\omega. \tag{4.18}$$

On the half-line the delta function can be represented by

$$\delta(t - y) = \frac{2}{\pi} \int_0^\infty \sin \omega t \sin \omega y \, d\omega$$

$$= -\frac{1}{2\pi} \int_0^\infty [e^{i\omega(t+y)} + e^{-i\omega(t+y)}] \, d\omega + \frac{1}{2\pi} \int_0^\infty [e^{i\omega(t-y)} + e^{-i\omega(t-y)}] \, d\omega. \quad (4.19)$$

Adding (4.18) and (4.19) and noting that $\bar{S}(\omega) = S(-\omega) = S(\omega)^{-1}$, there results,

$$F(t + y) + \delta(t - y) = \frac{1}{2\pi} \int_0^\infty H(\omega, t)\bar{H}(\omega, y) \, d\omega, \quad (4.20)$$

where

$$H(\omega, t) = \exp (i\omega t) - S(-\omega) \exp (-i\omega t). \quad (4.20a)$$

Given any integrable complex function $B(x, t)$, consider the double integral

$$\int_x^\infty B(x, t) \, dt \int_x^\infty [\delta(t - y) + F(t + y)]\bar{B}(x, y) \, dy$$

$$= \frac{1}{2\pi} \int_0^\infty \left[\int_x^\infty H(\omega, t)B(x, t) \, dt \right] \left[\int_x^\infty \bar{H}(\omega, y)\bar{B}(x, y) \, dy \right] d\omega$$

$$= \frac{1}{2\pi} \int_0^\infty \left| \int_x^\infty H(\omega, t)B(x, t) \, dt \right|^2 d\omega. \quad (4.21)$$

The right side of (4.21) is real and positive for all x unless $B(x, t) \equiv 0$, which completes the proof. By the same argument we can prove that the dual kernel, $\delta(x - y) + F^*(x + y)$, is also positive definite. These results imply that the coefficient matrices of (4.16) and (4.17) are nonsingular. Actually, a stronger statement can be made.

Lemma 3. Denote the coefficient matrices of (4.16) and (4.17) by $I \mp R$, respectively, where I is the identity matrix with elements

$$\delta_{\nu\mu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu \end{cases} \quad (4.22)$$

and R is the matrix whose elements are

$$R_{\nu\mu} = \rho_\nu \frac{\exp [i(\kappa_\nu + \kappa_\mu)x]}{\kappa_\nu + \kappa_\mu}, \qquad \nu, \mu = 1, 2, \cdots, n. \quad (4.23)$$

Then det $(I \mp R)$ is real and positive for all $x \geq 0$.

Proof. Let

$$B(x, t) = \sum_{\mu=1}^{n} g_\mu(x) \exp (i\kappa_\mu t), \quad (4.24)$$

where the $g_\mu(x)$ are arbitrary complex functions. Using (4.11) the left side of (4.21) can be written

$$\sum_{\alpha,\mu,\nu=1}^{n} g_\mu(x)\overline{g_\nu}(x)A_{\nu\alpha}[\delta_{\alpha\mu} - R_{\alpha\mu}] > 0, \qquad x \geq 0, \quad (4.25)$$

where

$$A_{\nu\alpha} = -\frac{\exp [i(\kappa_\alpha - \bar{\kappa}_\nu)x]}{i(\kappa_\alpha - \bar{\kappa}_\nu)}, \qquad \nu, \alpha = 1, 2, \cdots, n. \tag{4.26}$$

Now the matrix $A = [A_{\nu\alpha}]$ is positive definite since for arbitrary constants b_α,

$$\sum_{\nu,\alpha=1}^{n} b_\alpha \overline{b_\nu} A_{\nu\alpha} = \int_x^\infty \sum_{\nu,\alpha=1}^{n} b_\alpha \overline{b_\nu} \exp [i(\kappa_\alpha - \bar{\kappa}_\nu)t] \, dt$$
$$= \int_x^\infty \left| \sum_{\alpha=1}^{n} b_\alpha \exp (i\kappa_\alpha t) \right|^2 dt > 0, \tag{4.27}$$

unless the $b_\alpha \equiv 0$. Equation (4.25) implies that the matrix $A(I - R)$ is also positive definite. Therefore, by the product rule for determinants, det $(I - R)$ is real and positive for all $x \geq 0$. By Lemma 2 the proof holds for the dual matrix $(I + R)$.

