Corollary. If the c_i and F_h are continuous in Theorem 4, then the network problem has a solution.

In conclusion, it may be of interest to describe the mechanical analog of the network problem, whose potential (strain) energy function is $V(\mathbf{u})$. In this analog, the nodes are represented by smooth *rods*, whose displacements from given positions are the u_i. These rods are constrained to slide along the u-axis, a constraint which can be imposed by inserting the rods in a long smooth tube. These rods can then be imagined as joined by one or more *springs*, for each link a_i whose total *stress* c_i (Δu_i) depends on the relative displacement of the rods joined by the link a_i. This acts equally and in opposite directions as a force on these rods, as prescribed by the ϵ_{ki} (thus *stress* or *force* is the mechanical analog of *current*). At internal nodes, the condition for equilibrium is that the sum (resultant) of the forces be zero. At boundary nodes, one can suppose externally attached springs exert a force F_h which depends on the displacement of the rod A_h.

This model is very similar to that suggested by Duffin [2], provided the junction points are constrained to move in one dimension. However, Duffin's "force functions" $f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$ were assumed to be defined only for non-negative arguments, to satisfy $f(\mathbf{0}) = \mathbf{0}$, and to be unbounded. This would correspond roughly to assuming the $c_i(\Delta u_i)$ to be *odd* functions tending to infinity with $|\Delta u_i|$, and these assumptions seem to be unnecessary for the existence and uniqueness theorems stated above. Duffin also considers only the Dirichlet problem (all F_h vanish).

Duffin's hypotheses are, of course, appropriate for networks of springs under tension in more than one dimension.

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AN INTEGRAL EQUATION OCCURRING IN PLASMA OSCILLATIONS*

By R. S. B. ONG (*The University of Michigan*)

The stability of single hump velocity distributions in a collisionless hot plasma without magnetic field is usually shown by considering the linearized Landau-Vlasov self-consistent set of equations as an initial value problem. The Laplace—or the one-sided Fourier transform with respect to the time variable is then commonly used in the analysis. In this method, one needs the analytic continuation of the function considered beyond its original domain of definition in order to evaluate its inverse transform by complex contours. This introduces a certain degree of artificiality in the analysis. The purpose of this article is to show that one can deduce the stability of single hump velocity distributions by means of an argument which avoids this difficulty.

The one-dimensional linearized problem of longitudinal oscillations of a collisionless hot plasma without an external magnetic field is described by the following well-known self-consistent set of equations:

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$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{df_0(u)}{du} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = -\frac{e}{\epsilon_0}\int_{-\infty}^{\infty} f\, du, \tag{2}$$

where: $f_0(u)$ is the initial equilibrium velocity distribution function of the electrons, $f(x, u, t)$ the perturbation in the electron velocity distribution function, e the electron charge, m the electron mass, and ϵ_0 the dielectric constant in a vacuum.

There is no need to discuss the background of equations (1) and (2); this may be found in numerous places in the literature [1]. It is assumed here that the ions form a static uniformly smeared background which accounts for the electric neutrality of the plasma.

As the linear differential equations (1) and (2) do not contain the coordinate explicitly, the velocity distribution function $f(x, u, t)$ may be expanded into a Fourier integral with respect to the coordinates. The differential equation can then be written for every Fourier component separately, i.e., we consider a solution of the form $f_k(u, t)e^{ikx}$. Further, we shall omit the index k in f_k so that $f(u, t)$ will denote the Fourier component of the distribution function in question. Equations (1) and (2) then become

$$\frac{\partial f}{\partial t} + ikuf = \frac{eE}{m}\frac{df_0(u)}{du}, \tag{3}$$

$$ikE = -\frac{e}{\epsilon_0}\int_{-\infty}^{\infty} f\, du. \tag{4}$$

After eliminating the electric field E we obtain

$$\frac{\partial f}{\partial t} + ikuf = \frac{i}{k}\frac{e^2}{\epsilon_0 m}\frac{df_0(u)}{du}\int_{-\infty}^{\infty} f\, du. \tag{5}$$

The solution of this equation may be written in the form

$$f(u, t) = f(u, 0)e^{-ikut} + e^{-ikut}\left\{\int_0^t e^{iku\tau}\frac{i}{k}\frac{e^2}{\epsilon_0 m} f_0'(u)\int_{-\infty}^{\infty} f(u, \tau)\, du\, d\tau\right\}, \tag{6}$$

where $f(u, 0)$ is the perturbation distribution function at time $t = 0$.