5. Verification of the solution for the case of rational Y(ω). It will be instructive to show without reference to Marchenko's theory that $u(x, \omega)$ given by (4.14), where $f_\nu(x)$ satisfies (4.16), is a solution of (2.8) and that this solution also satisfies the boundary condition at $x = 0$ established by the rational function $u(0, \omega)$. Some of the discussion which follows parallels the work of Kay [3] for the case of real eigenfunctions $f_\nu(x)$ defined over the whole line.

First we show that $u(x, \omega)$ satisfies

$$\frac{d^2u(x, \omega)}{dx^2} + [\omega^2 - P(x)]u(x, \omega) = 0, \qquad x \geq 0, \tag{5.1}$$

where

$$P(x) = -2 \frac{d}{dx} \sum_{\nu=1}^{n} f_\nu(x) \exp (i\kappa_\nu x). \tag{5.2}$$

Apply the operator

$$L = \frac{d^2}{dx^2} + \omega^2 - P(x) \tag{5.3}$$

to (4.14). There results,

$$-Lu(x,\omega) = \left\{ \sum_{\nu=1}^{n} [M_\nu f_\nu(x)] \frac{\exp (i\kappa_\nu x)}{i(\kappa_\nu + \omega)} + 2 \frac{d}{dx} \sum_{\nu=1}^{n} f_\nu(x) \exp (i\kappa_\nu x) + P(x) \right\} \exp (i\omega x), \tag{5.4}$$

where

$$M_\nu = \frac{d^2}{dx^2} + \kappa_\nu^2 - P(x), \qquad \nu = 1, 2, \cdots, n. \tag{5.5}$$

Clearly (5.4) is zero for $x \geq 0$ if (5.2) holds and if

$$M_\nu f_\nu(x) = 0, \qquad \nu = 1, 2, \cdots, n. \tag{5.6}$$

In order to verify (5.6) apply the operator M_ν to the ν-th equation of (4.16). Then using (5.2) there results,

$$M_\nu f_\nu - \rho_\nu \sum_{\mu=1}^{n} [M_\mu f_\mu(x)] \frac{\exp [i(\kappa_\nu + \kappa_\mu)x]}{\kappa_\nu + \kappa_\mu} = 0, \qquad \nu = 1, 2, \cdots, n \tag{5.7}$$

or

$$\sum_{\mu=1}^{n} (\delta_{\nu\mu} - R_{\nu\mu}) M_\mu f_\mu(x) = 0, \qquad \nu = 1, 2, \cdots, n. \tag{5.8}$$

Since the matrix $(I - R)$ is nonsingular by Lemma 3, the only solution to (5.8) is given by (5.6). It follows that $u(x, \omega)$ in (4.14) is a solution of (5.1).

In order to demonstrate that $u(x, \omega)$ satisfies the boundary condition at $x = 0$, it will be sufficient to show that the ρ_ν and κ_ν in (4.16) are the residues and poles, respectively, of $1 - u(0, -\lambda)/u(0, \lambda)$ in the upper half-plane. That is, from (4.6),

$$1 - \frac{u(0, -\lambda)}{u(0, \lambda)} = \sum_{\nu=1}^{n} \frac{\rho_\nu}{\lambda - \kappa_\nu} + \sum_{\nu=1}^{n} \frac{\sigma_\nu}{\lambda - \mu_\nu}. \tag{5.9}$$

Since $u(0, \lambda)$ has no poles or zeros in the upper half-plane, there exists a set of n complex numbers g_μ such that

$$u(0, \lambda) = 1 + i \sum_{\mu=1}^{n} \frac{g_\mu}{\lambda + \kappa_\mu}. \tag{5.10}$$

Then for $\lambda = \kappa_\nu$ this may be written

$$\rho_\nu\left[\sum_{\mu=1}^{n} \frac{g_\mu}{\kappa_\nu + \kappa_\mu} - i\right] = -i\rho_\nu u(0, \kappa_\nu), \qquad \nu = 1, 2, \cdots, n, \tag{5.11}$$

where, from (5.9), the ρ_ν are given by

$$\rho_\nu = -\lim_{\lambda \to \kappa_\nu} (\lambda - \kappa_\nu)\left[\frac{u(0, -\lambda)}{u(0, \lambda)}\right], \qquad \nu = 1, 2, \cdots, n. \tag{5.12}$$

Therefore, (5.11) becomes

$$\begin{aligned} \rho_\nu\left[\sum_{\mu=1}^{n} \frac{g_\mu}{\kappa_\nu + \kappa_\mu} - i\right] &= i \lim_{\lambda \to \kappa_\nu} (\lambda - \kappa_\nu) u(0, -\lambda) \\ &= -i \lim_{\lambda \to -\kappa_\nu} (\lambda + \kappa_\nu) u(0, \lambda), \qquad \nu = 1, 2, \cdots, n. \end{aligned} \tag{5.13}$$

But from (5.10) the right side (5.13) is g_ν. It follows from (4.16) and the uniqueness of the solution for that system of equations that $g_\nu = f_\nu(0)$, $\nu = 1, 2, \cdots, n$. Therefore, $u(x, \omega)$ as given by (4.14) satisfies (5.9) as well as (5.1) at $x = 0$, and it is the only such solution which also satisfies the asymptotic condition (3.2a). A similar statement can be made for $v(x, \omega)$.

On the basis of the above results we can establish by means of Theorem 2 realizability conditions for the class of infinite lines considered in this section. We state this result as a theorem.

Theorem 3. Denote by $\mathfrak{M}$ the class of infinite lines having as solutions to (2.8) and (2.10), respectively, functions of the form

$$u(x, \omega) = \exp(i\omega x) - \sum_{\nu=1}^{n} f_\nu(x) \frac{\exp[i(\kappa_\nu + \omega)x]}{i(\kappa_\nu + \omega)}, \qquad x \geq 0, \tag{5.14}$$

$$v(x, \omega) = \exp(i\omega x) - \sum_{\nu=1}^{n} f_\nu^*(x) \frac{\exp[i(\kappa_\nu + \omega)x]}{i(\kappa_\nu + \omega)}, \qquad x \geq 0, \tag{5.15}$$

where $f_\nu(x)$ and $f_\nu^*(x)$ satisfy (4.16) and (4.17), respectively. In order that a function $Y(\omega)$ be realizable as the input admittance for a line in the class $\mathfrak{M}$ such that $Y(\omega) = v(0, \omega)/u(0, \omega)$, it is necessary and sufficient that $Y(\lambda)$ satisfy the conditions given in Theorem 2.

6. Construction of the characteristic impedance. For the class $\mathfrak{M}$ of lines considered in Section 5 it is possible to obtain a closed-form expression for $Z_0(x)$. Before deriving this expression we give a result due to Kay [6].

Lemma 4. Given the functions $f_\nu(x)$ and $f_\nu^*(x)$, $\nu = 1, 2, \cdots, n$, which are solutions of (4.16) and (4.17) respectively, then

$$\frac{d}{dx} \ln \{\det [I - R(x)]\} = \sum_{\nu=1}^{n} f_\nu(x) \exp (i\kappa_\nu x), \qquad x \geq 0 \tag{6.3a}$$

and

$$\frac{d}{dx} \ln \{\det [I + R(x)]\} = \sum_{\nu=1}^{n} f_\nu^*(x) \exp (i\kappa_\nu x), \qquad x \geq 0, \tag{6.3b}$$

where the matrices I and $R(x)$ are defined by (4.22) and (4.23).

Proof. As suggested by Kay, expand $\det [I - R(x)]$ by Cramer's rule. Thus, if $C_{\nu\mu}$ is the $\nu\mu$ cofactor of $I - R(x)$,

$$\begin{aligned} \frac{d}{dx} \det [I - R(x)] &= \sum_{\nu,\mu=1}^{n} \left[\frac{d}{dx} (\delta_{\nu\mu} - R_{\nu\mu})\right] C_{\nu\mu} \\ &= -i \sum_{\nu,\mu=1}^{n} \rho_\nu C_{\nu\mu} \exp [i(\kappa_\nu + \kappa_\mu)x]. \end{aligned} \tag{6.4}$$

But from (4.16) the $f_\mu(x)$ are given by

$$f_\mu(x) = -i \sum_{\nu=1}^{n} \rho_\nu \frac{C_{\nu\mu}}{\det (I - R)} \exp (i\kappa_\nu x). \tag{6.5}$$

Therefore,

$$\sum_{\mu=1}^{n} e^{i\kappa_\mu} f_\mu(x) = -i \sum_{\nu,\mu=1}^{n} \rho_\nu \frac{C_{\nu\mu}}{\det (I - R)} \exp [i(\kappa_\nu + \kappa_\mu)x], \tag{6.6}$$

which when substituted in (6.4) yields (6.3a). Equation (6.3b) follows by Lemma 2.