Integrating equation (6) with respect to u from $-\infty$ to $+\infty$ and interchanging the order of integration, which may be justified by Fubini's theorem, we get:

$$h(t) = \int_{-\infty}^{\infty} f(u, 0)e^{-iktu}\, du + \int_0^t \left\{\frac{i}{k}\frac{e^2}{\epsilon_0 m}\int_{-\infty}^{\infty} f_0'(u)e^{-iku(t-\tau)}\, du\right\}h(\tau)\, d\tau, \tag{7}$$

where

$$h(t) \equiv \int_{-\infty}^{\infty} f(u, t)\, du.$$

Let

$$g(t) \equiv \int_{-\infty}^{\infty} f(u, 0)e^{-iktu}\, du,$$

and

$$k(t) = \frac{i}{k}\frac{e^2}{\epsilon_0 m}\int_{-\infty}^{\infty} f_0'(u)e^{-iktu}\, du.$$

Equation (7) may then be written as:

$$h(t) = g(t) + \int_0^t k(t - \tau)h(\tau)\, d\tau. \tag{8}$$

This is a Volterra type integral equation of the second kind to be satisfied by the function $h(t)$ which according to (4), is proportional to the electric field E. It is well-known that the solution of the integral equation (8) may be written in the form:

$$h(t) = g(t) + \int_0^t m(t - \tau)g(\tau)\, d\tau, \tag{9}$$

where the resolvent kernel $m(t)$ is determined by

$$m(t) = k(t) + \int_0^t k(t - \tau)m(\tau)\, d\tau. \tag{10}$$

The problem now is to determine the class of kernels $k(t)$ such that the solution $h(t)$ to the integral equation (8), and consequently the electric field E, remains bounded for all $t \in [0, \infty]$ for given $g(t)$. This will enable us to find the class of initial velocity distribution functions $f_0(u)$ which are stable to initial perturbations $f(u, 0)$.

First we observe that $| g(t) | \leq \int_0^\infty | f(u, 0) | \, du$ and it is reasonable to assume that we may restrict ourselves physically to initial perturbations such that $| g(t) | \leq M < \infty$ for $t \in [0, \infty]$. We may then write

$$h(t) \leq | g(t) | + \left| \int_0^t m(t - \tau)g(\tau)\, d\tau \right|$$

or

$$h(t) \leq M + \int_0^t | m(t - \tau) | M\, d\tau = M\left(1 + \int_0^t | m(\tau) | \, d\tau\right).$$

Hence, a *sufficient* condition for $h(t)$ to be bounded for all $t \in [0, \infty]$ is that

$$\int_0^\infty | m(\tau) | \, d\tau < \infty .$$

To transform this into a condition involving the kernel $k(t)$ explicitly we make use of the following theorem by Paley and Wiener [2].

Theorem. Consider the integral equation

$$m(t) = k(t) + \int_0^t k(t - \tau)m(\tau)\, d\tau.$$

A *necessary and sufficient* condition for $\int_0^\infty | m(t) | \, dt < \infty$ is that

$$\int_0^\infty k(t)e^{-st}\, dt \neq 1; \qquad R(s) \geq 0.$$

Consequently, we arrive at the following theorem.

Theorem. A *sufficient* condition for the solution $h(t)$ of the integral equation

$$h(t) = g(t) + \int_0^t k(t - \tau)h(\tau)\, d\tau \tag{11}$$

with $g(t)$ bounded for all $t \varepsilon [0, \infty]$ is that the kernel $k(t)$ satisfies the relation

$$\int_0^\infty e^{-st}k(t)\,dt \neq 1; \qquad \text{Re}\,(s) \geq 0.$$

Interpreting this result in terms of the definition of $k(t)$, we arrive at the following sufficient condition for stability of the oscillations:

$$\int_0^\infty e^{-st}\,\frac{i}{k}\,\frac{e^2}{\epsilon_0 m}\int_{-\infty}^\infty f_0'(u)e^{-iktu}\,du\,dt \neq 1; \qquad \text{Re}\,(s) \geq 0.$$

Interchanging the order of integration, we obtain

$$\frac{i}{k}\,\frac{e^2}{\epsilon_0 m}\int_{-\infty}^\infty f_0'(u)\,\frac{1}{s + iku}\,du \neq 1; \qquad \text{Re}\,(s) \geq 0 \tag{12}$$

However, this is identical to the well-known *dispersion relation* in plasma oscillations.