A synthesis formula for infinite lines in the class $\mathfrak{M}$ can now be derived. This result is contained in the following theorem.

Theorem 4. Let $Y(\lambda)$ be any function which satisfies the conditions of Theorem 2 and let $Z_0(\infty)$ be any positive constant. Then there is one and only one line in the class $\mathfrak{M}$ having $Y(\lambda)$ for its input admittance and for which the limiting value of its characteristic impedance is

$$\lim_{x \to \infty} Z_0(x) \equiv Z_0(\infty) < \infty. \tag{6.7}$$

The characteristic impedance of this line is given by

$$Z_0(x) = Z_0(\infty)\left\{\frac{\det [I - R(x)]}{\det [I + R(x)]}\right\}^2, \qquad x \geq 0, \tag{6.8}$$

where the matrices I and $R(x)$ are defined in Lemma 3. Furthermore, $Z_0(x)$ is a real, bounded, and continuous function which is positive for all $x \geq 0$.

Proof. The existence of a solution follows from Theorem 3 since the restriction on the limiting value of $Z_0(x)$ is not inconsistent with (3.2). Because of the fact that (2.8) and (2.10) are equivalent to the system of equations (2.6), the functions $u(x, \omega)$ and

$v(x, \omega)$ given by (4.14) and (4.15) must also satisfy (2.6). Substituting (4.14) and (4.15) in (2.6), this will be true if and only if, for $\nu = 1, 2, \cdots, n$,

$$\frac{df_\nu(x)}{dx} + p(x)f_\nu(x) + i\kappa_\nu f_\nu^*(x) = 0, \tag{6.9a}$$

$$\frac{df_\nu^*(x)}{dx} - p(x)f_\nu^*(x) + i\kappa_\nu f_\nu = 0, \tag{6.9b}$$

where

$$p(x) = \sum_{\nu=1}^{n} [f_\nu(x) - f_\nu^*(x)] \exp(i\kappa_\nu x). \tag{6.10}$$

Equation (6.8) follows from (2.7) and (6.10) after applying Lemma 4. The constant of integration is given by (6.7) since

$$\lim_{x\to\infty} \det [I \pm R(x)] = 1. \tag{6.11}$$

There can be only one line for which $Z_0(x)$ satisfies (6.7) because $p(x)$ is uniquely determined by $f_\nu(x)$ and $f_\nu^*(x)$. The last statement in the theorem follows from Lemma 3 and (6.8).

In Theorem 4 it would have been possible to specify $Z_0(0)$ instead of $Z_0(\infty)$. The connection between these two constants can be shown as follows. Set $\omega = 0$ in (2.6) and solve for $p(x)$. There results,

$$2p(x) = \frac{d}{dx} \ln \frac{v(x, 0)}{u(x, 0)}. \tag{6.12}$$

It follows from (2.7) and (3.6) that

$$Z_0(x) = Z_0(\infty)y(x, 0). \tag{6.13}$$

In particular,

$$Z_0(0) = Z_0(\infty)Y(0). \tag{6.14}$$

7. An example. Consider the input admittance,

$$Y(is) = \frac{s^2 + s + 4}{s^2 + 2s + 1}. \tag{7.1}$$

Taking the real part of (7.1) with $s = -i\omega$,

$$G(\omega) = \frac{\omega^4 - 3\omega^2 + 4}{\omega^4 + 2\omega^2 + 1}. \tag{7.2}$$

It is easily verified that the conditions given in Theorem 2 are satisfied for this case. Factoring (7.2) according to (3.14) we find

$$u(0, \omega) = \frac{4(\omega + i)^2}{(2\omega + 7^{1/2} + i)(2\omega - 7^{1/2} + i)}. \tag{7.3}$$

The poles of $1 - u(0, -\lambda)/u(0, \lambda)$ in the upper half-plane are

$$\kappa_1 = \tfrac{1}{2}(7^{1/2} + i); \qquad \kappa_2 = \tfrac{1}{2}(-7^{1/2} + i), \tag{7.4}$$

and, referring to (4.6), the residues at these poles are

$$\rho_1 = \frac{-1 + i7^{1/2}}{7^{1/2}}\left(\frac{3 - i7^{1/2}}{1 - i3\cdot 7^{1/2}}\right); \qquad \rho_2 = \frac{1 + i7^{1/2}}{7^{1/2}}\left(\frac{3 + i7^{1/2}}{1 + i37^{1/2}}\right). \tag{7.5}$$

After forming the matrix $R(x) = [R_{\nu\mu}]$, where the elements are defined in (4.23), the synthesis formula (6.8) yields,

$$Z_0(x) = Z_0(\infty)\left[\frac{4\cdot 7^{1/2} - 7^{1/2}\exp(-2x) + (3\sin 7^{1/2}x + 7^{1/2}\cos 7^{1/2}x)\exp(-x)}{4\cdot 7^{1/2} - 7^{1/2}\exp(-2x) - (3\sin 7^{1/2}x + 7^{1/2}\cos 7^{1/2}x)\exp(-x)}\right]^x, \quad x \geq 0, \tag{7.6}$$

where $Z_0(\infty)$ is an arbitrary constant. The square root of this function is plotted in Fig. 1 for the case where $Z_0(\infty) = 1$.

It is interesting to compare this exact solution to the synthesis problem with the approximate result obtained by classical perturbation analysis. It can be shown that for the infinite line the input reflection coefficient is approximately given by [1]

$$\Gamma(\omega) = \int_0^\infty p(x)\exp(2i\omega x)\,dx, \tag{7.7}$$

where an $\exp(-i\omega t)$ time dependence has been assumed and the distance variable is normalized according to (2.4). Assuming that $p(x) = 0,\ x < 0$, the Fourier inversion theorem yields

$$p(x) = \frac{1}{\pi}\int_{-\infty}^\infty \Gamma(\omega)\exp(-2i\omega x)\,d\omega. \tag{7.8}$$

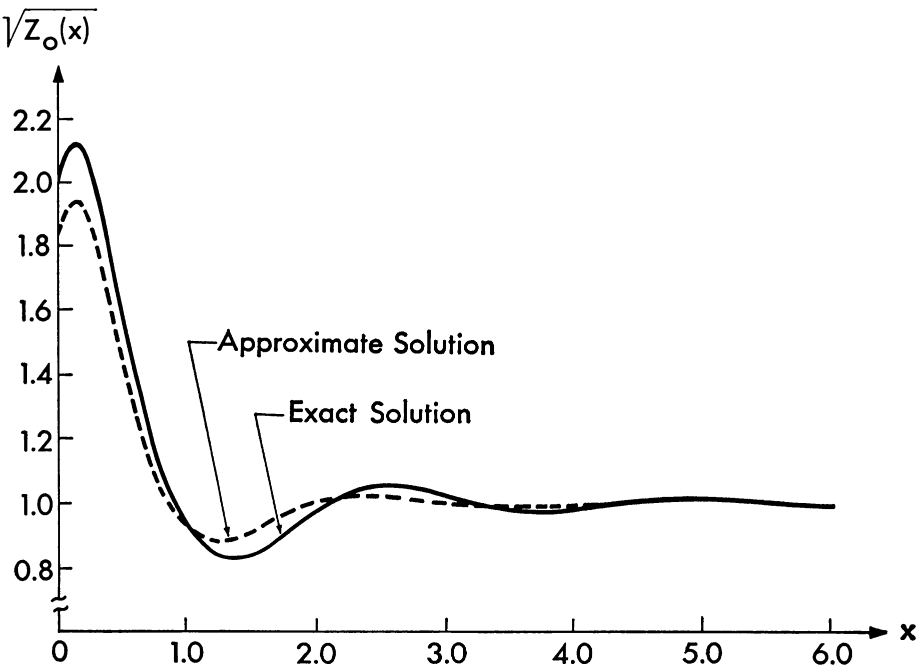

Fig. 1. Characteristic impedance versus distance for a given input admittance.

Since $Y(\omega)$ as defined in (3.11) is normalized with respect to the characteristic admittance at the input, the reflection coefficient is simply

$$\Gamma(\omega) = \frac{1 - Y(\omega)}{1 + Y(\omega)}. \tag{7.9}$$

The integral in (7.8) is readily evaluated by contour integration. For the admittance function given in (7.1) we obtain, after integration with respect to x,

$$Z_0(x) = Z_0(\infty) \exp\left\{\left[\frac{6}{5}\cos\frac{31^{1/2}}{2}x + \frac{38}{5\cdot 31^{1/2}}\sin\frac{31^{1/2}}{2}x\right]\exp\left(-\frac{3}{2}x\right)\right\}. \tag{7.10}$$

The square root of this function is also plotted in Figure 1 for purposes of comparison. It can be seen that the general behavior of the two functions is similar, more so than might be expected from their analytical forms.

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