It is now not difficult to show that all initial velocity distribution functions $f_0(u)$ having a single hump satisfy the condition expressed by (12) and hence they lead to bounded solutions $h(t)$ to the integral equation (11). These distribution functions are therefore called stable since the resulting electric field E will then be bounded for all time. To show this, we let $f_0(u)$ have a maximum at some finite velocity U such that $f_0'(u) \geq 0$ for $u < U$ and $f_0'(u) \leq 0$ for $u > U$ with $f_0'(U) = 0$ and $f_0'(u) \not\equiv 0$. By letting $s = x + iy$, with $x > 0$, we can split the left-hand side of equation (12) into real and imaginary parts and then the stability condition may be written as

$$\int_{-\infty}^\infty f_0'(u)\,\frac{1}{x^2 + (y + ku)^2}\,du \neq 0 \qquad (x > 0) \tag{13.a}$$

and

$$1 - \frac{1}{k}\,\frac{e^2}{\epsilon_0 m}\int_{-\infty}^\infty f_0'(u)\,\frac{y + ku}{x^2 + (y + ku)^2}\,du \neq 0. \qquad (x > 0) \tag{13.b}$$

Suppose that for fixed y there exists an x such that

$$\int_{-\infty}^\infty f_0'(u)\,\frac{1}{x^2 + (y + ku)^2}\,du = 0. \tag{14}$$

Then for this same choice of (x, y) it is impossible that the left-hand side of (13b) can vanish. For if Eq. (14) is true for some (x, y) then any multiple of the left-hand side will still be zero; in particular, instead of (13b), we may write

$$1 - \frac{1}{k}\,\frac{e^2}{\epsilon_0 m}\int_{-\infty}^\infty f_0'(u)\,\frac{y + ku}{x^2 + (y + ku)^2}\,du + \frac{1}{k}\,\frac{e^2}{\epsilon_0 m}\int_{-\infty}^\infty f_0'(u)\,\frac{y + kU}{x^2 + (y + ku)^2}\,du \neq 0,$$

that is,

$$1 - \frac{e^2}{\epsilon_0 m}\int_{-\infty}^\infty f_0'(u)\,\frac{u - U}{x^2 + (y + ku)^2}\,du \neq 0$$

or

$$1 - \frac{e^2}{\epsilon_0 m}\int_{-\infty}^U f_0'(u)\,\frac{u - U}{x^2 + (y + ku)^2}\,du - \frac{e^2}{\epsilon_0 m}\int_U^\infty f_0'(u)\,\frac{u - U}{x^2 + (y + ku)^2}\,du \neq 0.$$

Since by hypothesis $f_0'(u) \not\equiv 0$ and also $f_0'(u) \geq 0$ in the first integral and $f_0'(u) \leq 0$ in the second integral therefore both integrals in the above equation are negative and so the inequality is always satisfied.

Thus, we have shown that all single hump initial velocity distribution functions are stable. This is the same result which is usually obtained by means of a Laplace- or a one-sided Fourier transform with respect to the time variable. A notable feature of the present analysis is that it need not use any of the artificial mathematical assumptions which are, for example, present in regard to the process of analytic continuation when the method of integral transforms is used.

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A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING*

By RICHARD BELLMAN AND ROBERT KALABA (*The Rand Corporation, Santa Monica, Cal.*)

1. Introduction. Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian $H = H(q, p)$. The equations of motion are

$$\dot{q} = H_p , \tag{1.1}$$

$$-\dot{p} = H_q , \qquad 0 \leq t \leq T. \tag{1.2}$$

As boundary conditions we specify

$$q(0) = 0, \tag{1.3}$$

$$p(T) = c. \tag{1.4}$$

If our aim is to solve the system of equations (1.1) and (1.2), subject to the boundary conditions in (1.3) and (1.4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time T, $q(T)$, so that we shall have a complete set of conditions at time T. This would enable us to integrate the system (1.1) and (1.2) numerically, subject to initial values. Toward this end we introduce the function

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